CHAPTER 6

Conormal distributions at a submanifold

Next we develop the theory of conormal distributions associated to a p-submanifold of a manifold with corners. We start by considering the space $I(X,Y)$ of distributions which are 'smooth up to the boundary' of X and are conormal with respect to the interior p-submanifold, Y. This discussion is subsequently generalized to the spaces of distributions conormal to Y with polyhomogeneous expansions at the boundary of X and is also generalized to the allow Y to be a boundary p-submanifold or even a system of cleanly intersecting p-submanifolds. The effect of the basic operations of taking products, push-forward and pull-back, on these conormal distributions is then analyzed. In this chapter the composition formula for pseudodifferential operators on a compact manifold without boundary is derived, as an application of the results on operations and as a guide to the discussion in later chapters of other rings of pseudodifferential operators.

6.1. Conormal distributions in the interior

We shall first recall the properties of conormal distributions associated to an embedded submanifold of a (compact) manifold without boundary. These were formalized by Hörmander [?], see [?] for a full treatment or [?] for a discussion more closely parallel to the remarks here. In fact since rather little extra effort is involved we shall actually work with Y a closed interior p-submanifold of a compact manifold with corners.

Set

$$\mathcal{V}_E(X,Y) = \{V \in \mathcal{C}^\infty(X,TX); V \text{ is tangent to } Y\}\,. \tag{6.1.1}$$

Conormal distributions with respect to Y, smooth up to the boundary of X, are those distributions having stable regularity under the repeated application of elements of $\mathcal{V}_E(X,Y)$:

$$\begin{gathered} I^{[s]}(X,Y) = \left\{u \in H^s(X); \mathcal{V}_E(X,Y)^k u \subset H^s(X)\ \forall\ k\right\} \\ I^*(X,Y) = \bigcup_{s\in\mathbb{R}} I^{[s]}(X,Y). \end{gathered} \tag{6.1.2}$$

Notice that if $Y = X$ then $\mathcal{V}_E(X,X) = \mathcal{V}_E(X)$ and so $I^{[s]}(X,X) = \mathcal{C}^\infty(X)$ for every s. Since the definition (6.1.2) is local this shows

$$\text{sing supp}(u) \subset Y \quad \forall\ u \in I^{[*]}(X,Y). \tag{6.1.3}$$

Exercise 6.1.30. Let X be a compact manifold with boundary and $2X$ its double, so ∂X is a closed submanifold of $2X$. Show that for any $s \in \mathbb{R}$ the restriction map (3.3.6) gives a surjective map

$$I^{[s]}(2X,\partial X) \twoheadrightarrow \mathcal{A}^{[s]}(X). \tag{6.1.4}$$

The spaces $I^{[s]}(X,Y)$ are clearly $\mathcal{C}^\infty$modules, i.e.

$$\mathcal{C}^\infty(X)\cdot I^{[s]}(X,Y)=I^{[s]}(X,Y)\quad\forall\ s \tag{6.1.5}$$

and the action of differential operators is the same as for the Sobolev spaces:

LEMMA 6.1.1. *For any closed interior p-submanifold of a compact manifold with corners*

$$\mathrm{Diff}^m(X)\cdot I^{[s]}(X,Y)\subset I^{[s-m]}(X,Y)\quad\forall\ m,s. \tag{6.1.6}$$

PROOF. It suffices to show that

$$\mathrm{Diff}^m(X)\cdot\mathcal{V}_E(X,Y)=\mathcal{V}_E(X,Y)\cdot\mathrm{Diff}^m(X) \tag{6.1.7}$$

and to use the mapping property for Sobolev spaces

$$\mathrm{Diff}^m(X)\cdot H^s(X)\subset H^{s-m}(X)\quad\forall\ m,s.$$

Moreover (6.1.7) follows from the case $m=1$ by iteration and in case $m=1$ is just the fact that $\mathcal{V}_E(X)$ is a Lie algebra. □

Since we assume Y to be an interior p-submanifold, near any point $p\in Y$ we can always find local coordinates (x,z,y), $x=(x_1,\dots,x_k)$, $z=(z_1,\dots,z_d)$, $y=(y_1,\dots,y_N)$, $n=k+d+N$, in some open neighbourhood Ω of p such that the local boundary hypersurfaces through p are given by $x_i=0$, $i=1,\dots,k$ and

$$Y\cap\Omega=\{z=0\}. \tag{6.1.8}$$

The vector fields

$$\frac{\partial}{\partial x_i},\ \frac{\partial}{\partial y_j}\ \text{and}\ z_l\frac{\partial}{\partial z_k}\ \text{span}\ \mathcal{V}_E(X,Y) \tag{6.1.9}$$

in the coordinate patch.

Recall from Chapter 4 that $a\in\mathcal{C}^\infty(\mathbb{R}^N_y\times\mathbb{R}^d_\zeta)$ is a symbol of order M if

$$\begin{gathered}\text{For each } K\subset\subset\mathbb{R}^N,\ \alpha\in\mathbb{N}^N,\ \beta\in\mathbb{N}^d \text{ and } \gamma\in\mathbb{N}^k\ \exists\ C_{k,\alpha,\beta,\gamma}\text{ s.t.}\\ |D^\gamma_x D^\alpha_y D^\beta_\zeta a(x,y,\zeta)|\le C_{k,\alpha,\beta,\gamma}(1+|\zeta|)^{M-|\beta|}\ \text{on}\ K\times\mathbb{R}^d.\end{gathered} \tag{6.1.10}$$

PROPOSITION 6.1.1. *For any $\phi\in\mathcal{C}^\infty_c(\Omega)$ the distribution obtained in local coordinates by inverse Fourier transformation of a is conormal:*

$$u=(2\pi)^{-d}\phi(z,y)\int e^{iz\cdot\zeta}a(x,y,\zeta)d\zeta\in I^{[s]}(X,Y)\quad\forall\ s<-M-\frac{d}{2} \tag{6.1.11}$$

where Y is given by (6.1.8). *Conversely for an interior p-submanifold, Y, in local product coordinates* (6.1.8)

$$\begin{gathered}u\in I^{[s]}(X,Y)\cap\mathcal{C}^{-\infty}_c(\Omega)\Longrightarrow a(x,y,\zeta)=\int e^{-iz\cdot\zeta}u(x,y,z)dz\\ \textit{is a symbol of order } M\quad\forall\ M>-s-\frac{d}{2}.\end{gathered} \tag{6.1.12}$$

PROOF. The conditions, (6.1.10), defining a symbol imply in particular that

$$(1+|\zeta|^2)^{-s/2}a(x,y,\zeta)\in L^2(\mathbb{R}^d)\ \text{provided}\ s<-M-\frac{d}{2} \tag{6.1.13}$$

depends smoothly on x and y. Thus the inverse Fourier transform, (6.1.11), does define an element of $H^s_c(\Omega) \subset H^s(X)$. This is the condition in (6.1.2) for $k = 0$. Again from (6.1.10) it follows that with u defined by (6.1.11)

$$D^\gamma_x z^\nu D^\mu_z D^\alpha_y u \in H^s(X) \text{ if } |\nu| \geq |\mu|.$$

In view of (6.1.9) this gives the conditions in (6.1.2) for all k proving (6.1.11).

To prove the converse consider first the case $p = n$, so that Y is just a point and (6.1.8) reduces it to $\{0\} \subset \mathbb{R}^d$. Then the condition on u in (6.1.12) can be written

$$u \in H^s(\mathbb{R}^n) \text{ with } D^\beta_z z^\alpha u \in H^s(\mathbb{R}^n) \ \forall \ |\beta| \leq |\alpha| \tag{6.1.14}$$

Taking the Fourier transform to define a as in (6.1.12) these conditions imply that

$$\zeta^\beta D^\alpha_\zeta a(\zeta) \in (1 + |\zeta|^2)^{-s/2} L^2(\mathbb{R}^d), \text{ if } |\alpha| \geq |\beta|. \tag{6.1.15}$$

In particular $a \in \mathcal{C}^\infty(\mathbb{R}^d)$ but the growth rate at infinity is more important.

Making the stereographic compactification of $\mathbb{R}^d$ to $\mathbb{B}^d$, as discussed in ?? it follows that $L^2(\mathbb{R}^d)$ becomes $\rho^{-d/2} L^2_b(\mathbb{B}^d)$ and the conditions (6.1.15) just reduce to

$$\mathcal{V}_b(\mathbb{B}^d)^k a \in \rho^{-d/2} L^2_b(\mathbb{B}^d) \ \forall \ k \Longleftrightarrow a \in r^{-d/2 + \frac{1}{2}} \mathcal{A}^{[0]}(\mathbb{B}^d). \tag{6.1.16}$$

From ?? we conclude that $a \in S^M(\mathbb{R}^d)$ provided $M > -s - d/2$. This proves (6.1.12) for $d = n$.

The general case follows by considering the additional variables x and y as smooth parameters. □

All the local stuff should go here.

6.2. Symbol filtration

The significant difference between the way we treat conormal distributions at Y and the way conormal distributions at the boundary are treated in Chapter 4 is that we wish to filter the former by the order of the symbol in (6.1.11). To make sure that this is sensible we need to check the invariance under coordinate transformations. To do so we consider first the special case of Y as the zero section of its own normal bundle NY in X. Then we have a global Fourier transform

$$\mathcal{F} : \mathcal{C}^{-\infty}_c(NY) \longrightarrow \mathcal{C}^{-\infty}(N^*Y; \Omega_{\text{fibre}}) \tag{6.2.1}$$

where the range can be characterized using the Paley-Wiener theorems. Here Ω_{fibre} is the bundle of fibre densities on N^*Y and (6.2.1) is just the Fourier transform on each fibre of NY. In local coordinates this just reduces to (6.1.12) and the inverse takes the form (6.1.11). We shall use stereographic projection to compactify N^*Y so that it can be identified with a d-ball bundle, P^*Y, over Y. This is a compact manifold with corners (with one more boundary component than Y) and we note that the space of symbols of order M is identified with the restriction to P^*Y of $\mathcal{A}^{-M}(2_X P^*Y)$, the double over all boundaries of X. Since the fibre Fourier transform (6.2.1) is invertible on fibre-tempered distributions and these are identified with extendible distributions on P^*Y the inverse Fourier transform gives a map

$$\mathcal{F}^{-1}_c : \mathcal{C}^{-\infty}(P^*Y; \Omega_{\text{fibre}}) \longrightarrow \mathcal{C}^{-\infty}(NY). \tag{6.2.2}$$

DEFINITION 6.2.19. If NY is the normal bundle to a closed interior p-submanifold of a compact manifold with corners and P^*Y is the stereographic compactification of its dual then, regarding Y as the zero section, we set

$$\begin{aligned} I^{[s]}(NY,Y) &= \mathcal{C}^\infty(NY) + I_c^{[s]}(NY,Y) \\ I^m(NY,Y) &= \mathcal{C}^\infty(NY) + \mathcal{F}^{-1}\left[\mathcal{A}^{-m-\frac{n}{4}+\frac{d}{2}}(2_X P^*Y;\Omega_{\text{fibre}}) \restriction_{P^*Y}\right] \end{aligned} \tag{6.2.3}$$

and

$$\begin{aligned} I^m_{\text{KN}}(NY,Y) &= \mathcal{C}^\infty(NY) + \mathcal{F}^{-1}\left[\mathcal{A}_{\text{phg}}^{-m-\frac{n}{4}+\frac{d}{2},0}(P^*Y;\Omega_{\text{fibre}})\right] \\ &\Longrightarrow I^m\text{KN}(NY,Y) \subset I^m(NY,Y) \end{aligned} \tag{6.2.4}$$

where the suffix KN (for 'Hörmander, Kohn-Nirenberg') is used in place of polyhomogeneous.

It is straightforward to check that the local inclusion (6.1.11) can almost be reversed, leading to the following sandwiching result:

$$I_c^{[-m-\frac{n}{4}+\epsilon]}(NY,Y) \subset I_c^m(NY,Y) \subset I_c^{[-m-\frac{n}{4}-\epsilon]}(NY,Y) \quad \forall\ \epsilon > 0. \tag{6.2.5}$$

We use this to check the coordinate invariance of the filtration by order.

LEMMA 6.2.1. *For each $m \in \mathbb{R}$ the spaces $I^m(NY,Y)$ and $I^m_{\text{KN}}(NY,Y)$ are invariant under any diffeomorphism which fixes the zero section to first order.*

PROOF. Recall from the discussion at the end of §2.10 that a diffeomorphism fixing the zero section satisfies (2.10.11), with W_t as in (2.10.12). Since the vector fields W_j are tangent to Y we easily conclude, by expanding their coefficients in Taylor series at Y, that

$$\begin{aligned} W_j &: I^m(NY,Y) \longrightarrow I^m(NY,Y) \\ W_j &: I^m_{\text{KN}}(NY,Y) \longrightarrow I^m_{\text{KN}}(NY,Y). \end{aligned} \tag{6.2.6}$$

Then multiplication by ψ_j turns a distribution of order m into one of order $m-1$ in each case. It therefore follows from (2.10.11) that

$$\frac{d}{dt} f_t^* u = f_t^* v_t,\ \ v_t \in I^{m-1}(NY,Y). \tag{6.2.7}$$

From (6.2.5), $v_t \in I^{[-m+1-\frac{n}{4}-\epsilon]}(NY,Y)$. Since these Sobolev based conormal spaces are certainly invariant under diffeomorphism, using (6.2.5) in the opposite directions shows that

$$f^*u - u = \int_0^1 \frac{d}{dt} f_t^* u \in I^{m-1+\epsilon}(NY,Y) \subset I^m(NY,Y). \tag{6.2.8}$$

For the polyhomogeneous case we use iteration to conclude that

$$f^* : I^m_{\text{KN}}(NY,Y) \longrightarrow I^m_{\text{KN}}(NY,Y) + I^{m-q}(NY,Y) \quad \forall\ q. \tag{6.2.9}$$

Since $\bigcap_q I^{m-q}(NY,Y) = \mathcal{C}^\infty(NY)$ invariance again follows. □

Note that this proof actually shows

$$\begin{aligned} f^*u - u &\in I^{m-1}(NY,Y) \quad \forall\ u \in I^m(NY,Y) \\ f^*u - u &\in I^{m-1}_{\text{KN}}(NY,Y) \quad \forall\ u \in I^m_{\text{KN}}(NY,Y) \end{aligned} \tag{6.2.10}$$

This allows us to define the conormal spaces in case Y is a closed interior p-submanifold by setting, for *any* normal fibration f

$$I^m(X,Y) = \{f^{-1^*}u; u \in I^m(NY,Y) \cap \mathcal{C}_c^{-\infty}(\Gamma)\} + \mathcal{C}^\infty(X)$$
$$I^m_{\mathrm{KN}}(X,Y) = \{f^{-1^*}u; u \in I^m_{\mathrm{KN}}(NY,Y) \cap \mathcal{C}_c^{-\infty}(\Gamma)\} + \mathcal{C}^\infty(X) \quad (6.2.11)$$

It is also straightforward to check that the spaces on NY, and hence in the general case, are invariant under any diffeomorphism preserving Y.

Every conormal distribution associated to Y is of the form (6.1.11) in some local coordinate system. In fact (6.1.11) holds in *any* local coordinates in which (6.1.8) holds. This allows each $I^m(X,Y)$ to be topologized by the seminorms of $\mathcal{C}^\infty(X \setminus Y)$ and the best constants in (6.1.10) in a covering by local coordinates, with the ϕ_i being a partition of unity subordinate to this covering near Y. With this topology $I^m(X,Y)$ is a Fréchet space. Moreover

$$\mathcal{C}^\infty(X) \hookrightarrow I^m(X,Y) \text{ is dense in the topology of } I^{m+\epsilon}(X,Y)\ \forall\ \epsilon > 0. \quad (6.2.12)$$

Now from (6.2.5)

$$I^*(X,Y) = \bigcup_{m\in\mathbb{R}} I^m(X,Y)$$
$$I^{-\infty}(X,Y) = \bigcap_{m\in\mathbb{R}} I^m(X,Y) = \mathcal{C}^\infty(X). \quad (6.2.13)$$

Moreover these spaces admit asymptotic summation. Suppose $u_i \in I^{m_i}(X,Y)$ is a sequence with $m_i \longrightarrow -\infty$ as $i \longrightarrow \infty$. Then there exists an element $u \in I^M(X,Y)$, $M = \sup m_i$, such that

$$u - \sum_{i\le j} u_i \in I^{M(j)}(X,Y),\ M(j) = \sup_{i>j} m_i \quad \forall\ j. \quad (6.2.14)$$

Clearly this relation, written,

$$u \sim \sum_i u_i, \quad (6.2.15)$$

determines u up to a $\mathcal{C}^\infty$term. In the polyhomogeneous case of course we need to assume that $u_i \in I^{m_i}_{\mathrm{KN}}(X,Y)$ with $m - m_i \in \mathbb{Z}$.

Notice that (6.2.10) implies that there is a well-defined *symbol map:*

$$I^m(NY,Y) \xrightarrow{\sigma_m} \mathcal{A}^{\{-m-\frac{n}{4}+\frac{d}{2}\}}(P^*Y;\Omega_{\mathrm{fibre}}) \quad (6.2.16)$$

where $\mathcal{A}^{\{a\}}(P^*Y) = \mathcal{A}^a(P^*Y)/\mathcal{A}^{a+1}(P^*Y)$. In fact this extends directly to the general spaces

$$I^m(X,Y) \xrightarrow{\sigma_m} \mathcal{A}^{\{-m-\frac{n}{4}+\frac{d}{2}\}}(P^*Y;\Omega_{\mathrm{fibre}}). \quad (6.2.17)$$

There is also no difficulty in extending the discussion to conormal sections of vector bundles, the symbol map becomes

$$I^m(X,Y;G) \xrightarrow{\sigma_m} \mathcal{A}^{\{-m-\frac{n}{4}+\frac{d}{2}\}}(P^*Y;\Omega_{\mathrm{fibre}} \otimes \pi^*G). \quad (6.2.18)$$

where π^*G is the lift to P^*Y of the restriction of G to Y, the zero section. A standard simplifying assumption is to work with half-densities. In the present case we choose to work instead with b-half-densities. The b-half-density bundle on X pulled-back to N^*Y naturally decomposes as

$$\pi^*\Omega_b^{\frac12} \cong \Omega_b^{\frac12}(Y) \otimes \Omega_{\mathrm{fibre}}^{\frac12} \otimes \Omega_{\mathrm{fibre}}^{-1}. \quad (6.2.19)$$

Using this and then replacing the translation-invariant bundle Ω_{fibre} by the b-density bundle on the fibres of P^*Y gives

$$I^m(X,Y;\Omega_b^{\frac12}) \xrightarrow{\sigma_m} \mathcal{A}^{\{-m-\frac{n}{4}\}}(P^*Y;\Omega_b^{\frac12}). \tag{6.2.20}$$

As always we wish to characterize the distributions which have expansions, in this case $I^m_{\text{KN}}(X,Y)$ inside $I^m(X,Y)$ (or $I^{[s]}(X,Y)$.) We can do this using 'radial' vector fields. On the normal bundle to Y there is a well-defined radial vector field, the generator of the $\mathbb{R}^+$-action on the fibres (to which it is tangent.) Using a normal fibration this radial vector field can be transferred to a neighbourhood of Y in X and then it can be modified away from Y and extended smoothly to X. The resulting vector field $V \in \mathcal{V}_b(X,Y) = \mathcal{V}_b(X) \cup \mathcal{V}_E(X,Y)$ is not unique, but choice of a different fibration or extension modifies it to V' where

$$V' - V \in \mathcal{I}_Y \cdot \mathcal{V}_b(X,Y). \tag{6.2.21}$$

Another way of characterizing a radial vector field for Y up to an error as in (6.2.21) is as $V \in \mathcal{V}_b(X,Y)$ which vanishes at Y (as an element of $\mathcal{V}_b(X)$) and has linear part

$$L_V : N_yY \longrightarrow N_yY,\ L_V v = [V,\tilde v](y),\ L_V = \text{Id} \quad \forall\ y \in Y. \tag{6.2.22}$$

Here $\tilde v$ is a vector field representing $v \in N_yY$.

Proposition 6.2.1. *For any closed interior p-submanifold Y in a compact manifold with corners X, any $m,p \in \mathbb{R}$, $J \in \mathbb{N}$, with $J > m-p$ and any $V \in \mathcal{V}_b(X,Y)$ satisfying* (6.2.22)

$$\begin{gathered} u \in \mathcal{A}(X),\ \prod_{j=0}^{k}(V + m + \frac{n}{4} + \frac{d}{2} - j)u \in I^{p+k-J}(X,Y) \quad \forall\ 0 \le k \le J \\ \Longrightarrow u \in I^m_{\text{KN}}(X,Y) + I^p(X,Y). \end{gathered} \tag{6.2.23}$$

Proof. First we check that the conditions in (6.2.23), taken together, are independent of the choice of V. The properties of V which follow from (6.2.22) include

$$\begin{gathered} V : \mathcal{C}^\infty(X) \longrightarrow \mathcal{I}_Y \\ Vu - u \in \mathcal{I}_Y^2 \quad \forall\ u \in \mathcal{I}_Y \\ [V,W] \in \mathcal{I}_Y \cdot \mathcal{V}_b(X,Y) \quad \forall\ W \in \mathcal{V}_b(X,Y). \end{gathered} \tag{6.2.24}$$

Let $Q_k = (V + m + \frac{n}{4} + \frac{d}{2}) \cdots (V + m + \frac{n}{4} + \frac{d}{2} + k)$ be the operator appearing in (6.2.23) and let Q'_k be the corresponding operator with V replaced by V'. A straightforward inductive argument based on (6.2.24) and (6.2.21) shows that

$$\begin{gathered} Q'_k = Q_k + \sum_{j<k} C_{k,j}Q_j \\ C_{k,j} = \sum_{\alpha} a_{p,j,k} \prod_{r=1}^{p} W_{r,j,k},\ a_{\alpha,j,k} \in \mathcal{I}_Y^{k-j},\ W_{r,j,k} \in \mathcal{V}_b(X,Y). \end{gathered} \tag{6.2.25}$$

This shows that these conditions are independent of the choice of V. A similar argument shows that these conditions can be localized.

Thus we can work locally to prove (6.2.23). In local coordinates in which (6.1.8) holds we can take $V = z \cdot \partial_z$ to be the radial vector field. Taking the fibre Fourier transform (6.1.12) we find that (6.2.23) implies

$$\prod_{k=0}^{J}(\zeta\cdot\partial_\zeta - m - \frac{n}{4} + \frac{d}{2} - k)a = b \tag{6.2.26}$$

is a symbol or order $p+\frac{n}{4}-\frac{d}{2}$. The assumption that $m-p$ is not an integer means that (6.2.26) has a solution a which is a symbol of order $p+\frac{n}{4}-\frac{d}{2}$ up to a rapidly vanishing error. It follows that the general (symbolic) solution is a sum of homogeneous terms, plus such a remainder, proving (6.2.23). The converse is more obvious. □

6.3. Conormality at the boundary

Next we wish to extend the discussion from conormal distributions 'smooth up to the boundary' to those having conormal singularities at ∂X as well; this is straightforward. First we replace (6.1.2) by

$$I^{[s],[s]}(X,Y) = \{u \in H^s(X); \mathcal{V}_b(X,Y)^k u \subset H^s(X) \quad \forall\ k\} \tag{6.3.1}$$

where $\mathcal{V}_b(X,Y) = \mathcal{V}_b(X) \cap \mathcal{V}_E(X,Y)$ is the Lie algebra of vector fields tangent to Y and all the boundary hypersurfaces of X. If μ is a power set for X or $\mathcal{E}$ is an index family we can replace (6.2.3) by

$$\begin{aligned} I^{m,\mu}(NY,Y) &= \mathcal{A}^{\mu}(NY) + \mathcal{F}^{-1}\left[\mathcal{A}^{-m-\frac{n}{4}+\frac{d}{2},\mu}(P^*Y;\Omega_{\text{fibre}})\right] \\ I^{m,\mathcal{E}}_{\text{KN}}(NY,Y) &= \mathcal{A}^{\mu}(NY) + \mathcal{F}^{-1}\left[\mathcal{A}^{-m-\frac{n}{4}+\frac{d}{2},\mathcal{E}}_{\text{phg}}(P^*Y;\Omega_{\text{fibre}})\right] \end{aligned} \tag{6.3.2}$$

Then

$$I^{*,*}(X,Y) = \bigcup_s I^{[s],[s]}(X,Y) = \bigcup_{m,\mu} I^{m,\mu}(X,Y). \tag{6.3.3}$$

The proof of invariance in Lemma 6.2.1 extends directly and hence so does the general definition, as in (6.2.11)

$$\begin{aligned} I^{m,\mu}(X,Y) &= \mathcal{A}^{\mu}(X) + \{(f^{-1})^*u; u \in I^{m,\mu}(NY,Y) \cap \mathcal{C}_c^{-\infty}(\Gamma)\} \\ I^{m,\mathcal{E}}_{\text{KN}}(X,Y) &= \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) + \{(f^{-1})^*u; u \in I^{m,\mathcal{E}}_{\text{KN}}(NY,Y) \cap \mathcal{C}_c^{-\infty}(\Gamma)\} \end{aligned} \tag{6.3.4}$$

for any $Y \subset X$ a closed interior p-submanifold and $f : \Gamma \longrightarrow X$ a normal fibration. The discussion of the principal symbol applies essentially unchanged, so we get

$$\begin{aligned} I^{m,\mu}(X,Y;\Omega_b^{\frac12}) &\xrightarrow{\sigma_m} \mathcal{A}^{\{-m-\frac{n}{4}\},\mu}(P^*Y;\Omega_b^{\frac12}) \\ I^{m,\mathcal{E}}_{\text{KN}}(X,Y;\Omega_b^{\frac12}) &\xrightarrow{\sigma_m} \mathcal{A}^{\{-m-\frac{n}{4}\},\mathcal{E}}_{\text{phg}}(P^*Y;\Omega_b^{\frac12}). \end{aligned} \tag{6.3.5}$$

Since the definition in (6.3.2) is in terms of the Fourier transform we can easily discuss any operation which commutes with it. In particular consider a bundle isomorphism covering a diffeomorphism of Y :

$$\begin{tikzcd} NY \arrow[r, "F"] \arrow[d] & NY \arrow[d] \\ Y \arrow[r, "f"] & Y. \end{tikzcd} \tag{6.3.6}$$

Thus F is a linear isomorphism from N_yY to $N_{f(y)}Y$ for each $y \in Y$. Let $F' : NY \longrightarrow NY$ be the dual map, covering f^{-1}. The invariance of the Fourier transform shows that pull-back under F gives

$$F^* : I^{m,\mu}(NY, Y) \longrightarrow I^{m,\mu}(NY, Y),\ \sigma_m(F^*u) = [(F')^*]^{-1}\sigma_m(u). \tag{6.3.7}$$

Combining this with the extension of Lemma 6.2.1 to this case we find:

PROPOSITION 6.3.1. *If $f : X \longrightarrow X$ is a diffeomorphism of a compact manifold with corners and $Y \subset X$ is a closed p-submanifold then by extension from $\mathcal{C}^\infty$,*

$$f^* : I^{m,\mu}(X, f(Y)) \longrightarrow I^{m,\mu}(NY, Y),\ \sigma_m(f^*u) = (F^{-1})^*\sigma_m(u) \tag{6.3.8}$$

where $F = f^* : N^*[f(Y)] \longrightarrow N^*Y$.

Asymptotic summation carries over in the obvious way, as do the other results. In particular Proposition 6.2.1 and Proposition 4.14.2 can be combined to give:

PROPOSITION 6.3.2. *For any closed interior p-submanifold Y in a compact manifold with corners X, any $m, p \in \mathbb{R}$, $J \in \mathbb{N}$, with $J > m - p$, any $V \in \mathcal{V}_b(X, Y)$ satisfying (6.3.3), any index family $\mathcal{E}$ and power set μ*

(6.3.9)
$$\begin{gathered} u \in I^{s,\mu}(X, Y),\ \prod_{j=0}^{k}(V + m + \frac{n}{4} + \frac{d}{2} - j)u \in I^{p+k-J,\mu}(X, Y) \quad \forall\ 0 \le k \le J \\ P_{M,j}u \in I^{s,\mu(j,N)}(X) \quad \forall\ M, \quad \text{with } \mu(j, M) = (\mu_1, \dots, \mu_{j-1}), M, \mu_{j+1}, \dots, \mu_N) \\ \Longrightarrow u \in I^{m,\mathcal{E}}_{\mathrm{KN}}(X, Y) + I^{p,\mu}(X, Y). \end{gathered}$$

In particular if k is arbitrary then the conclusion of (6.3.9) is that $u \in I^{m,\mathcal{E}}(X, Y)$.

6.4. Symbol sequence and ellipticity

The symbol map (6.3.5) has certain important properties. It captures the 'leading singularity' at Y of the conormal distributions associated to Y. This can be expressed in terms of the exactness of the sequences

(6.4.1)
$$\begin{gathered} 0 \hookrightarrow I^{m-1,\mu}(X, Y; \Omega_b^{\frac12}) \hookrightarrow I^{m,\mu}(X, Y; \Omega_b^{\frac12}) \xrightarrow[\sigma_m]{\mathcal{A}} {}^{\{-m+\frac{n}{4}\},\mu}(P^*Y; \Omega_b^{\frac12}) \longrightarrow 0 \\ 0 \hookrightarrow I^{m-1,\mathcal{E}}_{\mathrm{KN}}(X, Y; \Omega_b^{\frac12}) \hookrightarrow I^{m,\mathcal{E}}_{\mathrm{KN}}(X, Y; \Omega_b^{\frac12}) \xrightarrow[\sigma_m]{\mathcal{A}} {}^{\{-m+\frac{n}{4}\},\mathcal{E}}_{\mathrm{phg}}(P^*Y; \Omega_b^{\frac12}) \longrightarrow 0 \end{gathered}$$

Secondly, suppose $P \in \mathrm{Diff}^r_b(X)$ is a b-differential operator with $\mathcal{C}^\infty$ coefficients. The symbol of P is a polynomial on the fibres of ${}^bT^*X$ and is homogeneous of degree r. In local coordinates, $\sigma_r(P)$ is just given by the polynomial formed by the leading coefficients of P :

$$\begin{gathered} P = \sum_{|\alpha|+|\beta| \le r} p_{\alpha,\beta}(x, y)(xD_x)^\alpha D_y^\beta \Longrightarrow \\ \sigma_r(P)(x, y, \lambda, \eta) = \sum_{|\alpha|+|\beta|=r} p_{\alpha,\beta}(x, y)\lambda^\alpha\eta^\beta, \end{gathered} \tag{6.4.2}$$

with (x, y, λ, η) the canonical coordinates defined in ${}^bT^*X$ by the coordinates x, y in X.

Now it is always the case that for any closed interior p-submanifold

$$\begin{aligned}
&\mathrm{Diff}^r_b(X)\cdot I^{m,\mu}(X,Y)\subset I^{m+r,\mu}(X,Y)\\
&\mathrm{Diff}^r_b(X)\cdot I^{m,\mathcal{E}}_{\mathrm{KN}}(X,Y)\subset I^{m+r,\mathcal{E}}_{\mathrm{KN}}(X,Y)\\
&\sigma_{m+r}(Pu)=\sigma_r(P)\restriction_{{}^bN^*Y}\cdot\sigma_m(u).
\end{aligned}\tag{6.4.3}$$

There is also a 'negative order' version of (6.4.3). Suppose that $f\in\mathcal{C}^\infty(X)$ is a function which vanishes on Y. As a function on ${}^bT^*X$ it therefore vanishes on ${}^bN^*Y$. The b-Hamilton vector field (see §2.10), on ${}^bN^*Y$, of f is fixed in local coordinates in which (6.1.8) holds by

$${}^bH_f=-\sum_{i=1}^k x_i\frac{\partial f}{\partial x_i}\frac{\partial}{\partial\lambda_i}-\sum_{J>k}\frac{\partial f}{\partial y_j}\frac{\partial}{\partial\eta_j}.\tag{6.4.4}$$

One can see that this is independent of coordinates by noting that it is tangent to the fibres of ${}^bN^*Y$ and hence can be identified with a section of ${}^bN^*Y$, namely $-df$. Then,

$$\begin{aligned}
&f\in\mathcal{C}^\infty(X),\ f=0\text{ on }Y\Longrightarrow f\cdot I^{m,\mathcal{E}}_{\mathrm{KN}}(X,Y)\subset I^{m-1,\mathcal{E}}_{\mathrm{KN}}(X,Y)\\
&\sigma_{m-1}(fu)=\frac{1}{i}{}^bH_f\sigma_m(u).
\end{aligned}\tag{6.4.5}$$

The most basic use of the symbol map and formula (6.4.3) is in the solution, modulo conormal errors, of the differential equation

$$Pu=f,\quad P\in\mathrm{Diff}^r_b(X).\tag{6.4.6}$$

Suppose that Y is non-characteristic for P, i.e.

$$\sigma_r(P)\neq 0\quad\text{on}\quad{}^bN^*Y\backslash 0.\tag{6.4.7}$$

Then multiplication by the symbol, interpreted as an element of $\mathcal{A}^{-r,0}_{\mathrm{phg}}(P^*Y)$ gives an isomorphism

$$\sigma_m(P)\cdot\mathcal{A}^{M,\mathcal{E}}_{\mathrm{phg}}(P^*Y)\longleftrightarrow\mathcal{A}^{M+r,\mathcal{E}}_{\mathrm{phg}}(P^*Y)\quad\forall\ M.\tag{6.4.8}$$

Using the exactness in (6.4.1) we conclude that, if (6.4.7) holds,

$$\forall\ f\in I^{m,\mathcal{E}}_{\mathrm{KN}}(X,Y)\ \exists\ u\in I^{m-r,\mathcal{E}}_{\mathrm{KN}}(X,Y)\text{ s.t. }Pu-f\in I^{m-1,\mathcal{E}}_{\mathrm{KN}}(X,Y).\tag{6.4.9}$$

Asymptotic summability of series shows that, by iteration, (6.4.9) can be improved to

$$\forall\,f\in I^{m,\mathcal{E}}_{\mathrm{KN}}(X,Y)\ \exists\ u\in I^{m-r,\mathcal{E}}_{\mathrm{KN}}(X,Y)\text{ s.t. }Pu-f\in\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X).\tag{6.4.10}$$

If $\mathcal{E}=0$ this gives existence modulo $\mathcal{C}^\infty$. Of course a similar construction works for the spaces $I^{m,\mu}(X,Y)$. One small extension of all these results involves sections of vector bundles. If W is any vector bundle over X then the distributional sections of W can be defined in two equivalent ways:

$$\begin{aligned}
\mathcal{C}^{-\infty}(X;W)&=[\mathcal{C}^\infty(X,\Omega\otimes W)]'\\
&\simeq\mathcal{C}^{-\infty}(X)\otimes_{\mathcal{C}^\infty(X)}\mathcal{C}^\infty(X;W).
\end{aligned}\tag{6.4.11}$$

The second representation, and the fact that each $I^m(X,Y)$ is a $\mathcal{C}^\infty(X)$-module, allows us simply to define

$$I^m(X,Y;W)=I^m(X,Y)\otimes_{\mathcal{C}^\infty(X)}\mathcal{C}^\infty(X;W).\tag{6.4.12}$$

The analogues of the results discussed above are then easily deduced. For example the symbol map becomes

$$\sigma_m : I^m(X,Y;W) \longrightarrow S^{\{M\}}(N^*(Y) \otimes \Omega_{\text{fibre}} \otimes W \quad M = m + \frac{n}{4} - \frac{d}{2} \tag{6.4.13}$$

where W is really the lift of $W|_Y$ to N^*Y and the tensor product is still with respect to $\mathcal{C}^\infty(Y)$.

One particular case which we meet frequently is that of half-densities. These behave a little better than other bundles since the restriction to Y of the density bundle over X can be identified as the tensor product of the density bundle on Y with the densities on the fibres of the normal bundle to Y in X :

$$\Omega_Y X \equiv \Omega Y \otimes \Omega_{\text{fibre}} NY. \tag{6.4.14}$$

Inserting this into (6.4.15) after taking the square-root and noting that

$$\Omega_{\text{fibre}} NY \equiv \Omega^{-1}_{\text{fibre}} N^*Y \tag{6.4.15}$$

shows that

$$I^m(X,Y;\Omega^{\frac12}X) \xrightarrow{\sigma_m} S^{\{M\}}(N^*Y) \otimes \Omega^{\frac12}(N^*Y)), \quad M = m + \frac{n}{4} - \frac{r}{2} \tag{6.4.16}$$

which is nice and functorial!

6.5. Products

There are two further, related, generalizations of these spaces we shall consider. The first is just the space $I^{m,\mu}(X,Y)$, or its polyhomogeneous counterpart, for Y a closed boundary p-submanifold. We do not consider a 'Kohn-Nirenberg' space since no Fourier transform is involved. For the space without filtration we can take the same definition, (6.3.1), as for an interior p-submanifold. From Proposition 5.3.1 it follows that the blow-down map for blow-up of Y gives an isomorphism

$$\beta_Y^* : I^{*,*}(X,Y) \longleftrightarrow \mathcal{A}(X_Y). \tag{6.5.1}$$

Definition 6.5.20. If $Y \subset \partial X$ is closed p-submanifold and $\beta_Y : X_Y \longrightarrow X$ is the blow-down map we set, for $m \in \mathbb{R}$, μ a power set or $\mathcal{E}$ an index family for X :

$$\begin{aligned} I^{m,\mu}(X,Y) &= (\beta_Y^*)^{-1}\mathcal{A}^{m,\mu}(X_Y) \\ I^{m,\mathcal{E}}_{\text{phg}}(X,Y) &= (\beta_Y^*)^{-1}\mathcal{A}^{m,\mathcal{E}}_{\text{phg}}(X_Y) \end{aligned} \tag{6.5.2}$$

using the fact that β_Y is an isomorphism of extendible distributions and interpreting m,μ and $m,\mathcal{E}$ as a power set and index family for X_Y.

More seriously we also consider the space $I^{m,m_1,m_2,\mathcal{E}}(X,Y_1 \cup Y_2)$ for Y_1, Y_2 a pair of normal closed p-submanifolds. Set

$$\mathcal{V}_b(X,Y_1 \cup Y_2) = \{V \in \mathcal{V}_b(X); V \text{ is tangent to both } Y_1 \text{ and } Y_2\}. \tag{6.5.3}$$

Then put

$$I^{[s]}_c(X,Y_1 \cup Y_2) = \{u \in H^s_c(X); \mathcal{V}_b(X,Y_1 \cup Y_2)^j u \subset H^s_c(X) \quad \forall\, j \in \mathbb{N}\}. \tag{6.5.4}$$

As usual we drop the suffix if X is compact. To analyze these spaces we start, as before, by choosing a normal fibration of X around $Y = Y_1 \cap Y_2$. Thus we consider $X = NY$, with Y_1 and Y_2 reduced to complementary subbundles of NY. Let $Y_i^* \subset N^*Y$ be the dual subbundles, so

$$N^*Y = Y_1^* \oplus Y_2^*. \tag{6.5.5}$$

Consider the stereographic compactification P^*Y of N^*Y. Certainly the Y_i^* are interior p-submanifolds of P^*Y, let $Y_i^\infty = \partial_{\text{fibre}} P^*Y \cap Y_i^*$, $i = 1, 2$ be their intersections with the boundary hypersurface of P^*Y formed by the spheres at infinity. The invariant Fourier transform (6.2.1) or (6.2.2) can then be applied to compactly supported distributions and we find:

LEMMA 6.5.1. *If Y_1 and Y_2 are complementary subbundles of NY the invariant Fourier transform* (6.2.1) *maps conormal distributions on NY with respect to Y_1 and Y_2 into conormal distributions on P^*Y with respect to $Y_1^\infty \cap Y_2^\infty$:*

$$\mathcal{F}_c : I_c^{[s]}(NY, Y_1 \cup Y_2) \longrightarrow I^{*,*,*}(P^*Y, Y_1^\infty \cup Y_2^\infty). \tag{6.5.6}$$

PROOF. Notice that Y_1^∞ and Y_2^∞ are disjoint, so the space on the right is defined by Proposition 6.5.20. □

In this case we fix the orders of conormal distributions by defining

(6.5.7)

6.6. Pull-back

To analyze the pull-back of conormal distributions associated to an interior p-submanifold we begin, as usual with a simple case.

LEMMA 6.6.1. *Let $f : X_1 \longrightarrow X_2$ be a b-map of compact manifolds with boundary and let $p \in Y$ be point in a compact manifold without boundary then for any power set $\mathfrak{a}$ for X_2 and any $m \in \mathbb{R}$ pull-back under the product map $g = f \times \mathrm{Id} : X_1 \times Y \longrightarrow X_2 \times Y$ extends by continuity from $\mathcal{C}^\infty(X_2 \times Y)$ to*

$$\begin{gathered} g^* : I^{m,\mathfrak{a}}(X_2 \times Y, X_2 \times \{p\}) \longrightarrow I^{M,\mathfrak{b}}(X_1 \times Y, X_1 \times \{p\}), \\ M = m + \frac{1}{4}(\dim X_2 - \dim X_1),\ \mathfrak{b} = f^\# \mathfrak{a}; \end{gathered} \tag{6.6.1}$$

if $\mathcal{E}$ is an index set for X_2 then

$$\begin{gathered} g^* : I_{\mathrm{KN}}^{m,\mathcal{E}}(X_2 \times Y, X_2 \times \{p\}) \longrightarrow I^{M,\mathcal{F}}(X_1 \times Y, X_1 \times \{p\}), \\ M = m + \frac{1}{4}(\dim X_2 - \dim X_1),\ \mathcal{E} = f^\# \mathcal{F}. \end{gathered} \tag{6.6.2}$$

PROOF. Using Proposition 4.11.1, and more particularly (4.11.9), we see that (6.6.1) holds if $m = -\infty$. Thus we only need to consider $u \in I^{m,\mathfrak{a}}(X_2 \times Y, X_2 \times \{p\})$ with support in $X_2 \times \Omega$ for any pre assigned neighbourhood Ω of p. In particular we can use a normal fibration of p in Y (i.e. local coordinates) to transfer the problem to $X_2 \times \mathbb{R}^d$. Then the conormal distributions can be characterized by Fourier transformation. Then (6.6.1) follows directly from (4.11.9) applied to the product map $g^* = f \times \mathrm{Id} : X_1 \times \mathbb{B}^d \longrightarrow X_2 \times \mathbb{B}^d$. The change in order m is a consequence of the normalization in (6.2.3). Similar (6.6.2) follows from Lemma 4.15.1. □

Let $f : X_1 \longrightarrow X_2$ be a b-map of compact manifolds with corners and suppose $Y_2 \subset X_2$ is a closed interior p-submanifold which is transversal to f :

$$\forall\ p \in Y_1 = f^{-1}(Y_2),\ f_*(T_pX_1) + T_{f(p)}Y_2 = T_{f(p)}X_2. \tag{6.6.3}$$

Then $Y_1 \subset X_1$ is a closed p-submanifold of X_1. The dual statement to (6.6.3), that the map

$$f^* : N^*_{f(p)}Y_2 \longrightarrow T_p^*Y_1 \quad \forall\ p \in Y_1 \tag{6.6.4}$$

is injective, shows that f^* is an isomorphism from $N^*_p Y_1$ to $N^*_{f(p)} Y_2$ for all $p \in Y_1$. In fact $f : Y_1 \longrightarrow Y_2$ is a local isomorphism.

PROPOSITION 6.6.1. *Suppose $f : X_1 \longrightarrow X_2$ is a b-map which is transversal to a closed interior p-submanifold $Y_2 \subset X_2$ then if $Y_1 = f^{-1}(Y_2)$ pull-back of $\mathcal{C}^\infty$functions extends by continuity to*

$$
\begin{gathered}
f^* : I^{m,\mu}(X_2, Y_2) \longrightarrow I^{M,\mu'}(X_1, Y_1), \\
M = m + \frac{1}{4}(\dim X_2 - \dim X_1),\ \mu' = f^\# \mu
\end{gathered}
\tag{6.6.5}
$$

and moreover if $f^\# : N^*Y_1 \longrightarrow N^*_{f(Y_1)} Y_2$ is the bundle isomorphism which is given by $f^\#(y, df^* g(y)) = (f(y), dg)$ then

$$
\sigma_M(f^* u) = (f^\#)^* \sigma_m(u). \tag{6.6.6}
$$

PROOF. $\square$

For Kohn-Nirenberg distributions similar arguments give, under the conditions of Proposition 6.6.1

$$
\begin{gathered}
f^* : I^{m,\mathcal{E}}_{HKN}(X_2, Y_2) \longrightarrow I^{M,\mathcal{F}}_{HKN}(X_1, Y_1), \\
M = m + \frac{1}{4}(\dim X_2 - \dim X_1),\ \mathcal{F} = f^\# \mathcal{E}.
\end{gathered}
\tag{6.6.7}
$$

6.7. Push-forward

The similar, simpler, result for push-forward is for projections, Thus suppose $g : X_1 \times X_2 \longrightarrow X_2$ is projection off the first factor. If $Y_1 \subset X_1 \times X_2$ is an embedded submanifold such that

$$
T_{(p_1,p_2)}(X_1 \times \{p_2\}) \cap T_{(p_1,p_2)} Y_1 = \{0\} \quad \forall\ (p_1, p_2) \in Y_1 \tag{6.7.1}
$$

then $g : Y_1 \longrightarrow X_2$ is an immersion. We assume additionally that

$$
g : Y_1 \longrightarrow X_2 \text{ is an embedding (i.e. 1-1).} \tag{6.7.2}
$$

PROPOSITION 6.7.1. *Suppose $g : X_1 \times X_2 \longrightarrow X_2$ is the projection and $Y_1 \subset X_1 \times X_2$ is an embedded submanifold satisfying (6.7.1) and (6.7.2), then if $Y_2 = g(Y_1)$*

$$
g_* : I^M(X_1 \times X_2, Y_1; \Omega) \longrightarrow I^M(X_2, Y_2; \Omega),\ M = m - \frac{1}{4}\dim X_1, \tag{6.7.3}
$$

and (if (6.7.2) is not an isomorphism)

$$
\sigma_M(g_* u) = (g^\#)^* [\sigma_m(u) \restriction [N^*Y_1 \cap g^*(T^*X_2)]] \tag{6.7.4}
$$

where $g^\# : N^*Y_2 \longrightarrow N^*Y_1$ is the bundle map which is an isomorphism onto the subbundle $N^*Y_1 \cap g^*(T^*X_2)$, the space of conormal directions to Y_1 also conormal to the fibres.

PROOF. By assumption Y_2 is just Y_1 embedded in X_2 by (6.7.2). Since it suffices to work locally we can assume that $X_2 = Y_2' \times X_2'$ with $Y_2 = Y_2' \times \{p_2'\}$ and $Y_1 = \{p_1\} \times Y_2' \times \{p_2\} \subset X_1 \times X_2$. Proposition 6.6.3 allows Y_2' to be treated as a parameter space, reducing the problem to the special case $Y = \{p_1\} \times \{p_2\}$ is a point. Then (6.7.1) and (6.7.2) are automatic and push-forward is just integration in some of the z variables in (6.1.11). The Fourier inversion formula immediately gives

(6.7.3) and (6.7.4) in this case and the general case case follows by reintroducing the parameters. $\square$

Note that the special case excluded in (6.7.4) is trivial, but of major importance:

$$\text{(6.7.5)}\qquad \begin{gathered}\text{If } g : Y_1 \longrightarrow Y_2 \text{ in (6.7.2) is an isomorphism then}\\ g_* : I^m(X_1 \times X_2, Y_1; \Omega) \longrightarrow \mathcal{C}^\infty(X; \Omega) \quad \forall\ m.\end{gathered}$$

In Chapter 5 we have developed a theory of spaces analogues to these, when Y_1 and Y_2 are boundary faces of a manifold with corners. The reason that we do not need a complete symbolic theory (although this is readily done, see [?]), is that an appropriate push-forward takes us back to standard conormal distributions.

Thus suppose $Y_1, Y_2 \subset X$ are embedded submanifolds meeting transversally. Let $f : X \longrightarrow X'$ be a projection with the following additional conditions on its relation to Y_1, Y_2 and $Y_1 \cap Y_2$:

$$\text{(6.7.6)}\qquad \begin{gathered} f : Y_1, Y_2 \longrightarrow X' \text{ are submersions}\\ f : Y_1 \cap Y_2 \longrightarrow X' \text{ is an embedding.}\end{gathered}$$

The second of these implies, or really states, that $f(Y_1 \cap Y_2)$ is an embedded submanifold of X'. Lifting under f^* gives a bundle map

$$\text{(6.7.7)}\qquad f^* : N^*f(Y_1 \cap Y_2) \longrightarrow N^*(Y_1 \cap Y_2)$$

which, as already noted, is an isomorphism onto $N^*(Y_1 \cap Y_2) \cap f^*(T^*X')$, the intersection with the fibre-normals. We can write this as

$$\text{(6.7.8)}\qquad f^\# : N^*f(Y_1 \cap Y_2) \longleftrightarrow \left[N^*_{Y_1\cap Y_2}Y_1 \oplus N^*_{Y_1\cap Y_2}Y_2\right] \cap f^*(T^*X').$$

The sum is direct because of the assumed transversality. Now, if a is a function on N^*Y_1 and b a function on N^*Y_2 this gives a function $f^\#(ab)$ on $N^*f(Y_1 \cap Y_2)$. More generally we note that there is also a natural identification of density bundles at $Y_1 \cap Y_2$

$$\text{(6.7.9)}\qquad \Omega_y(Y_1) \otimes \Omega_{\text{fibre}}(N^*Y_2) \simeq \Omega_y(Y_1 \cap Y_2),\ y \in Y_1 \cap Y_2.$$

Indeed this just follows from the fact that, at $Y_1 \cap Y_2$, the normal bundle to Y_2 in X can be naturally identified with the normal bundle to $Y_1 \cap Y_2$ in Y_1 so

$$\text{(6.7.10)}\qquad \Omega_y(Y_1) \simeq \Omega_y(Y_1 \cap Y_2) \otimes \Omega_{\text{fibre},y}(NY_1).$$

The point of (6.7.9) is that, using (6.7.8), we get a bilinear map

$$\text{(6.7.11)}\qquad \begin{gathered}(f^\#)^* : [\mathcal{C}^\infty(N^*Y_1) \otimes \Omega(Y_1)] \times [\mathcal{C}^\infty(N^*Y_2) \otimes \Omega_{\text{fibre}}(N^*Y_2)]\\ \longrightarrow \mathcal{C}^\infty(N^*Z) \otimes \Omega(Z),\ Z = f(Y_1 \cap Y_2).\end{gathered}$$

Now we can state the result!

Proposition 6.7.2. *Suppose $f : X \longrightarrow X'$ is a projection of compact manifolds and Y_1, $Y_2 \subset X$ are two embedded submanifolds meeting transversally, for which* (6.7.6) *holds, then the bilinear map defined using* (6.7.15) *gives*

$$\text{(6.7.12)}\qquad \begin{gathered} f_*\left[I^m(X, Y_1; \Omega) \cdot I^{m'}(X, Y_2)\right] \subset I^M(X', Z; \Omega),\\ Z = f(Y_1 \cap Y_2),\ M = m + m' + \frac{1}{4}\dim X'\end{gathered}$$

and using (6.7.11),

$$\text{(6.7.13)}\qquad \sigma_M(f_*(uv)) = (f^\#)^*(\sigma_m(u) \cdot \sigma_{m'}(v)).$$

PROOF. As usual we successively simplify the situation until the proof becomes a direct computation. First note that the variables in Z, or $Y_1 \cap Y_2$, are parameters. Removing these locally, using Proposition 6.6.3 repeatedly, we can assume

$$X = Y_1' \times Y_2',\ Y_1 = Y_1' \times \{p_2'\},\ Y_2 = \{p_1'\} \times Y_2', \tag{6.7.14}$$

with $Y_1 \cap Y_2$ and Z a point. This makes the last condition in (6.7.6) trivial.

There is a rather obvious symmetry between Y_1 and Y_2, so we choose Y_1 to have codimension greater than, or equal to, that of Y_2. The assumption (6.7.6) on Y_1 now means that we can introduce local coordinates, near $p \in Y_1 \cap Y_2$, (y', y'', x) such that

$$f(y', y'', x) = y', \qquad Y_1 = \{x = 0\}. \tag{6.7.15}$$

By assumption $\{(0,0,0)\} = Y_1 \cap Y_2$. Moreover the transversality means that, locally,

$$Y_2 = \{y' = Y'(x), y'' = Y''(x)\} \tag{6.7.16}$$

where the condition on Y_2 in (6.7.6) means that rank of $\partial_x Y'$ is equal to the number of y' variables. Since we can freely change the x variables, amongst themselves, and the y'' variables arbitrarily we can choose the coordinates so that (6.7.15) holds and (6.7.16) becomes

$$Y_2 = \{y' = x', y'' = x''\},\ x = (x', x'', x'''). \tag{6.7.17}$$

Now we can factor the projection f into two projections $f = f_2 \cdot f_1$, $f_1(y, x) = y$, $f_2(y) = y'$. The advantage of doing so is that the hypotheses of the proposition still hold for f_1 and if we prove it in that case then the general case follows by applying Proposition 6.7.1 to f_2.

Thus we can assume (6.7.15) and (6.7.17) hold with no y'' variables; changing notation slightly,

$$f(y, x, z) = y, Y_1 = \{x = z = 0\}, Y_2 = \{y = x\}. \tag{6.7.18}$$

Following the proof of Proposition 6.7.6 we can use the formula (6.7.13) for the expansion of the product. If N is large enough the second sum and final remainder term contribute terms to $(f_1)_*\,(uv\nu)$ which becomes increasingly smooth. Thus it suffices to consider $(f_1)_*(uv\mu)$ where u and v are of the form

$$u = (2\pi)^{-n_1} \int e^{ix\cdot\xi + iz\cdot\zeta} a(\xi, \zeta) d\xi d\zeta |dx||dz||dy|$$
$$v = (2\pi)^{-n_2} \int e^{i(y-x)\cdot\eta} b(\eta) d\eta.$$

The cancellation (6.7.9) is $|dx||d\xi||dz||d\zeta| = 1$ and the Fourier inversion formula gives

$$\int uv dx dz |dy| = (2\pi)^{n_2} \int e^{iy\cdot\eta} a(\eta, 0) b(\eta) d\eta |dy|.$$

In this simple case, (6.7.13) becomes

$$\sigma_m((f_1)_*(uv\nu)) = a(y, 0) b(\eta) \tag{6.7.19}$$

so this completes the proof of the proposition. □

6.8. Pseudodifferential operarators, $\partial X = \emptyset$

If X is a compact manifold without boundary the Schwartz kernel theorem identifies continuous linear operators

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^{-\infty}(X) \tag{6.8.1}$$

with their kernels

$$K_A \in \mathcal{C}^{-\infty}(X \times X; \pi_R^*\Omega). \tag{6.8.2}$$

The identification is fixed by the usual push-forward and pull-back formula:

$$(A\phi, \psi) = \int \pi_L^*\phi \cdot K_A \cdot \pi_R^*\psi \quad \forall\ \phi \in \mathcal{C}^\infty(X),\ \psi \in \mathcal{C}^\infty(X;\Omega). \tag{6.8.3}$$

This formula uses the identification

$$\pi_L^*\Omega \otimes \pi_R^*\Omega \simeq \Omega \qquad \text{over } X \times X. \tag{6.8.4}$$

In this context the diagonal, as an embedded submanifold of the product,

$$\Delta = \{(x,x) \in X \times X\} \hookrightarrow X \times X \tag{6.8.5}$$

plays an important rôle since the associated conormal distributions give rise to the spaces of pseudodifferential operators.

$$\Psi^m(X) = I^m(X \times X, \Delta; \pi_R^*\Omega). \tag{6.8.6}$$

There is a natural identification of Δ with X and a further identification

$$N^*\Delta \longleftrightarrow T^*X,\ (x, (\xi, -\xi)) \longleftrightarrow (x, \xi). \tag{6.8.7}$$

It is also the case that, over the diagonal, there is a natural trivialization of the bundle $\Omega_{\text{fibre}}(N^*\Delta) \otimes \pi_R^*\Omega$. This allows us to identify the symbol of a pseudodifferential operator as an equivalence class of functions on T^*X

$$\sigma_m : \Psi^m(X) \longrightarrow S^{\{m\}}(T^*X). \tag{6.8.8}$$

For any $r \in \mathbb{N}$

$$\mathrm{Diff}^r(X) \subset \Psi^r(X)$$

and the map (6.8.8) is consistent with (??).

Amongst the important mapping properties of pseudodifferential operators we note

$$\begin{gathered} A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \\ A : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X) \qquad \exists\ A \in \Psi^m(X) \\ A : H^s(X) \longrightarrow H^{s-m}(X) \end{gathered} \tag{6.8.9}$$

and for any embedded submanifold, Y, of X

$$\left.\begin{aligned} A : I^r(X,Y) \longrightarrow I^{r+m}(X,Y) \\ \sigma_{m+r}(Au) = \sigma_m(A) \cdot \sigma_r(u) \end{aligned}\right. \quad \forall\ A \in \Psi^m(X). \tag{6.8.10}$$

Of course the most significant properties of pseudodifferential operators are concerned with composition. We shall discuss the proof of the basic result

$$\begin{gathered} \Psi^m(X) \cdot \Psi^{m'}(X) \subset \Psi^{m+m'}(X) \\ \sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_{m'}(B) \end{gathered} \tag{6.8.11}$$

in some detail. It is closely connected to (6.8.10). The composition property will be deduced from a more general push-forward result for conormal distributions.

Simple extensions of this latter result lead to the composition formulae for the more general classes of operators discussed below.

6.9. Composition formula

As we shall be generalizing such arguments we briefly recall the proofs of (6.9.22) and (6.9.23). In fact we start with the first mapping property in (6.9.21). The mapping properties are naturally connected with the two projections

$$X \xleftarrow[\pi_L]{X} {}^2 \xrightarrow[\pi_R]{X} . \tag{6.9.1}$$

If ν is a $\mathcal{C}^\infty$ density on X and $A \in \Psi^m(X)$ is a pseudodifferential operator with kernel K_A then the defining formula, (6.9.15), becomes

$$A\phi \cdot \nu = (\pi_L)_* [\pi_L^* \nu \cdot K_A \cdot \pi_R^* \phi] , \forall \; \phi \in \mathcal{C}^\infty(X). \tag{6.9.2}$$

Here $\pi_L^*\nu \in \mathcal{C}^\infty(X^2, \pi_L^*\Omega X)$ and $K_A \in I^m(X^2, \Delta; \pi_R^*\Omega X)$. Since there is a natural identification

$$\pi_L^*\Omega(X) \otimes \pi_R^*\Omega(X) \simeq \Omega(X^2) \tag{6.9.3}$$

we can interpret (6.9.2) as the push-forward of a distributional density. Now

$$\pi_L^* v \cdot K_A \cdot \pi_R^* \phi \in I^m(X^2, \Delta; \Omega)$$

and we can apply Proposition 6.7.1 since the projection $\pi_L : \Delta \longrightarrow X$ is actually a diffeomorphism. It follows, from (6.7.5) that $A\phi \cdot \nu \in \mathcal{C}^\infty(X; \Omega)$, proving the first part of (6.7.2).

Next we prove (6.9.22). Notice that if $Y \subset X$ is an embedded submanifold and $\phi \in I^r(X, Y)$ then, applying (6.6.1),

$$\pi_R^*\phi \in I^{r-\frac{n}{4}}(X^2, X \times Y), \quad n = \dim X. \tag{6.9.4}$$

The projection $\pi_L : X^2 \longrightarrow X$ is a submersion and

$$\Delta \cap (X \times Y) = \{(y, y), y \in Y\} \simeq Y \tag{6.9.5}$$

so $\pi_L : \Delta \cap (X \times Y) \longrightarrow X$ embeds Y in X. The hypotheses of Proposition 6.7.2 therefore hold for Δ and $X \times Y$ in X^2. Thus

$$(\pi_L)_* [\pi_L^* \cdot K_A \cdot \pi_R^* \phi] \in I^{m+r}(X, Y; \Omega) \tag{6.9.6}$$

proving (6.9.22).

Now consider (6.9.23). Let ν be a smooth positive density on X. Suppose $A \in \Psi^m(X)$, $B \in \Psi^{m'}(X)$ and we write the three projections off the left, central and right factors as

$$\begin{tikzcd} X^2 & X^3 \arrow[l, "\pi_S"'] \arrow[r, "\pi_F"] \arrow[d, "\pi_C"] & X^2 \\ & X^2. & \end{tikzcd} \tag{6.9.7}$$

where the subscripts refer to the first, composite and second operator. Then we want to analyze the product

$$\pi_L^*\nu \cdot \pi_S^* K_A \cdot \pi_F^* K_B \in \mathcal{C}^{-\infty}(X^3, \pi_L^*\Omega \otimes \pi_M^*\Omega \otimes \pi_R^*\Omega) \tag{6.9.8}$$

where $\pi_L, \pi_M, \pi_R : X^3 \longrightarrow X$ are the three projections onto the left, middle and right factors of X. We are using (6.7.15) to define the product, and the fact that

$$\begin{aligned} \pi_S^* K_A &\in I^{m-\frac{n}{4}}(X^3, \Delta_S; \pi_M^*\Omega) \\ \pi_F^* K_B &\in I^{m'-\frac{n}{4}}(X^3, \Delta_F; \pi_L^*\Omega) \end{aligned} \tag{6.9.9}$$

with $\Delta_S = \{(x, x, x'), x, x' \in X\}$, $\Delta_F = \{(x', x, x), x, x' \in X\}$ the lifts of the diagonals. Then (6.7.7) holds because

$$\Delta_S \pitchfork \Delta_F = \Delta_T = \{(x, x, x); x \in X\} \tag{6.9.10}$$

is the triple diagonal.

By continuity from the case of smooth kernels if $C = A \cdot B$ then

$$K_C \cdot \pi_L^* \nu = (\pi_C)_* \left[\pi_L^* v \cdot \pi_S^* K_A \cdot \pi_F^* K_B\right]. \tag{6.9.11}$$

The hypotheses of Proposition 6.7.2 apply, since $\pi_C : \Delta_S \longrightarrow X^2$, $\pi_F : \Delta_F \longrightarrow X^2$ are diffeomorphisms and $\pi_C : \Delta_T \longrightarrow \Delta \hookrightarrow X^2$ embeds the triple diagonal as the diagonal. Thus (6.7.12) implies

$$K_C \in I^M(X^2, \Delta; \pi_R^*\Omega),\ M = m + m', \tag{6.9.12}$$

proving (6.9.23).

If X is a compact manifold without boundary the Schwartz kernel theorem identifies continuous linear operators

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^{-\infty}(X) \tag{6.9.13}$$

with their kernels

$$K_A \in \mathcal{C}^{-\infty}(X \times X; \pi_R^*\Omega). \tag{6.9.14}$$

The identification is fixed by the usual push-forward and pull-back formula:

$$(A\phi, \psi) = \int \pi_L^*\phi \cdot K_A \cdot \pi_R^*\psi \quad \forall\ \phi \in \mathcal{C}^\infty(X),\ \psi \in \mathcal{C}^\infty(X; \Omega). \tag{6.9.15}$$

This formula uses the identification

$$\pi_L^*\Omega \otimes \pi_R^*\Omega \simeq \Omega \qquad \text{over } X \times X. \tag{6.9.16}$$

In this context the diagonal, as an embedded submanifold of the product,

$$\Delta = \{(x, x) \in X \times X\} \hookrightarrow X \times X \tag{6.9.17}$$

plays an important rôle since the associated conormal distributions give rise to the spaces of pseudodifferential operators.

$$\Psi^m(X) = I^m(X \times X, \Delta; \pi_R^*\Omega). \tag{6.9.18}$$

There is a natural identification of Δ with X and a further identification

$$N^*\Delta \longleftrightarrow T^*X,\ (x, (\xi, -\xi)) \longleftrightarrow (x, \xi). \tag{6.9.19}$$

It is also the case that, over the diagonal, $\Omega_{\text{fibre}}(N^*\Delta) \otimes \pi_R^*\Omega$ is naturally trivial. This allows us to identify the symbol of a pseudodifferential operator as an equivalence class of functions on T^*X

$$\sigma_m : \Psi^m(X) \longrightarrow S^{\{m\}}(T^*X). \tag{6.9.20}$$

For any $r \in \mathbb{N}$

$$\mathrm{Diff}^r(X) \subset \Psi^r(X)$$

and the map (6.9.20) is consistent with (6.4.2).

Amongst the important mapping properties of pseudodifferential operators we note

$$
\begin{aligned}
A &: \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \\
A &: \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X) \qquad \exists\, A \in \Psi^m(X) \\
A &: H^s(X) \longrightarrow H^{s-m}(X)
\end{aligned}
\tag{6.9.21}
$$

and for any embedded submanifold, Y, of X

$$
\begin{aligned}
A &: I^r(X,Y) \longrightarrow I^{r+m}(X,Y) \\
\sigma_{m+r}(Au) &= \sigma_m(A)\cdot\sigma_r(u)
\end{aligned}
\quad \forall\, A \in \Psi^m(X).
\tag{6.9.22}
$$

Of course the most significant properties of pseudodifferential operators are concerned with composition. We shall discuss the proof of the basic result

$$
\begin{aligned}
\Psi^m(X)\cdot\Psi^{m'}(X) &\subset \Psi^{m+m'}(X) \\
\sigma_{m+m'}(A\cdot B) &= \sigma_m(A)\cdot\sigma_{m'}(B)
\end{aligned}
\tag{6.9.23}
$$

in some detail. It is closely connected to (6.9.22). The composition property will be deduced from a more general push-forward result for conormal distributions. Simple extensions of this latter result lead to the composition formulae for the more general classes of operators discussed below.

6.10. Conormal distributions in the interior

We briefly recall the properties of conormal distributions associated to an embedded submanifold of a (compact) manifold without boundary. These were formalized by Hörmander [?], see [?] for a full treatment or [?] for a discussion more closely parallel to the remarks here.

If $\partial X = \emptyset$ and $Y \hookrightarrow X$ is an embedded submanifold set

$$
\mathcal{V}(X,Y) = \{V \in \mathcal{C}^\infty(X, TX); V \text{ is tangent to } Y\}.
\tag{6.10.1}
$$

Conormal distributions (with respect to Y) are precisely those distributions having stable regularity under the repeated application of elements of $\mathcal{V}(X,Y)$:

$$
\begin{aligned}
I^{[s]}(X,Y) &= \left\{u \in H^s(X); \mathcal{V}(X,Y)^k u \subset H^s(X)\ \forall\, k\right\} \\
I^*(X,Y) &= \bigcup_{S\in\mathbb{R}} I^{[s]}(X,Y).
\end{aligned}
\tag{6.10.2}
$$

Exercise 6.10.31. Let X be a compact manifold with boundary and $2X$ its double. Show that for any $s \in \mathbb{R}$ the restriction map (3.3.6) gives a surjective map

$$
I^{[s]}(2X, \partial X) \twoheadrightarrow \mathcal{A}^{[s]}(X).
\tag{6.10.3}
$$

The spaces $I^{[s]}(X,Y)$ are clearly $\mathcal{C}^\infty$modules, i.e.

$$
\mathcal{C}^\infty(X)\cdot I^{[s]}(X,Y) = I^{[s]}(X,Y) \quad \forall\, s
\tag{6.10.4}
$$

and the action of differential operators is the same as for the Sobolev spaces

$$
\mathrm{Diff}^m(X)\cdot I^{[s]}(X,Y) \subset I^{[s-m]}(X,Y) \quad \forall\, m, s.
\tag{6.10.5}
$$

Near any point $p \in Y$ we can always find local coordinates $x = (z, y), z = (z_1, \dots, z_d)$, $y = (y_1, \dots, y_N)$, $n = d + N$, in some open neighbourhood Ω of p such that

$$Y \cap \Omega = \{z = 0\}. \tag{6.10.6}$$

Suppose that $a \in \mathcal{C}^\infty(\mathbb{R}^N_y \times \mathbb{R}^d_\zeta)$ is a symbol of order M :

$$\begin{gathered} \text{For each } K \subset\subset \mathbb{R}^N,\ \alpha \in \mathbb{N}^N \text{ and } \beta \in \mathbb{N}^d\ \exists\ C_{k,\alpha,\beta} \text{ s.t.} \\ |D^\alpha_y D^\beta_z a(y,\zeta)| \le C_{k,\alpha,\beta}(1+|\zeta|)^{M-|\beta|} \text{ on } K \times \mathbb{R}^d. \end{gathered} \tag{6.10.7}$$

Then for any $\phi \in \mathcal{C}^\infty_c(\Omega)$ the distributions obtained in local coordinates by inverse Fourier transformation of a is conormal:

$$u = (2\pi)^{-d}\phi(z,y)\int e^{iz\cdot\zeta}a(y,\zeta)d\zeta \in I^{[s]}(X,Y) \quad \forall\ s < -M - \frac{d}{2}. \tag{6.10.8}$$

Definition 6.10.21. For each $m \in \mathbb{R}$ the space $I^m(X,Y)$ is the linear span of $\mathcal{C}^\infty(X)$ and the distributions of the form (6.10.8) (in various local coordinate systems in which (6.10.6) holds) with a of order at most $m + \frac{n}{4} - \frac{d}{2}$.

The local inclusion (6.10.8) can almost be reversed, leading to the following sandwiching result:

$$I^{[-m-\frac{n}{4}+\epsilon]}(X,Y) \subset I^m(X,Y) \subset I^{[-m-\frac{n}{4}-\epsilon]}(X,Y) \quad \forall\ \epsilon > 0. \tag{6.10.9}$$

Thus *every* conormal distribution associated to Y is of the form (6.10.8) in some local coordinate system. In fact (6.10.8) holds in *any* local coordinates in which (6.10.6) holds. This allows each $I^m(X,Y)$ to be topologized by the seminorms of $\mathcal{C}^\infty(X \setminus Y)$ and the best constants in (6.10.7) in a covering by local coordinates, with the ϕ_i being a partition of unity subordinate to this covering near Y. With this topology $I^m(X,Y)$ is a Fréchet space. Moreover

$$\mathcal{C}^\infty(X) \hookrightarrow I^m(X,Y) \text{ is dense in the topology of } I^{m+\epsilon}(X,Y)\ \forall\ \epsilon > 0. \tag{6.10.10}$$

It is necessary to weaken the topology in this way for the same reason that $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is *not* dense in $L^\infty(\mathbb{R}^n)$.

Now from (6.10.9)

$$\begin{gathered} I^*(X,Y) = \bigcup_{m\in\mathbb{R}} I^m(X,Y) \\ I^{-\infty}(X,Y) = \bigcap_{m\in\mathbb{R}} I^m(X,Y) = \mathcal{C}^\infty(X). \end{gathered} \tag{6.10.11}$$

Moreover these spaces admit asymptotic summation. Suppose $u_i \in I^{m_i}(X,Y)$ is a sequence with $m_i \longrightarrow -\infty$ as $i \longrightarrow \infty$. Then there exists an element $u \in I^M(X,Y)$, $M = \sup m_i$, such that

$$u - \sum_{i\le j} u_i \in I^{M(j)}(X,Y),\ M(j) = \sup_{i>j} m_i \quad \forall\ j. \tag{6.10.12}$$

Clearly this relation, written,

$$u \sim \sum_i u_i, \tag{6.10.13}$$

determines u up to a $\mathcal{C}^\infty$term.

The symbol of u, a in (6.10.8), depends on the local coordinates (even the localization factor ϕ) but its leading part does not. Thus if the local coordinates are used to identify the conormal bundle to Y as

$$N^*_\Omega Y = \bigcup_{y\in Y\cap\Omega} N^*_y Y = \{(y,\zeta), \zeta\in\mathbb{R}^d\} \tag{6.10.14}$$

then a defines a fibre density on N^*Y

$$\phi(0,y)\cdot a(y,\zeta)|d\zeta| \in S^{m+\frac{n}{4}-\frac{d}{2}}(N^*Y)\otimes\Omega_{\text{fibre}}. \tag{6.10.15}$$

Here the tensor product is over $\mathcal{C}^\infty(Y)$ (since $|d\zeta|$ is translation-invariant on the fibres.) With this identification the symbol is independent of choices, up to a lower order term. Writing

$$S^{\{M\}}(N^*Y) = S^M(N^*Y)/S^{m-1}(N^*Y)$$

this means that if u has support in a coordinate patch then (6.10.8) holds with $\phi\equiv 1$ on $\operatorname{supp}(u)$ and the symbol

$$[a|d\zeta|] \in S^{\{M\}}(N^*Y)\otimes\Omega_{\text{fibre}},\ M = m+\frac{n}{4}-\frac{d}{2} \tag{6.10.16}$$

is independent of the choice of local coordinates. Using a partition of unity to extend this by linearity leads to the symbol map

$$I^m(X,Y)\xrightarrow[\sigma_m]{} S^{\{M\}}(N^*Y)\otimes\Omega_{\text{fibre}},\ M = m+\frac{n}{4}-\frac{d}{2}. \tag{6.10.17}$$

6.11. Operations on conormal distributions

Conormal distributions are smooth along the submanifold with which they are associated. This self-evident statement can be given a precise form in terms of a product decomposition.

PROPOSITION 6.11.1. *Suppose $X = X_1\times X_2$ is a product of compact manifolds without boundary and $Y_2\subset X_2$ is a embedded submanifold then if $Y = X_1\times Y_2\subset X$*

$$I^m(X,Y) = \mathcal{C}^\infty(X_1; I^M(X_2,Y_2)),\ M = m+\frac{n_1}{4}. \tag{6.11.1}$$

Note that the splitting gives N^*Y a product decomposition, $N^*Y = X_1\times N^*Y_2$ and then under the identification (6.11.1)

$$\sigma_m(u) = \sigma_M(\tilde u) \qquad u\leftrightarrow\tilde u. \tag{6.11.2}$$

One direct consequence of (6.11.1) is the behaviour of conormal distributions with respect to pull-back by an embedding. Let $f: X_1\longrightarrow X_2$ be an embedding of compact manifolds, i.e. f_* is injective and f is $1-1$. Suppose $Y_2\subset X_2$ is an embedded submanifold which is transversal to f :

$$\forall\, p\in Y_1 = f^{-1}(Y_2),\ f_*(T_pX_1)+T_{f(p)}Y_2 = T_{f(p)}X_2. \tag{6.11.3}$$

Then $Y_1\subset X_1$ is an embedded submanifold of X_1.

PROPOSITION 6.11.2. *Suppose $f\colon X_1\longrightarrow X_2$ is an embedding which is transversal to the submanifold $Y_2\subset X_2$ then if $Y_1 = f^{-1}(Y_2)$ pull-back of $\mathcal{C}^\infty$ functions extends by continuity to*

$$f^*: I^m(X_2,Y_2)\longrightarrow I^M(X_1,Y_1),\ M = m+\frac{1}{4}(\dim X_2-\dim X_1) \tag{6.11.4}$$

and moreover if $f^\#: N^*Y_1 \longrightarrow N^*_{f(Y_1)}Y_2$ *is the bundle isomorphism,* $f^\#(y, df^*g(y)) = (f(y), dg)$ *then*

$$\sigma_M(f^*u) = (f^\#)^*\sigma_m(u). \tag{6.11.5}$$

PROOF. This result is local so it suffices to consider the case where $X_2 = X_1 \times X_2'$ with f the identification of X_1 with $X_1 \times \{p\}$ for some $p \in X_2'$ and $Y_2 = Y_1 \times X_2'$. Then (6.11.4), with X_1, X_2 and Y_2 replaced by X_2', X_1, Y_1 gives (6.11.4) by freezing coefficients of ϕ; (6.11.5) follows from (6.11.2). □

The similar, simpler, result for push-forward is for projections, Thus suppose $g : X_1 \times X_2 \longrightarrow X_2$ is projection off the first factor. If $Y_1 \subset X_1 \times X_2$ is an embedded submanifold such that

$$T_{(p_1,p_2)}(X_1 \times \{p_2\}) \cap T_{(p_1,p_2)}Y_1 = \{0\} \quad \forall\ (p_1, p_2) \in Y_1 \tag{6.11.6}$$

then $g : Y_1 \longrightarrow X_2$ is an immersion. We assume additionally that

$$g : Y_1 \longrightarrow X_2 \text{ is an embedding (i.e. 1-1).} \tag{6.11.7}$$

PROPOSITION 6.11.3. *Suppose* $g : X_1 \times X_2 \longrightarrow X_2$ *is the projection and* $Y_1 \subset X_1 \times X_2$ *is an embedded submanifold satisfying* (6.11.6) *and* (6.11.7)*, then if* $Y_2 = g(Y_1)$

$$g_* : I^M(X_1 \times X_2, Y_1; \Omega) \longrightarrow I^M(X_2, Y_2; \Omega),\ M = m - \frac{1}{4}\dim X_1, \tag{6.11.8}$$

and (if (6.11.7) *is not an isomorphism)*

$$\sigma_M(g_*u) = (g^\#)^*\left[\sigma_m(u) \restriction [N^*Y_1 \cap g^*(T^*X_2)]\right] \tag{6.11.9}$$

where $g^\# : N^*Y_2 \longrightarrow N^*Y_1$ *is the bundle map which is an isomorphism onto the subbundle* $N^*Y_1 \cap g^*(T^*X_2)$*, the space of conormal directions to* Y_1 *also conormal to the fibres.*

PROOF. By assumption Y_2 is just Y_1 embedded in X_2 by (6.11.7). Since it suffices to work locally we can assume that $X_2 = Y_2' \times X_2'$ with $Y_2 = Y_2' \times \{p_2'\}$ and $Y_1 = \{p_1\} \times Y_2' \times \{p_2\} \subset X_1 \times X_2$. Proposition 6.11.1 allows Y_2' to be treated as a parameter space, reducing the problem to the special case $Y = \{p_1\} \times \{p_2\}$ is a point. Then (6.11.6) and (6.11.7) are automatic and push-forward is just integration in some of the z variables in (6.10.8). The Fourier inversion formula immediately gives (6.11.8) and (6.11.9) in this case and the general case case follows by reintroducing the parameters. □

Note that the special case excluded in (6.11.9) is trivial, but of major importance:

$$\begin{gathered} \text{If } g : Y_1 \longrightarrow Y_2 \text{ in (6.11.7) is an isomorphism then} \\ g_* : I^m(X_1 \times X_2, Y_1; \Omega) \longrightarrow \mathcal{C}^\infty(X; \Omega) \quad \forall\ m. \end{gathered} \tag{6.11.10}$$

The next basic result involves the pairing of conormal distributions. Recall that we have topologized $I^m(X, Y)$ by symbol seminorms and the seminorms on $\mathcal{C}^\infty(X\backslash Y)$; we make strong use of the density of $\mathcal{C}^\infty(X)$ in $I^m(X, Y)$ in the weaker topology of $I^{m+\epsilon}(X, Y), \epsilon > 0$.

PROPOSITION 6.11.4. *If $Y_1, Y_2 \subset X$ are embedded submanifolds which meet transversally,*

$$T_pY_1 + T_pY_2 = T_pX \qquad \forall\, p \in Y_1 \cap Y_2 \tag{6.11.11}$$

then $\forall\, m, m'$

$$I^m(X, Y_1; \Omega) \hookrightarrow [I^{m'}(X, Y_2)]', \tag{6.11.12}$$

with the injection continuous in the strong dual topology.

PROOF. The transversality condition (6.11.11) implies that $Z = Y_1 \cap Y_2$ is an embedded submanifold. Locally we can therefore decompose the space as a product

$$X = Y_1' \times Z \times Y_2',\ Y_2 = Y_1' \times Z \times \{p_2'\},\ Y_2 = \{p_1'\} \times Z \times Y_2'. \tag{6.11.13}$$

As usual the factor Z can be suppressed, as a parameter space, so we can assume $X = Y_1' \times Y_2'$, etc. Let $\nu = \nu_1 \cdot \nu_2$ be a product of positive smooth densities, so

$$I^m(X, Y_1; \Omega) = I^m(X; Y_1) \cdot \nu. \tag{6.11.14}$$

We want to define the pairing $\langle u \cdot \nu, v\rangle$, $u \in I^m(X, Y_1)$, $v \in I^{m'}(X, Y_2)$. Since Y_1' is a parameter space for the conormal distributions for $Y_1 = Y_1' \times \{p_2'\}$ we can expand in Taylor series around p_1' :

$$u = \sum_{|\alpha|<N} g_\alpha u_\alpha + g_{(N)} u_{(N)} \tag{6.11.15}$$

where $g_\alpha, g_{(N)} \in \mathcal{C}^\infty(Y_1')$ vanish to order $|\alpha|, N$ respectively at p_1', $u_\alpha \in I^{m+\frac{n_1}{4}}(Y_2', \{p_2'\})$ and $u_{(N)} \in I^m(X, Y_1)$. The pairing (for the moment with $v \in \mathcal{C}^\infty(X)$) can be expanded using (6.11.15) to

$$\langle u\nu, v\rangle = \sum_{|\alpha|<N} \langle u_\alpha \nu_1, g_\alpha v \nu_2\rangle + \langle u_{(N)}\nu_1, g_{(N)} v \nu_2\rangle. \tag{6.11.16}$$

The last term is well-defined by continuity if N is large enough. Thus if $N > m + m' + \frac{n}{2}$ then

$$g_{(N)} \cdot I^{m'}(X, Y_2) \subset I^{m'-N}(X, Y_2) \subset H^{(-m'+N-\frac{n}{4}-\epsilon)}(X) \tag{6.11.17}$$

by (6.10.9). Then (6.11.12) and the duality of Sobolev spaces shows that the pairing makes sense. It therefore suffices to show that each term in the sum in (6.11.16) also extends by continuity.

We can similarly expand v in Taylor series, in the parameters in Y_2', around p_2'. This replaces (6.11.16) by a double sum, since the remainder is again defined by Sobolev pairing, of the form

$$\begin{aligned} \langle uv, v\rangle = &\sum_{|\alpha|,|\beta|<N} \langle h_\beta u_\alpha \nu_1, g_\alpha v_\beta \nu_2\rangle \\ &+ \sum_{|\alpha|<N} \langle h_{(N)} u_\alpha \nu_1, g_\alpha v_{(N)} \nu_2\rangle + \langle u_{(N)}\nu_1, g_{(N)} v \nu_2\rangle. \end{aligned} \tag{6.11.18}$$

Now all the terms in the double sum are well-defined since they are just exterior products paired with 1 :

$$\mathcal{C}^{-\infty}(Y_1', \Omega_1) \cdot \mathcal{C}^{-\infty}(Y_2', \Omega_2) \subset \mathcal{C}^{-\infty}(Y_1' \times Y_2', \Omega). \tag{6.11.19}$$

This proves the proposition. □

One important way of interpreting the pairing (6.11.12) is that it gives a separately continuous extension of the multiplication of $\mathcal{C}^\infty$ functions:

$$I^m(X,Y_1)\cdot I^{m'}(X,Y_2)\subset\mathcal{C}^{-\infty}(X)\quad\forall\ m,m', \tag{6.11.20}$$

provided of course (6.11.11) holds. This product actually takes us outside the realm of conormal distributions associated to embedded submanifolds into the wider class of 'product-type' distributions. We do not need to develop the theory here but we note the following

EXERCISE 6.11.32. If $Y_1 \pitchfork Y_2$ (i.e. (6.11.11) holds) let $\mathcal{V}(X,Y_1\cup Y_2)$ be the space of vector fields on X tangent to *both* Y_1 and Y_2 :

$$\mathcal{V}(X,Y_1\cup Y_2)=\mathcal{V}(X,Y_1)\cap\mathcal{V}(X,Y_2). \tag{6.11.21}$$

Define the spaces of conormal distributions, $I^{[s]}(X,Y_1\cup Y_2)$ associated to $Y_1\cup Y_2$ by direct extension of (6.10.2), replacing $\mathcal{V}(X,Y)$ by $\mathcal{V}(X,Y_1\cup Y_2)$. Show that for some s, depending on m,m' and dimensions, (6.11.20) can be improved to

$$I^m(X,Y_1;\Omega)\cdot I^{m'}(X,Y_2)\subset I^{[s]}(X,Y_1\cup Y_2;\Omega). \tag{6.11.22}$$

In Chapter 5 we have developed a theory of spaces analogues to these, when Y_1 and Y_2 are boundary faces of a manifold with corners. The reason that we do not need a complete symbolic theory (although this is readily done, see [?]), is that an appropriate push-forward takes us back to standard conormal distributions.

Thus suppose $Y_1,Y_2\subset X$ are embedded submanifolds meeting transversally. Let $f:X\longrightarrow X'$ be a projection with the following additional conditions on its relation to Y_1,Y_2 and $Y_1\cap Y_2$:

$$\begin{aligned}&f:Y_1,Y_2\longrightarrow X' \text{ are submersions}\\&f:Y_1\cap Y_2\longrightarrow X' \text{ is an embedding.}\end{aligned} \tag{6.11.23}$$

The second of these implies, or really states, that $f(Y_1\cap Y_2)$ is an embedded submanifold of X'. Lifting under f^* gives a bundle map

$$f^*:N^*f(Y_1\cap Y_2)\longrightarrow N^*(Y_1\cap Y_2) \tag{6.11.24}$$

which, as already noted, is an isomorphism onto $N^*(Y_1\cap Y_2)\cap f^*(T^*X')$, the intersection with the fibre-normals. We can write this as

$$f^\#:N^*f(Y_1\cap Y_2)\longleftrightarrow\left[N^*_{Y_1\cap Y_2}Y_1\oplus N^*_{Y_1\cap Y_2}Y_2\right]\cap f^*(T^*X'). \tag{6.11.25}$$

The sum is direct because of the assumed transversality. Now, if a is a function on N^*Y_1 and b a function on N^*Y_2 this gives a function $f^\#(ab)$ on $N^*f(Y_1\cap Y_2)$. More generally we note that there is also a natural identification of density bundles at $Y_1\cap Y_2$

$$\Omega_y(Y_1)\otimes\Omega_{\text{fibre}}(N^*Y_2)\simeq\Omega_y(Y_1\cap Y_2),\ y\in Y_1\cap Y_2. \tag{6.11.26}$$

Indeed this just follows from the fact that, at $Y_1\cap Y_2$, the normal bundle to Y_2 in X can be naturally identified with the normal bundle to $Y_1\cap Y_2$ in Y_1 so

$$\Omega_y(Y_1)\simeq\Omega_y(Y_1\cap Y_2)\otimes\Omega_{\text{fibre},y}(NY_1). \tag{6.11.27}$$

The point of (6.11.26) is that, using (6.11.25), we get a bilinear map

$$\begin{aligned}(f^\#)^*:[\mathcal{C}^\infty(N^*Y_1)\otimes\Omega(Y_1)]&\times[\mathcal{C}^\infty(N^*Y_2)\otimes\Omega_{\text{fibre}}(N^*Y_2)]\\&\longrightarrow\mathcal{C}^\infty(N^*Z)\otimes\Omega(Z),\ Z=f(Y_1\cap Y_2).\end{aligned} \tag{6.11.28}$$

Now we can state the result!

PROPOSITION 6.11.5. *Suppose $f : X \longrightarrow X'$ is a projection of compact manifolds and Y_1, $Y_2 \subset X$ are two embedded submanifolds meeting transversally, for which* (6.11.23) *holds, then the bilinear map defined using* (6.11.20) *gives*

$$f_* \left[I^m(X, Y_1; \Omega) \cdot I^{m'}(X, Y_2) \right] \subset I^M(X', Z; \Omega),$$
$$Z = f(Y_1 \cap Y_2),\ M = m + m' + \frac{1}{4} \dim X' \tag{6.11.29}$$

and using (6.11.28),

$$\sigma_M(f_*(uv)) = (f^\#)^*(\sigma_m(u) \cdot \sigma_{m'}(v)). \tag{6.11.30}$$

PROOF. As usual we successively simplify the situation until the proof becomes a direct computation. First note that the variables in Z, or $Y_1 \cap Y_2$, are parameters. Removing these locally, using Proposition 6.11.1 repeatedly, we can assume

$$X = Y_1' \times Y_2',\ Y_1 = Y_1' \times \{p_2'\},\ Y_2 = \{p_1'\} \times Y_2', \tag{6.11.31}$$

with $Y_1 \cap Y_2$ and Z a point. This makes the last condition in (6.11.23) trivial.

There is a rather obvious symmetry between Y_1 and Y_2, so we choose Y_1 to have codimension greater than, or equal to, that of Y_2. The assumption (6.11.23) on Y_1 now means that we can introduce local coordinates, near $p \in Y_1 \cap Y_2$, (y', y'', x) such that

$$f(y', y'', x) = y', \qquad Y_1 = \{x = 0\}. \tag{6.11.32}$$

By assumption $\{(0, 0, 0)\} = Y_1 \cap Y_2$. Moreover the transversality means that, locally,

$$Y_2 = \{y' = Y'(x), y'' = Y''(x)\} \tag{6.11.33}$$

where the condition on Y_2 in (6.11.23) means that rank of $\partial_x Y'$ is equal to the number of y' variables. Since we can freely change the x variables, amongst themselves, and the y'' variables arbitrarily we can choose the coordinates so that (6.11.32) holds and (6.11.33) becomes

$$Y_2 = \{y' = x', y'' = x''\},\ x = (x', x'', x'''). \tag{6.11.34}$$

Now we can factor the projection f into two projections $f = f_2 \cdot f_1$, $f_1(y, x) = y$, $f_2(y) = y'$. The advantage of doing so is that the hypotheses of the proposition still hold for f_1 and if we prove it in that case then the general case follows by applying Proposition 6.11.3 to f_2.

Thus we can assume (6.11.32) and (6.11.34) hold with no y'' variables; changing notation slightly,

$$f(y, x, z) = y, Y_1 = \{x = z = 0\}, Y_2 = \{y = x\}. \tag{6.11.35}$$

Following the proof of Proposition 6.11.4 we can use the formula (6.11.18) for the expansion of the product. If N is large enough the second sum and final remainder term contribute terms to $(f_1)_*\ (uv\nu)$ which becomes increasingly smooth. Thus it suffices to consider $(f_1)_*(uv\mu)$ where u and v are of the form

$$u = (2\pi)^{-n_1} \int e^{ix\cdot\xi + iz\cdot\zeta} a(\xi, \zeta) d\xi d\zeta |dx||dz||dy|$$
$$v = (2\pi)^{-n_2} \int e^{i(y-x)\cdot\eta} b(\eta) d\eta.$$

The cancellation (6.11.26) is $|dx||d\xi||dz||d\zeta| = 1$ and the Fourier inversion formula gives

$$\int uvdxdz|dy| = (2\pi)^{n_2}\int e^{iy\cdot\eta}a(\eta,0)b(\eta)d\eta|dy|.$$

In this simple case, (6.11.30) becomes

$$\sigma_m((f_1)_*(uv\nu)) = a(y,0)b(\eta) \tag{6.11.36}$$

so this completes the proof of the proposition. □

6.12. Conormal distributions at a clean submanifold

Let $Y \subset X$ be a clean submanifold of a compact manifold with corners, with $\partial Y = Y \cap \partial X$. Let $\mathcal{V}_b(X,Y) \subset \mathcal{V}_b(X)$ be the space of $\mathcal{C}^\infty$ vector fields tangent to the boundary of X and to Y. Thus at any $p \in \partial Y$ there are local coordinates in which

$$Y = \{x_i = 0\ k < i \le k+N\}. \tag{6.12.1}$$

In terms of these local coordinates

(6.12.2)

$$\mathcal{V}_b(X,Y) = \operatorname{sp}\{x_i\partial_i,\ 1 \le i \le k,\ x_l\partial_p,\ k < l, p \le k+N,\ \partial_q,\ q > k+N\}.$$

As before, we define the conormal distributions, now singular at Y and ∂X by iterative regularity conditions with respect to these vector fields:

$$\begin{gathered}\dot{\mathcal{A}}(X,Y) = \bigcup_{s\in\mathbb{R}} \dot{\mathcal{A}}^{[s]}(X,Y),\\ \dot{\mathcal{A}}^{[s]}(X,Y) = \left\{u \in \dot{\mathcal{H}}^s(X); \mathcal{V}_b(X,Y)^k u \subset \dot{\mathcal{H}}^s(X)\ \forall\ k \in \mathbb{N}\right\}\\ \mathcal{A}(X,Y) = \bigcup_{s\in\mathbb{R}} \mathcal{A}^{[s]}(X,Y),\\ \mathcal{A}^{[s]}(X,Y) = \left\{u \in \mathcal{H}^s(X); \mathcal{V}_b(X,Y)^k u \subset \mathcal{H}^s(X)\ \forall\ k \in \mathbb{N}\right\}\end{gathered} \tag{6.12.3}$$

These are the largest spaces of conormal distributions associated to Y.

Proposition 6.12.1. *If $Y \subset X$ is a clean submanifold with $\partial Y = Y \cap \partial X$ then*

$$\dot{\mathcal{A}}(X,Y) = (\beta_Y)_*\left(\dot{\mathcal{A}}(X_Y)\right). \tag{6.12.4}$$

Proof. First we check that the push-forward does indeed give a map

$$(\beta_Y)_* : \dot{\mathcal{A}}(X_Y) \longrightarrow \dot{\mathcal{A}}(X,Y). \tag{6.12.5}$$

This follows from the lifting properties of vector fields. Namely each element $V \in \mathcal{V}_b(X,Y)$ lifts to $\tilde{V} \in \mathcal{V}_B(X_Y)$ and

$$(\beta_Y)_*[\tilde{V}v] = V(\beta_Y)_*[v]. \tag{6.12.6}$$

Since β_Y maps each Sobolev space into some Sobolev space on X_Y we conclude that $(\beta_Y)_*$ has iterative regularity with respect to $\mathcal{V}_b(X,Y)$ if $v \in \dot{\mathcal{A}}(X_Y)$. This shows (6.12.5).

Thus we need to show the surjectivity of (6.12.5). Under the blow-up of the clean submanifold Y we have an isomorphism

$$(\beta_Y)^* : \dot{\mathcal{C}}^{-\infty}(X)/\{u \in \dot{\mathcal{C}}^{-\infty}(X); \operatorname{supp}(u) \subset Y\} \longleftrightarrow \dot{\mathcal{C}}^{-\infty}_{\beta_Y^*(\partial X)}(X_Y), \tag{6.12.7}$$

lifting a distribution on X, modulo a term supported on Y, to a distribution on X_Y which is supported across all of the boundary except the front face(produced by blowing up Y.) Moreover this lift maps each Sobolev space into some Sobolev space. Note that under β_Y the vector fields in $\mathcal{V}_b(X,Y)$ lift to span $\mathcal{V}_b(X_Y)$. Thus, reversing the argument above, (6.12.7) becomes

$$(\beta_Y)^* : \dot{\mathcal{A}}(X,Y)/\{u \in \dot{\mathcal{A}}(X,Y); \operatorname{supp}(u) \subset Y\} \longleftrightarrow \dot{\mathcal{A}}_{\beta_Y^*(\partial X)}(X_Y). \tag{6.12.8}$$

Since extendible conormal distributions can always be extended to supported conormal distributions, given $u \in \dot{\mathcal{A}}(X,Y)$ we can in fact find $v \in \dot{\mathcal{A}}(X_Y)$ such that $(\beta_Y)_* v - u - u'$ has support in Y. By (6.12.5) we know that $u' \in \dot{\mathcal{A}}(X,Y)$ so, in local coordinates (6.12.1) near $\partial X \cup Y$ it is of the form

$$u' = \sum_{|\alpha| \le q} u_\alpha(x', x''') D^\alpha_{x'} \delta(x'), \tag{6.12.9}$$

where $x' = (x_1, \dots, x_k)$ are defining functions for the boundary, the components of $x' = (x_{k+1}, \dots, x_{k+N}$ are defining functions for Y and $x''' = (x_{k+N+1}, \dots, x_n)$. The conormality of u' shows immediately that the coefficients $u_\alpha(x', x''')$ are conormal with respect to $x' = 0$. Thus we can also lift u' to be conormal on X_Y simply by lifting the delta functions to be conormal. This completes the proof of the proposition. $\square$

One obvious space of this type is:

$$I(X,Y) = \dot{\mathcal{A}}(X,Y) \cap \mathcal{B}(X), \tag{6.12.10}$$

with elements which are smooth up to the boundary. We certainly have

$$I(X,Y) \cap \dot{\mathcal{A}}(X) = \mathcal{C}^\infty(X) \text{ if } Y \cap \overset{\circ}{\to} X \neq \emptyset \tag{6.12.11}$$

so all the singularities are associated to Y. To make this more precise we first investigate lifts of these distributions under blow-up of Y and obtain the following refinement of Proposition 6.12.1.

PROPOSITION 6.12.2. *If $Y \subset X$ is a clean submanifold with $\partial Y = \partial X \cap Y$ then*

$$I(X,Y) = (\beta_Y)_* \left(\dot{\mathcal{A}}(X_Y) \cap \mathcal{B}_{\beta_Y^*(\partial X)}(X_Y) \right). \tag{6.12.12}$$

PROOF. The proof of Proposition 6.12.1 can be followed very closely. Thus to replace (6.12.5) we note that

$$(\beta_Y)_* : \mathcal{B}_{\beta_Y^*(\partial X)} \longrightarrow \mathcal{B}(X_Y) \tag{6.12.13}$$

so

$$(\beta_Y)_* : \dot{\mathcal{A}}(X_Y) \cap \mathcal{B}_{\beta_Y^*(\partial X)} \longrightarrow I(X,Y). \tag{6.12.14}$$

The lifting argument also gives a lift which is smooth up to $\beta_Y^*(\partial X)$ so the remainder of the argument goes through directly. $\square$

There is another obvious way to define the conormal distributions associated to Y and smooth up to the boundary, namely as the restrictions of conormal distributions for some extension. That this is actually an equivalent definition follows from Proposition 6.12.2.

PROPOSITION 6.12.3. *If $Y \hookrightarrow X$ is clean and $\tilde{X}$ extends X, $\partial\tilde{X} = \emptyset$, then t here is an embedded submanifold (possibly not closed) $\tilde{Y}$ extending Y and then*

$$I(X,Y) = I(\tilde{X},\tilde{Y})|_X. \tag{6.12.15}$$

PROOF. The existence of the extension $tildeY$ has already been shown. The naturality properties of blow-up show that $\tilde{X}_{\tilde{Y}}$ is an extension of X_Y across $\beta_Y^*(\partial X)$. We have shown in Chapter5 that the space on the right

$$\dot{\mathcal{A}}(X_Y) \cap \mathcal{B}_{\beta_Y^*(\partial X)}(X_Y) = \dot{\mathcal{A}}(\tilde{X}_{\tilde{Y}} \restriction X_Y. \tag{6.12.16}$$

Since the push-forward of $\dot{\mathcal{A}}(\tilde{X}_{\tilde{Y}})$ under the blow-down map is $I(\tilde{X},\tilde{Y})$ (6.12.15) follows. □

Now we wish to carry the definition of order across to these clean submanifolds by setting

$$I^M(X,Y) \subset I(\tilde{X},\tilde{Y})_{\restriction X}. \tag{6.12.17}$$

The only problem with doing this is that we have to make sure that the space is independent of the choice of extension. This is easily seen by noting that if X', Y' are also extension of X, Y across the boundary then there is a diffeomorphism between a neighbourhood of X in X' and a neighbourhood of X in $\tilde{X}$ which fixes X point wise and maps Y' to $\tilde{Y}$. Since such a diffeomorphism maps the conormal distributions and preserves their orders (6.12.17) is meaningful.

This reinterpretation allows us to easily extend the various properties of conormal distributions obtained above to this case of a clean submanifold. For example the symbol map is defined by restriction:

$$I^m(X,Y) \underset{\sigma_m}{\longrightarrow} S^{\{M\}}(N^*Y) \otimes \Omega_{\text{fibre}}. \tag{6.12.18}$$

Again we have to check that this is well-defined but this is a consequence of the fact, following directly from the definition of the symbol, that if $u \in I^M(\tilde{X},\tilde{Y})$ vanishes in X then its symbol vanishes over X. Clearly this leads to an exact symbol sequence precisely as in (??).

6.13. Operations on $I(X,Y)$ for Y an interior p-submanifold

The operations on conormal distributions discussed in §6.11 carry over directly to this more general case, of a clean submanifold, provided extra restrictions are placed on the geometry.

The simplest properties extend easily, for example the density of $\mathcal{C}^\infty(X)$ in $I^m(X,Y)$ in the topology of $I^{m+\epsilon}(X,Y)$ for any $\epsilon > 0$. The action of differential operators is also straight-forward provided we distinguish between supported and extendible distributions. Thus, because of the corresponding property (5.5.3) of $\mathcal{B}(X)$, we have

$$\begin{tikzcd} & \dot{\mathcal{C}}^{-\infty}(X) \arrow[dd] \\ I(X,Y) \arrow[ur] & \\ & \mathcal{C}^{-\infty}(X). \end{tikzcd} \tag{6.13.1}$$

Thus if we consider $I(X,Y)$ as a space of extendible distributions we have

$$\begin{aligned} \operatorname{Diff}^m(X)\cdot I^r(X,Y) &\subset I^{m+r}(X,Y) \text{ in } \mathcal{C}^{-\infty}(X) \\ \sigma_{m+r}(Pu) &= \sigma_m(P)_{\restriction N^*Y}\sigma_r(u). \end{aligned} \tag{6.13.2}$$

If however we wish to consider $I(X,Y)$ as a space of supported distributions then we must limit the differential operators to be totally characteristic and then we get

$$\begin{aligned} \operatorname{Diff}_b^m(X)\cdot I^r(X,Y) &\subset I^{m+r}(X,Y) \text{ in } \dot{\mathcal{C}}^{-\infty}(X) \\ \sigma_{m+r}(Pu) &= \sigma_m(P)_{\restriction N^*Y}\sigma_r(u). \end{aligned} \tag{6.13.3}$$

Notice that in either case (??) can hold and then the solvability modulo $\mathcal{C}^\infty$ follows as before,

$$\text{For either (6.13.2) or (6.13.3), (??)} \Longrightarrow \text{(??)}. \tag{6.13.4}$$

Next consider the extension of Proposition 6.11.1.

Proposition 6.13.1. *Suppose $X = X_1 \times X_2$ is a product of compact manifolds with corners and $Y_2 \subset X_2$ is a clean submanifold then if $Y = X_1 \times Y_2 \subset X$*

$$I^m(X,Y) = \mathcal{C}^\infty(X_1; I^M(X_2,Y_2)),\ M = m + \frac{n_1}{4}. \tag{6.13.5}$$

Proof. This follows immediately from Proposition 6.11.1 itself by choosing the extension of X to be a product of extensions of X_1 and X_2, with an extension of Y_2 in the latter. □

Since this result allows us to introduce parameters in a simple way we can easily extend the other operations.

Proposition 6.13.2. *Suppose $f : X_1 \longrightarrow X_2$ is an embedding of a manifold with corners as a clean submanifold with*

$$f(\partial X_1) = \partial X_2 \cap f(X_1) \tag{6.13.6}$$

and f is transversal to a clean submanifold, $Y_2 \subset X_2$, in the sense that (6.11.3) *holds and*

$$\forall\ p \in Y_1 \cap \partial X_1,\ \left(N^*_{f(p)}Y_2 + \operatorname{null} f^*_{f(p)}\right) \cap N^*\partial X_2 = \{0\} \tag{6.13.7}$$

then $Y_1 \subset X_1$ is a clean submanifold and (6.11.4) *and* (6.11.5) *hold.*

Proof. This result is local in X_1 and if $p \notin \partial X_1$ then by (6.13.6), $f(p) \notin \partial X_1$ and Proposition 6.11.2 applies directly. This suppose $p \in \partial X_1$. The additional hypothesis (6.13.7) shows that locally near $f(p)$ X_2 can be decomposed as a product

$$X_2 = X_1' \times Z_2,\ \partial Z_2 = \emptyset \tag{6.13.8}$$

in such a way that $X_1 = X_1' \times Z_1$, f is the product of the identity on X_1' and an embedding, f', of Z_1 in Z_2 and Y_2 is an embedded submanifold of Z_2. It follows that Z_1 is transversal to Y_2 in Z_2. Thus, once Proposition 6.13.1 is applied to remove the parameters in X_1, Proposition 6.11.2 can be applied to f' as an embedding of Z_1 in Z_2 and the parameters subsequently restored using Proposition 6.13.1. This proves the proposition. □

A similar argument leads to the following generalization of Proposition 6.11.3:

Proposition 6.13.3. *Suppose $g : X_1 \times X_2 \longrightarrow X_2$ is the projection of the first factor in a product of compact manifolds with corners. Let $Y_1 \subset X_1 \times X_2$ be a clean submanifold with $\partial Y_1 = Y_1 \cap (X_1 \times \partial X_2)$ and suppose that*

$$g : Y_1 \longrightarrow X_2 \text{ is an embedding} \tag{6.13.9}$$

and (6.11.6) *holds then if $Y_2 = g(Y_1)$*

$$g_* : I^M(X_1 \times X_2, Y_1; \Omega) \longrightarrow I^M(X_2, Y_2; \Omega),\ \ M = m - \frac{1}{4}\dim X_1, \tag{6.13.10}$$

and (if (6.13.9) *is not an isomorphism)*

$$\sigma_M(g_* u) = (g^{\#})^* \left[\sigma_m(u) \restriction [N^*Y_1 \cap g^*(T^*X_2)]\right] \tag{6.13.11}$$

where $g^{\#} : N^*Y_2 \longrightarrow N^*Y_1$ is the bundle map which is an isomorphism onto the subbundle $N^*Y_1 \cap g^*(T^*X_2)$, the space of conormal directions to Y_1 also conormal to the fibres.

Similarly we can extend Proposition 6.11.4 readily:

Proposition 6.13.4. *If Y_1, $Y_2 \subset X$ are clean submanifolds of a manifold with corners, with $\partial Y_i = Y_i \cup X$ for $i = 1, 2$ then provided they intersect transversally in the sense of* (6.11.11) *and*

$$\left(N_p^*Y_1 + N_p^*Y_2\right) \cap N_p^*X = \{0\} \quad \forall\ p \in Y_1 \cap Y_2 \cap \partial X, \tag{6.13.12}$$

then (6.11.12) *holds.*

This in turn leads to the important extension of Proposition 6.11.5:

Proposition 6.13.5. *Suppose that $f : X = X' \times X'' \longrightarrow X'$ is projection off the second fact of a product of compact manifolds with corners, that Y_1, $Y_2 \subset X$ are two embedded submanifolds satisfying* (6.11.11) *and* (6.13.12) *then provided* (6.11.23) *holds so does* (6.11.29).