CHAPTER 5

Blowing up submanifolds

In order to simplify various analytic constructions below it turns out to be both natural and helpful to introduce polar coordinates along submanifolds. Since it may be necessary to do this repeatedly we shall give a systematic treatment of the process, starting with the basic case of radial (or normal) blow up. After examining the general case of the blow up of a closed p-submanifold the commutation of blow ups and the lifting of maps and submanifolds under iterated blow up is considered. The logarithmic blow up of boundary hypersrufaces and it relation to ordinary blow up is also discussed.

5.1. The origin of a vector space

Blowing up of the origin in $\mathbb{R}^n$ simply amounts to the introduction of polar coordinates. We *define* $\mathbb{R}^n$ blown up at $\{0\}$ to be

$$[\mathbb{R}^n;\{0\}] = \mathbb{S}^{n-1}\times[0,\infty) \tag{5.1.1}$$

together with the associated blow-down map

$$\beta=\beta[\mathbb{R}^n;\{0\}],\ \beta\colon[\mathbb{R}^n;\{0\}]\longrightarrow\mathbb{R}^n,\ \beta(\omega,r)=r\omega. \tag{5.1.2}$$

This is a diffeomorphism of $[\mathbb{R}^n;\{0\}]\backslash\left(\{0\}\times\mathbb{S}^{n-1}\right)$ onto $\mathbb{R}^n\backslash\{0\}$ and has rank 1 at the boundary $\partial[\mathbb{R}^n;\{0\}]=\mathbb{S}^{n-1}\times\{0\}$, which projects to $\{0\}$.

In general for a vector space V we define

$$[V;\{0\}] = \left((V\backslash\{0\})/\mathbb{R}^+\right)\sqcup\left(V\backslash\{0\}\right). \tag{5.1.3}$$

Thus the blow-up of V at $\{0\}$ is the disjoint union of the (projective) sphere in V and the complement of $\{0\}$. The choice of a basis in V gives a linear isomorphism $V\longleftrightarrow\mathbb{R}^n$ which allows us to identify $[V;\{0\}]$ and $[\mathbb{R}^n;\{0\}]$. To show that the $\mathcal{C}^\infty$ structure, as a manifold with boundary, of $[V;\{0\}]$ is well defined we therefore need only:

Lemma 5.1.1. *The action of* $\mathrm{GL}(n)$ *on* $\mathbb{R}^n$ *lifts to a* $\mathcal{C}^\infty$ *action of* $\mathrm{GL}(n)$ *on* $[\mathbb{R}^n;\{0\}]$.

Proof. If $A\in\mathrm{GL}(n)$ the action, $\widetilde{A}$, on $[\mathbb{R}^n;\{0\}]$ is fixed by the commutativity of

$$\begin{array}{ccc} [\mathbb{R}^n;\{0\}] & \xrightarrow{\tilde{A}} & [\mathbb{R}^n;\{0\}] \\ \downarrow{\scriptstyle\beta} & & {\scriptstyle\beta}\downarrow \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \end{array} \tag{5.1.4}$$

where β is the blow-down map.

An invertible linear transformation $Ax_i = \sum_{j=1}^{n} a_{ij}x_j$ maps $\mathbb{R}^n \setminus \{0\}$ onto itself and can be written in terms of polar coordinates as

$$\tilde{A}(r,w) = (r',\omega'),\ \omega' = \frac{A\omega}{|A\omega|},\ r' = |A\omega|r. \tag{5.1.5}$$

Since $\omega \neq 0$ and A is invertible $|A\omega|$ depends smoothly on ω, so (5.1.5) extends to be $\mathcal{C}^\infty$ down to $r = 0$. The smooth dependence of $\tilde{A}$ on the coefficients of A also follows from (5.1.5). □

Notice that the smooth action of $\mathrm{GL}(n)$ on $[\mathbb{R}^n;\{0\}]$ means that the Lie algebra, $\mathfrak{gl}(n)$, lifts to $[\mathbb{R}^n;\{0\}]$. Since the exponentials of linear vector fields are linear transformations this means that for each i and j there are smooth vector fields V_{ij} on $[\mathbb{R}^n;\{0\}]$ such that

$$\beta_* V_{ij} = x_i D_j. \tag{5.1.6}$$

This in turn shows that *any* $\mathcal{C}^\infty$ vector field on $\mathbb{R}^n$ which vanishes at 0 lifts to a smooth vector field on $[\mathbb{R}^n;\{0\}]$:

$$\sum_{i,j=1}^{n} a_{ij}(x)x_i D_j \xrightarrow{\beta_*^{-1}} \sum_{i,j=1}^{n} a_{ij}(r\omega)V_{ij}. \tag{5.1.7}$$

Lemma 5.1.2. *The vector fields V_{ij}, fixed by (5.1.6), span $\mathcal{V}_b([\mathbb{R}^n;\{0\}])$ over $\mathcal{C}^\infty([\mathbb{R}^n;\{0\}])$ and any local diffeomorphism $f\colon \Omega \longrightarrow \Omega'$ where $0 \in \Omega \subset \mathbb{R}^n$ and $0 \in \Omega' \subset \mathbb{R}^n$ are open and $f(0) = 0$, lifts to a diffeomorphism*

$$\tilde{f}\colon \beta^{-1}(\Omega) \longrightarrow \beta^{-1}(\Omega').$$

Proof. The first part of the lemma follows from Lemma 2.1.2, so consider the lift of a diffeomorphism.

Away from the boundary, $\mathbb{S}^{n-1}$, of the blown-up space $[\mathbb{R}^n;\{0\}]$ the lift is immediate. Shrinking Ω and Ω', if necessary, f can be factored $f = f_1 \circ A$ where A is the Jacobian of f of 0. Thus $(f_1)_* = \mathrm{Id}$ at 0, i.e. $f(x) = x + f'(x)$ with $|f'(x)| = (|x|^2)$. Then $f_t(x) = x + tf'(x)$ is an homotopy near 0 satisfying

$$\begin{gathered} f_0 = \mathrm{Id},\ f_t|_{t=1} = f_1,\ f_t(0) = 0\ \forall\ t \in [0,1] \\ \text{and } (f_t)_* = \mathrm{Id} \text{ at } 0\ \forall\ t \in [0,1]. \end{gathered} \tag{5.1.8}$$

Such a 1-parameter family of maps is always given by the integration of a (parameter dependent) vector field W_t. That is, f_t is fixed locally by

$$\frac{d}{dt} f_t^* \phi = f_t^*[W_t\phi]\ \forall\ \phi \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{5.1.9}$$

and conversely this determines W_t, given f_t. Now, since $f_t(0) = 0$, W_t vanishes at 0. Thus W_t can be lifted to $[\mathbb{R}^n;\{0\}]$ using (5.1.7). Let the lift be $\tilde{W}_t$. The lifted diffeomorphism corresponding to f_t is determined by the analogue of (5.1.9)

$$\frac{d}{dt}\tilde{f}_t^* \psi = \tilde{f}_t^*[\tilde{W}_t\psi]\ \forall\ \psi \in \mathcal{C}^\infty([\mathbb{R}^n;\{0\}]),\ \tilde{f}_0 = \mathrm{Id}. \tag{5.1.10}$$

Finally then the lift of f_1 is $\tilde{f}_1 \circ \tilde{A}$, proving the lemma. □

5.2. Blowing up the zero section

If V is a vector bundle over the base (a manifold with corners) Y then we identify Y as the zero section of V and define V blown up along Y to be

$$[V;Y] = \bigsqcup_{y\in Y} [V_y;\{0\}],\ \beta = \beta[V;Y]\colon [V;Y] \longrightarrow V, \tag{5.2.1}$$

i.e. simply blow up the origin of each fibre.

Again $[V;Y]$ has a natural $\mathcal{C}^\infty$ structure as a manifold with corners. We define $\mathcal{C}^\infty([V;Y])$ by reference to local trivializations of V. Thus if V is trivialized over an open set $\Omega \subset Y$

$$\pi^{-1}(\Omega) \simeq \Omega \times V_{\bar y},\ \bar y \in \Omega \tag{5.2.2}$$

the space of $\mathcal{C}^\infty$ functions on $\beta^{-1}(\pi^{-1}(\Omega))$ is fixed by the product structure. To show that this is a $\mathcal{C}^\infty$ structure we need to check that a (local) bundle isomorphism

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d] & U \arrow[d] \\ Y \arrow[r, "\mathrm{Id}"'] & Y \end{tikzcd} \tag{5.2.3}$$

lifts to a diffeomorphism $\tilde\Phi\colon [V;Y] \longleftrightarrow [U;Y]$. This follows from Lemma 5.1.1, since Φ is linear on the fibres and depends smoothly in the local variables in the base.

We can easily extend this to the case of an $\mathbb{R}^n_k$-bundle over a manifold with corners. Thus the blow-up of $\{0\} \subset \mathbb{R}^n_k$ is just the closure in $[\mathbb{R}^n;\{0\}]$ of the preimage $\beta^{-1}(\mathbb{R}^n_k\backslash\{0\})$. This gives

$$\begin{aligned} \mathbb{R}^n_k;\{0\}t &= \left(\mathbb{S}^{n-1}\cap\mathbb{R}^n_k\right) \sqcup \left(\mathbb{R}^n_k\backslash\{0\}\right) \\ &\cong [0,\infty)\times\mathbb{S}^{n-1}_k \end{aligned} \tag{5.2.4}$$

where $\mathbb{S}^{n-1}_k = \mathbb{R}^n_k \cap \mathbb{S}^{n-1}$; $\mathbb{S}^{n-1}_k$ is a manifold with corners of maximum codimension k unless $k = n$ in which case the maximum codimension is $k-1$.

5.3. The general case

If $Y \subset X$ is a closed p-submanifold (see Definition 1.7.4) we define X blown up along Y to be the set

$$[X;Y] = S^+NY \sqcup (X\backslash Y). \tag{5.3.1}$$

Here S_+NY is the inward-pointing part of the spherical normal bundle to Y, in X. Thus

$$N^+_yY = T^+_yX/T_yY \tag{5.3.2}$$

is the quotient of the inward-pointing part of tangent space to X by the tangent space to Y. Then

$$S^+N_yY = (T^+_yX/T_yY\backslash\{0\})/\mathbb{R}^+. \tag{5.3.3}$$

The *full* normal space at $y \in Y$ would be T_yX/T_yY.

Naturally we wish to show that $\mathcal{C}^\infty([X;Y])$ is well defined, giving $[X;Y]$ the structure of a manifold with corners. This can be done in a straightforward manner by checking the invariance of the local coordinate definition. However we prefer to give a more global proof, which is essentially the same as Lemma 5.1.2. The blow-down map is denoted

$$\beta = \beta[X;Y] \colon [X;Y] \longrightarrow X \tag{5.3.4}$$

and the 'new boundary' or 'front face' of $[X;Y]$ will be denoted

$$\mathrm{ff}([X;Y]) = S^+NY \hookrightarrow [X;Y]. \tag{5.3.5}$$

To construct $\mathcal{C}^\infty([X;Y])$ we use the existence of a normal fibration, as shown in Proposition 2.10.1, to reduce the question to the vector bundle case already considered. Using such an identification of a neighbourhood of Y in X with a neighbourhood of Y as the zero section of N^+Y, the inward-pointing part of the normal bundle to Y in X, we can immediately define $\mathcal{C}^\infty([X;Y])$ by identifying it with $\mathcal{C}^\infty([N^+Y;Y])$ near $\mathrm{ff}(X_Y) = S^+NY$. It remains to show that this is independent of the choice of f. Any two normal fibrations can be connected by a smooth homotopy of normal fibrations so if f_0, and f_1 are two choices there is an homotopy f_t, defined in some neighbourhood of the zero section of NY. As in the proof of Lemma 5.1.2 consider the vector field which generates the homotopy $f_t^{-1}f_0 = F_t$

$$\frac{d}{dt}F_t^* = F_t^* W_t \text{ on } \mathcal{C}^\infty(N^+Y). \tag{5.3.6}$$

Then $W_t \in \mathcal{V}_b(N^+Y)$ is tangent to the zero section. Thus the proof of Lemma 5.1.2 applies directly as soon as one has the following replacement for (5.1.7):

LEMMA 5.3.1. *Any smooth vector field, on a vector bundle, which is tangent to the zero section lifts to a smooth vector field tangent to the front face when the zero section is blown up.*

PROOF. One only needs check this in local trivializations, i.e. for the case of a product $Y \times \mathbb{R}^n$ where Y is a manifold with corners. A vector field tangent to $Y \times \{0\}$ is of the form

$$W_z + \sum_{i,j=1}^{n} a_{ij}(y,z) z_i D_j \;\; z \in \mathbb{R}^n \tag{5.3.7}$$

where W_z is a $\mathcal{C}^\infty$ vector field on Y depending smoothly on $z \in \mathbb{R}^n$ (near 0). The first part lifts since the blown-up space is the product $Y \times [\mathbb{R}^n; \{0\}]$ and the second lifts because of (5.1.7). □

Not only does this complete the proof that $\mathcal{C}^\infty([X;Y])$ is naturally defined, but it also demonstrates the following basic lifting result.

PROPOSITION 5.3.1. *If $Y \hookrightarrow X$ is a closed p-submanifold then the blow-down map $\beta \colon [X;Y] \longrightarrow X$ is a b-map and is a blow-down map in the sense of § 2.3 under which the space of extendible smooth vector fields on X tangent to Y lifts to span, over $\mathcal{C}^\infty([X;Y])$, the space of extendible smooth vector fields on $[X;Y]$ tangent to $\mathrm{ff}([X;Y])$; similarly the subspace of $\mathcal{V}_b(X)$ consisting of vector fields tangent to Y lifts to span, over $\mathcal{C}^\infty([X;Y])$, the whole of $\mathcal{V}_b([X;Y])$.*

PROOF. The spanning result is local and the first part follows by combining the form, (5.3.7), of a vector field tangent to the zero section with Lemma 5.1.2. For the second part it is certainly the case that the elements of $\mathcal{V}_b(X)$ tangent to Y lift into $\mathcal{V}_b([X;Y])$, so it only remains to show that they span. Again this reduces to the product case and hence to the case of the blow up of $\{0\}$ in $\mathbb{R}^n_k$. The argument in

Lemma 2.1.2 can be repeated. Thus the linear vector fields in $\mathcal{V}_b(\mathbb{R}^n_k)$ span $\mathcal{V}_b(\mathbb{R}^n_k)$ away from 0 and hence by homogeneity considerations lift to span $\mathcal{V}_b([\mathbb{R}^n_k;\{0\}])$ over $\mathcal{C}^\infty([\mathbb{R}^n_k;\{0\}])$. Finally it follows from this that β is a blow-down map in the sense of §2.3. □

5.4. Local coordinates

If $Y \hookrightarrow X$ is a p-submanifold in X then there are local coordinates in X at each point $\bar{y}$ of Y such that (1.7.12) holds. Locally in $[X;Y]$ these induce coordinates. From a computational point of view the simplest of these are *projective coordinates.* Given a point $q \in \mathrm{ff}([X;Y])$, with $\beta(q) = \bar{y}$, the remaining local coordinates in (1.7.12) can be so numbered that $\beta^*(dx_L) \neq 0$ at q, where $L = 1$, or $L = k+1$. In these two cases.

$$(5.4.1)\qquad \begin{cases} x_1, \frac{x_2}{x_1}, \dots, \frac{x_{k'}}{x_1}, x_{k'+1}, \dots, x_k, \frac{x_{k+1}}{x_1}, \dots, \frac{x_{k+p}}{x_1}, x_{k+p+1}, \dots, x_n \\ \text{or} \\ \frac{x_1}{x_{k+1}}, \dots, \frac{x_{k'}}{x_{k+1}}, x_{k'+1}, \dots, x_k, x_{k+1}, \frac{x_{k+2}}{x_{k+1}}, \dots, \frac{x_{k+p}}{x_{k+1}}, x_{k+p+1}, \dots, x_k \end{cases}$$

give local coordinates in $[X;Y]$ near q. Notice that they are ordered here so as to be admissible, i.e. product, coordinates in that the boundary of $[X;Y]$ is given locally by the vanishing of the first k coordinates in the first case, or $k+1$ in the second. This reflects the fact that blowing up a p-submanifold may increase the maximum codimension of the boundary faces by one. One case of particular interest is the blow-up of boundary faces. Then only the first case in (5.4.1) can occur and the maximum codimension of a boundary faces does not increase at all.

5.5. Pull-back

For the blow-up of a p-submanifold there is a natural pull-back operation on smooth densities:

$$(5.5.1)\qquad \beta^* : \mathcal{C}^\infty(X;\Omega X) \longrightarrow \mathcal{C}^\infty([X;Y];\Omega([X;Y])).$$

Since ΩX is a trivial bundle it suffices to define β^* for any one non-vanishing section. In the basic case of $Y = \{0\}$, $X = \mathbb{R}^n$, the pull back is just the standard formula for the Lebesgue measure in polar coordinates:

$$(5.5.2)\qquad \beta^*(|dx|) = r^{n-1}|dr d\omega|.$$

The right hand side is the unique smooth measure on $[\mathbb{R}^n;\{0\}]$ which pushes forward to $|dx|$. This shows that (5.5.1) is well-defined. The push-forward of distributional densities gives a left inverse for this pull back:

$$(5.5.3)\qquad \beta_* \circ \beta^* = \mathrm{Id} \ \text{ on } \mathcal{C}^\infty(X;\Omega X).$$

In the special case that $Y \subset \partial X$ notice that the pull-back operation (3.6.1) restricts to give

$$(5.5.4)\qquad \beta^* : \dot{\mathcal{C}}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty([X;Y]) \quad Y \subset \partial X$$

since this is just a special case of (3.8.11).

Lemma 5.5.1. *If $Y \subset X$ is a closed boundary p-submanifold the pull-back operation* (5.5.4) *is an isomorphism.*

PROOF. We have reduced the definition of $\mathcal{C}^\infty([X;Y])$ to the product case, so it suffices to consider $\{0\} \subset \mathbb{R}^n_k$. Then the fact that (5.5.4) is an isomorphism is a consequence of the fact that a smooth function on $[0,\infty) \times \mathbb{S}^{n-1}$ which vanishes to all orders at $\{0\} \times \mathbb{S}^{n-1}$ is just the polar coordinate representation of a smooth function on $\mathbb{R}^n$, vanishing to infinite order at 0. □

Combining (5.5.1), or (5.5.2), with Proposition 5.5.1 shows that the pull-back behaves similarly on densities:

$$\beta^* : \dot{\mathcal{C}}^\infty(X;\Omega X) \longleftrightarrow \dot{\mathcal{C}}^\infty([X;Y];\Omega([X;Y])) \quad \text{if } Y \subset X \text{ is a boundary } p\text{-submanifold.} \tag{5.5.5}$$

5.6. Push-forward

These statements for the pull-back operation have immediate consequences, by duality, for push-forward. First, dually from (5.5.1), there is a well-defined push-forward operation on supported distributions:

$$\beta_* : \dot{\mathcal{C}}^{-\infty}([X;Y]) \twoheadrightarrow \dot{\mathcal{C}}^{-\infty}(X). \tag{5.6.1}$$

PROPOSITION 5.6.1. *The push-forward map* (5.6.1) *under blow-down is surjective and restricts to give a surjective map*

$$\beta_* : \dot{\mathcal{A}}([X;Y]) \twoheadrightarrow \dot{\mathcal{A}}(X) \text{ if } Y \in M(X). \tag{5.6.2}$$

PROOF. The map (5.6.1) is continuous so it maps each of the Sobolev space $\dot{\mathcal{H}}^s([X;Y])$ into some Sobolev space $\dot{\mathcal{H}}^S(X)$ where S depends on s. From Proposition 5.3.1 we know that if $V \in \mathcal{V}_b(X)$ then V lifts to $[X;Y]$ to $\tilde{V} \in \mathcal{V}_b([X;Y])$. From the naturality of β_*, assuming $Y \in M(X)$,

$$\beta_*(\tilde{V}u) = V(\beta_* u). \tag{5.6.3}$$

Thus if $u \in \dot{\mathcal{A}}^{[S]}([X;Y])$ then $\tilde{V}u \in \dot{\mathcal{H}}^S([X;Y])$, so $V(\beta_* u) \in \dot{\mathcal{H}}^s(X)$. Iterating this argument shows that β_* restricts to a map (5.6.2).

Now consider the surjectivity of (5.6.1). To prove this it is enough to show that the image of (5.5.1) is closed, since then the Hahn-Banach theorem can be applied. As usual because of the local product definition of $[X;Y]$ it is enough to check this closure property for $\{0\} \subset \mathbb{R}^n_k$. Since β is a diffeomorphism away from $\mathrm{ff}([\mathbb{R}^n_k;\{0\}])$ the range of (5.5.1) is just $r^{n-1}\beta^*\mathcal{C}^\infty(\mathbb{R}^n_k) \cdot |dr d\omega|$. Thus it suffices to show that the range of

$$\beta^* : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty([\mathbb{R}^n;\{0\}]) \tag{5.6.4}$$

is closed.

The conditions on $f \in \mathcal{C}^\infty([\mathbb{R}^n,\{0\}])$ such that $f = \beta^* g$, $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ are of the form:

$$\Sigma \frac{1}{\tilde{x}_l}\tilde{V}_{l,j_1} \dots \frac{1}{\tilde{x}_l}\tilde{V}_{l,j_p} f \text{ is bounded,} \tag{5.6.5}$$

where $\beta_* V_{l,j} = x_l D_j$ and $\tilde{x}_l = \beta^* x_l$, with p, l, j_1, j_p running over all indices. Notice that $\tilde{x}_l = r\omega_l$ with $d\omega_l \neq 0$ when $\omega_l = 0$. Since $\tilde{V}_{l,j}$ is always tangent to $r = 0$ and $\omega_l = 0$ the requirement (5.6.4) becomes

$$r^{-p}\omega_l^{-p} P_{p,l,r} f \text{ is bounded } \forall p, l \tag{5.6.6}$$

where $P_{p,l,r} \in \mathrm{Diff}^p_b([\mathbb{R}^n,\{0\}])$ is one of a finite number of operators for each p, l. Since $P_{p,l,r}f$ is $\mathcal{C}^\infty$ the boundedness conditions in (5.6.5) simply require that $P_{p,l,r}f$

vanishes to order p on $r = 0$ and $\omega_l = 0$ for a finite number of $\mathcal{C}^\infty$ different operators $P_{p,l,r}$. Each of the conditions (5.6.6) is closed in $\mathcal{C}^\infty$, so the range of (5.6.5) is closed.
The last step, the proof of the surjectivity of (5.6.2) is left as an exercise. □

By duality from (5.5.5) we easily deduce the first part of

PROPOSITION 5.6.2. *If $Y \subset X$ is a boundary p-submanifold then*

$$\beta_* : \mathcal{C}^{-\infty}([X;Y]) \longleftrightarrow \mathcal{C}^{-\infty}(X) \text{ is an isomorphism.} \tag{5.6.7}$$

If $Y \in M(X)$ then by restriction we obtain an isomorphism

$$\beta_* : \mathcal{A}([X;Y]) \longleftrightarrow \mathcal{A}(X). \tag{5.6.8}$$

PROOF. That (5.6.8) is an isomorphism follows, as in the first part of the proof of 5.6.1, from the lifting properties of $\mathcal{V}_b(X)$. □

The surjectivity of (5.6.1) and (5.6.2) and the isomorphisms (5.6.7), (5.6.8) extend to densities without any problem. Thus, if $Y \in M(X)$ is a boundary face

$$\left\{\begin{aligned} &\beta_* : \dot{\mathcal{C}}^{-\infty}([X;Y];\Omega) \twoheadrightarrow \dot{\mathcal{C}}^{-\infty}(X;\Omega X)\\ &\beta_* : \dot{\mathcal{A}}([X;Y];\Omega) \twoheadrightarrow \dot{\mathcal{A}}(X;\Omega X)\\ &\beta_* : \mathcal{C}^{-\infty}([X;Y];\Omega) \longleftrightarrow \mathcal{C}^{-\infty}(X;\Omega X)\\ &\beta_* : \mathcal{A}([X;Y];\Omega) \longleftrightarrow \mathcal{A}(X;\Omega X)\end{aligned}\right. \quad \text{for } Y \in M(X). \tag{5.6.9}$$

where, on the left, Ω is the density bundle of the blown-up space.

5.7. Lifting submanifolds

If $Z \subset X$ is a closed subset of a manifold with corners and $Y \subset X$ is a closed p-submanifold we shall define the lift of Z to $[X;Y]$ under the blow-down map β for Y in two distinct cases. First

$$\text{if } Z \subset Y,\ \beta^*(Z) = \beta^{-1}(Z) \subset \mathrm{ff}([X;Y]). \tag{5.7.1}$$

Secondly

$$\text{if } Z = \mathrm{cl}(Z\backslash Y) \text{ then } \beta^*(Z) = \mathrm{cl}\,\beta^{-1}(Z\backslash Y) \tag{5.7.2}$$

where in this second case the closure defining $\beta^*(Z)$ is, of course, in $[X;Y]$.
Following Bott we say that two submanifolds, Y and Z, intersect cleanly if their intersection is a submanifold and at each point $p \in Y \cap Z$

$$T_pY \cap T_pZ = T_p(Y \cap Z)\ \forall\ p \in Y \cap Z. \tag{5.7.3}$$

Although we shall use this notion of cleanness of intersection it is also useful, especially when considering the lift of a b-submanifold under the blow up of a p-submanifold which intersects it, to have a stronger notion which includes appropriate intersection properties with the boundary faces.

DEFINITION 5.7.16. Let $Y \subset X$ be a p-submanifold and $Z \subset X$ be a b-submanifold with $\partial Z \subset \partial X$. We say that the intersection of Y and Z is *free* if it is a clean intersection, $Y \cap Z$ is p-submanifold of Z and

$$\begin{gathered}[T_pZ + T_pY] \cap [T_pB + T_pY] = T_p(Z \cap B) + T_pY\\ \forall\ p \in Y \cap Z \cap B\ \forall\ B \in M(X).\end{gathered} \tag{5.7.4}$$

It is sometimes convenient to rewrite the conditions in (5.7.3) and (5.7.4) in dual form:

$$
\begin{gathered}
N_p^*Y + N_p^*Z = N_p^*(Y \cap Z),\ p \in Y \cap Z \\
\left[N_p^*Z \cap N_p^*Y\right] + \left[N_p^*B \cap N_p^*Y\right] = N_p^*(Z \cap B) \cap N_p^*Y \\
\forall\ p \in Y \cap Z \cap B\ \forall\ B \in M(X).
\end{gathered} \tag{5.7.5}
$$

Such free intersection is equivalent to the the existence of an appropriate simultaneous normal form at each point of intersection:

Proposition 5.7.1. *A p-submanifold Y and a b-submanifold Z, with $\partial Z \subset \partial X$, intersect freely if and only if at each point of $Y \cap Z$ there are local coordinates with respect to which*

(5.7.6)
$$
\begin{gathered}
Y = \{x_j = 0,\ 1 \le i \le s,\ y_j = 0,\ 1 \le j \le l_1\} \text{ and} \\
Z = \{A_t'(x') = 0,\ 1 \le t \le s',\ A_r''(x'') = 0,\ 1 \le r \le s'',\ y_j = 0,\ l' \le j \le l_2\}
\end{gathered}
$$

where $x' = (x_1, \dots, x_s)$, $x'' = (x_{s+1}, \dots, x_k)$ and the $A_t'(x')$ and $A_r''(x'')$ are independent, constant coefficient, linear forms.

As in (1.12.9) the $A_t'(x')$ and $A_r''(x'')$ can be replaced by single variables and forms with take both signs.

Proof. First let us check that (5.7.6) implies that the intersection of Y and Z is free in the sense of Definition 5.7.16. It follows directly from (5.7.6) that $Y \cap Z$ is a p-submanifold of Z and then (5.7.3) is a consequence of the simultaneous linearization of Y and Z. Similarly, since all the submanifolds are linear, (5.7.4) reduces to

$$
(Z + Y) \cap (B + Y) = (Z \cap B) + Y. \tag{5.7.7}
$$

All three spaces, B, Y and Z have product decompositions with respect to $\mathbb{R}_k^n = \mathbb{R}_k^k \times \mathbb{R}^{n-k}$, namely $Y = Y_1 \times Y_2$ and $Z = Z_1 \times Z_2$ and $B = B_1 \times \mathbb{R}^{n-k}$. The desired equality (5.7.7) splits into the two factors and in the second factor it is trivial, since B_2 is the whole space. Thus we can assume that $k = n$. Now consider the product decomposition into the first s and remaining $k - s$ variables in which Y is the second factor. Both $B = B_1 \times B_2$ and $Z = Z_1 \times Z_2$ have product decompositions. The addition of Y in (5.7.7) therefore just gives the second factor throughout, so it suffices to consider the first factor, where (5.7.7) becomes trivially true since $Y = \{0\}$. Thus (5.7.4) does follow from (5.7.6).

So now consider a p-submanifold Y and a b-submanifold Z which intersect freely. Let us first assume that Y is an interior p-submanifold of X. We can introduce local product coordinates for $Y \cap Z$ as a p-submanifold of Z, which it is by assumption. In terms of these coordinates x_i', y_j' in Z, for $1 \le i \le k'$ and $1 \le j \le l'$

$$
Y \cap Z = \left\{x_i = 0,\ 1 \le i \le k'',\ y_j' = 0,\ 1 \le j \le l''\right\} \text{ in } Z \tag{5.7.8}
$$

As a b-submanifold Z can be reduced to the local form (1.12.9) and by assumption $\partial Z \subset \partial X$ so $l_2 = 0$. Since they are not boundary defining functions the y_j' defining $Y \cap Z$ in Z can be extended off Z and used in place of some of the y_j variables. The assumption that Y is an interior p-submanifold implies that at each boundary point $p \in Y \cap B$, $B \in M'(X)$, $N_p^* Y \cap N_p^* B = \{0\}$. Thus the second set of conditions in (5.7.5) becomes

$$\begin{aligned} N_p^* Z \cap N_p^* Y = N_p^*(Y \cap Z) \cap N_p^* B \\ \forall\ p \in Y \cap Z \cap B\ \forall\ B \in M(X). \end{aligned} \tag{5.7.9}$$

Intersecting both sides of (5.7.9) with $N_p^* B$ shows that both sides must reduce to $\{0\}$. Thus $k'' = 0$ in (5.7.8) so $Y \cap Z$ is an interior p-submanifold of Z. Thus, with some renumbering of coordinates, Z and $Y \cap Z$ can be brought to the simultaneous normal form:

$$\begin{aligned} Z = \{A_r(x) = 0,\ 1 \le r \le k',\ y_j = 0,\ 1 \le j \le l_1, \}, \\ Y \cap Z = \{A_r(x) = 0,\ 1 \le r \le k,\ y_j = 0,\ 1 < j \le l_2\} \end{aligned} \tag{5.7.10}$$

where the $A_r(x)$ are independent linear forms in x and $l_2 \ge l_1$.

Next consider the form of Y in these coordinates. If Y_r are local defining functions for Y then, by the assumed cleannes of the intersection, the restrictions of the Y_r to Z must vanish on $Y \cap Z$ with the matrix of differentials having full rank. Thus, after rearrangement, the first l'' of the Y_r can be taken as local defining functions for $Z \cap Y$ in Z and the remainder can be taken to vanish on Z. This gives the simultaneous local normal form for Z and Y where Z (and $Y \cap Z$) are given by (5.7.10) and

$$Y = \{y_j = 0,\ l' \le j \le l_2\} \tag{5.7.11}$$

with $l' \le l_1$. Certainly this reduces to (5.7.6), with $s = s' = 0$:

$$\begin{aligned} Y = \{y_j = 0,\ 1 \le j \le l_1\} \text{ and} \\ Z = \{A_r(x) = 0,\ 1 \le r \le s'',\ y_j = 0,\ l' \le j \le l_2\}. \end{aligned} \tag{5.7.12}$$

It remains to consider the case where Y is a boundary p-submanifold. Let $F = \mathrm{Fa}(Y)$ be the smallest boundary face containing Y. From Lemma 1.12.4 it follows that $Z' = Z \cap F$ is a b-submanifold of F. If $B' \in M(F)$ is a boundary face then (1.12.10) shows that

$$T_p Z' \cap T_p B' = T_p Z \cap T_p F \cap T_p B' = T_p Z \cap T_p B',\ p \in Z' \cap B' \tag{5.7.13}$$

where B' can also be regarded as a boundary face of X. Thus it follows from (5.7.4) that Y and Z' intersect freely as submanifolds of F. The normal form given by (5.7.10) and (5.7.11) therefore applies to Y and Z'. Consider coordinates in which Z takes the form (1.12.9), the tangential coordinates can be chosen (up to renumbering) to restrict to F to the coordinates just selected there, so (5.7.12) still holds for Y in F and $Z' = Y \cap F$. By renumbering the new coordinates we obtain the local coordinate form:

(5.7.14)

$$\begin{gathered} Y = \{x_i = 0,\ 1 \le i \le s,\ y_j = 0,\ 1 \le j \le l_1\} \text{ and} \\ Z = \{A'_t(x') = 0,\ 1 \le t \le s',\ A''_r(x', x'') = 0,\ 1 \le r \le s'',\ y_j = 0,\ l' \le j \le l_2\} \end{gathered}$$

where the $A_r(x', x'')$ are linear forms in all k of the x coordinates such that the $A_r(0, x'')$ are independent. However taking $B = \{x_i = 0,\ i > s\}$ in the second set of conditions in (5.7.5) $N_p^* B \cap N_p^* Y = \{0\}$ shows that the $dA''_r(x', 0)$ must vanish, which means that the forms A''_r only depend on the x_i for $i > s$, for an appropriate choice of coordinates. This gives the normal form (5.7.6). $\square$

We can immediately use this normal form to analyze the lift of Z under the blow up of Y :

PROPOSITION 5.7.2. *If $Y \subset X$ is a closed p-submanifold and $Z \subset X$ is a closed b-submanifold satisfying $\partial Z \subset \partial X$ and meeting Y freely in the sense of Definition 5.7.16 then under the blow-up of Y, Z lifts to a b-submanifold $\beta^* Z \subset [X;Y]$ which is naturally isomorphic to $S^+N_Z Y$ or $[Z; Y \cap Z]$ as $Z \subset Y$ or not. If, in addition, Z is a closed p-submanifold then so is $\beta^* Z$.*

PROOF. Notice that the lift of Z under the blow-up of Y is defined since either $Z \subset Y$, in which case (5.7.1) applies or else $Z \cap Y$ has dimension less than Y and so $Z \setminus Y = Z \setminus (Y \cap Z)$ is dense in Z and (5.7.2) applies. In the first case Z is a b-submanifold of Y and $\beta^* Z \subset \mathrm{ff}([X;Y])$ is just $SN_Z^+ Y$, consisting of the fibres over Z of the inward-pointing sperical normal bundle to Y in X so it is also a b-submanifold (of the boundary of) $[X;Y]$. Thus we can assume that Z is not a subset of Y.

Consider first the case that $\partial X = \emptyset$; then none of the submanifolds have boundaries and the second part of (5.7.4) is void. As cleanly intersecting submanifolds Y and Z can be brought to the simultaneous local normal form:

$$Y = \{y_i = 0,\ 1 \le i \le k_1\},\ Z = \{y_i = 0,\ k_1' \le i \le k_2\} \tag{5.7.15}$$

where $k_1' \le k_1$ and $k_2 \ge k_1$, as follows from Proposition 5.7.1. With the two manifolds simultaneously linearized the blow-up of Y is just the introduction of polar coordinates in the variables $y' = r\omega$. The lift of Z is then locally defined by

$$\beta^* Z = \{\omega_i = 0,\ k_1' \le i \le k_1, y_j = 0,\ k_1 \le j \le k_2\}. \tag{5.7.16}$$

Since the differentials of any set of the functions ω_i on the sphere are independent at any point where they all vanish $\beta^* Z$ is, in this case, an interior p-submanifold.

If boundaries are present the same discussion applies, using the coordinate normal form (5.7.6). Blow up of Y is locally the introduction of polar coordinates

$$(x_1, \dots, x_s) = r(\omega_1, \dots, \omega_s),\ (y_1, \dots y_{l_1}) = r(\omega_{s+1} \dots, \omega_{s+l_1}),\ \omega \in \mathbb{S}^{s+l_1-1}.$$

Then

$$\begin{aligned}\beta^* Z = \big\{A_t'(\omega) = 0,\ 1 \le t \le s',\ A_r(x'') = 0,\ 1 \le r \le s'',\\ \omega_j = 0,\ s + l' \le j \le s + l_1,\ y_j = 0,\ l_1 < j \le l_2\big\}.\end{aligned} \tag{5.7.17}$$

It is therefore a b-submanifold of $[X;Y]$ and is diffeomorphic to $[Z; Y \cap Z]$ as stated.

If Z is a p-submanifold then the linear functions $A_r(x'')$ in (5.7.17) can always be taken to be individual boundary defining functions and it follows that $\beta^* Z$ is then a p-submanifold as well. □

EXERCISE 5.7.29. To see how blow-up can affect the intersection properties of submanifolds take $X = [0, \infty) \times [0, \infty)$ and consider the diagonal

$$\Delta = \{(x, x) \in [0, \infty) \times [0, \infty)\}.$$

Show that, while Δ is *not* a p-submanifold of X, Δ meets $\{0\} \subset X$ freely in the sense of Definition 5.7.16. Show further that the lift of Δ to $[X; \{0\}]$ *is* a p-submanifold. This is the basis for the construction of the stretched product in §7.

5.8. Commutativity of blow-up

If X is a manifold with corners, Y is a closed p-submanifold and $Z \subset X$ is a closed subset which lifts under the blow-down map for Y to a p-submanifold of $[X;Y]$ then we use the abbreviated notation

$$[X;Y;Z] = [[X;Y];\beta^*Z],\ \beta = \beta[X;Y] \tag{5.8.1}$$

for the iterated blow up. By the use of this notation we are asserting that β^*Z exists, i.e. is defined by (5.7.1) or (5.7.2) and is a p-submanifold. If Y and Z are both closed p-submanifolds it may well happen that Z lifts to a p-submanifold in $[X;Y]$ and Y lifts to a p-submanifold of $[X;Z]$. Then both iterated blow-ups $[X;Y;Z]$ and $[X;Z;Y]$ are defined. It is natural to ask when they are the same. This is a well-defined question since there can be at most one diffeomorphism making the following diagramme commute:

$$\begin{tikzcd}
{[X;Y;Z]} \arrow[r, leftrightarrow] \arrow[d, "{\beta[[X;Y];Z]}"'] & {[X;Z;Y]} \arrow[d, "{\beta[[X;Z];Y]}"] \\
{[X;Y]} \arrow[d, "{\beta[X;Y]}"'] & {[X;Z]} \arrow[d, "{\beta[X;Z]}"] \\
X \arrow[r, leftrightarrow, "\mathrm{Id}"'] & X.
\end{tikzcd} \tag{5.8.2}$$

PROPOSITION 5.8.1. *If $Z \hookrightarrow Y \hookrightarrow X$ form a two-element chain of closed p-submanifolds, in the sense of Definition 1.9.6, then the lift of Y under the blow-down map for Z is a p-submanifold of $[X;Z]$, the lift of Z under the blow-down map for Y is a p-submanifold of $[X;Y]$ and there is a natural isomorphism making* (5.8.2) *commutative.*

PROOF. Certainly this result is local near Z, or its inverse image in the blow-ups. Hence by the invariance of the blow-up procedure we can, using the normal fibration theorem, replace X by the normal bundle NZ to Z in X. We can also arrange that, near the zero section, Y is a subbundle of NZ.

Now the variables in Z are essentially parameters in all blow-ups and problem is reduced to the product case. Thus it suffices to consider

$$\begin{cases} X = \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \times \mathbb{R}^{n''}_{k''} \\ Y = \{0\} \times \mathbb{R}^{n'}_{k'} \times \mathbb{R}^{n''}_{k''} \\ Z = \{0\} \times \{0\} \times \mathbb{R}^{n''}_{k''}. \end{cases} \tag{5.8.3}$$

Again the last factor is preserved under all blow-ups, so it is enough to suppose that $n'' = 0$ in (5.8.3).

Let Y' and Z' be, respectively, the lift of Y under the blow up of Z and the lift of Z under the blow up of Y. The first blow-ups therefore give

$$\begin{gathered} = \mathbb{S}^{n-1}_k \times [0,\infty) \times \mathbb{R}^{n'}_{k'},\ Z' = \mathbb{S}^{n-1}_k \times \{0\} \times \{0\} \subset [X;Y] \text{ and} \\ [X;Z] = \mathbb{S}^{n+n'-1}_{k+k'} \times [0,\infty),\ Y' = \mathbb{S}^{n'-1}_{k'} \times [0,\infty) \subset [X;Z]. \end{gathered} \tag{5.8.4}$$

Here

$$\mathbb{S}^{n-1}_k = \{x \in \mathbb{R}^n_k; |x| = 1,\ x_i \geq 0 \text{ for } i \leq k\}. \tag{5.8.5}$$

Notice that the codimension of the boundary is indeed k unless $k = n$ in which case it is $k - 1$ since all n coordinates cannot vanish simultaneously. As a submanifold

$$\mathbb{S}^j_l = \left\{(0, \omega) \in \mathbb{R}^{j'} \times \mathbb{R}^{j+1}; \omega_i \geq 0,\ i \leq l,\ |\omega| = 1\right\} \subset \mathbb{S}^{j+j'}_{l+l'}. \tag{5.8.6}$$

The natural diffeomorphism between open dense sets of these first blow-ups is given by the blow-down maps:

$$\begin{aligned} \setminus (\partial[X;Y] \cup Z') \ni (\xi, r, x') \longmapsto (r\xi, x') = \\ R\omega \longleftarrow (\omega, R) \in [X;Z] \setminus (\partial[X;Z] \cup Y'). \end{aligned} \tag{5.8.7}$$

Now from (5.8.4) we can find the double blow-ups:

$$\begin{gathered} = [[X;Y];Z'] = \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'}_{k'+1} \times [0, \infty) \\ [X;Y;Z] \ni (\xi, \eta, \rho) \longmapsto (\rho\eta_1\xi, \rho\eta_1, \rho\eta') = (r\xi, r, x') \in \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \\ \rho^2 = r^2 + |x'|^2 = |x|^2 + |X'|^2 = R^2, \end{gathered} \tag{5.8.8}$$

since blowing up Z' in $[X;Y]$ just amounts to blowing up $\{0\} \times \{0\}$ in $[0, \infty) \times \mathbb{R}^{n'}_{k'} = \mathbb{R}^{n'+1}_{k'+1}$. Similarly

$$[X;Z;Y] = [[X;Z];Y'] = [\mathbb{S}^{n+n'-1}_{k+k'}; \mathbb{S}^{n'-1}_{k'}] \times [0, \infty) \tag{5.8.9}$$

Since ρ in (5.8.8) is the same as R in (5.8.7) it suffices to show that:

$$\mathbb{S}^{r'-1}_{k'} \times \mathbb{S}^{r+1}_{k+1} \cong [\mathbb{S}^{r'+r}_{k'+k}; \mathbb{S}^r_k] \tag{5.8.10}$$

in the strong sense that

$$\mathbb{S}^{r'-1}_{k'} \times \mathbb{S}^{r+1}_{k+1} \ni (\xi, \eta) \longmapsto (\eta_1\xi, \eta') \in \mathbb{S}^{r'+r}_{k'+k},\ \eta = (\eta_1, \eta') \tag{5.8.11}$$

is the blow-down map for $[\mathbb{S}^{r'+r}_{k'+k}; \mathbb{S}^r_k]$. Writing out the spheres explicitly:

$$\left\{ \begin{aligned} &\mathbb{S}^{r'+r}_{k'+k} = \{\omega = (\phi, \psi) \in \mathbb{R}^{r'} \times \mathbb{R}^{r+1};\ |\phi|^2 + |\psi|^2 = 1, \\ &\qquad\qquad \phi_j \geq 0,\ j \leq k',\ \psi_i \geq 0\ i \leq k\} \\ &\mathbb{S}^r_k = \{(0, \psi) \in \mathbb{R}^{r'} \times \mathbb{R}^{r+1}; |\psi|^2 = 1,\ \psi_i \geq 0\ i \leq k\} \end{aligned} \right. . \tag{5.8.12}$$

it follows immediately that (5.8.11) is a diffeomorphism of $\eta_1 > 0$ onto $\phi \neq 0$ with inverse

$$\xi = \frac{\phi}{|\phi|},\ \eta_1 = |\phi|,\ \eta' = \psi. \tag{5.8.13}$$

Thus it suffices to consider small neighbourhoods of $\eta_1 = 0$ and $\phi = 0$. The blow-up can be performed by setting $\mu = \phi/|\phi|$ with radial variable $|\phi|$ near $\phi = 0$ so (5.8.11) is indeed the blow-down map for $[\mathbb{S}^{r'+r}_{k'+k}; \mathbb{S}^r_k]$. $\square$

There is another important case in which there is commutativity of blow-ups, namely for trasversal intersection.

Proposition 5.8.2. *If Y_1 and Y_2 are two closed p-submanifolds of a manifold with corners which, as single element chains, meet transversally in the sense of Definition 1.9.7, then the natural isomorphism away from the front faces extends to an isomorphism $[X;Y_1;Y_2] \cong [X;Y_2;Y_1]$.*

PROOF. The notion of transversal intersection of p-submanifolds is fixed by Definition 1.9.6. Thus near any point of $Y_1 \cap Y_2$ there is a local decomposition of X as a product of manifolds with corners in terms of which:

$$X = X_1 \times X_2 \times X_3,\ Y_1 = \{p_1\} \times X_2 \times X_3,\ Y_2 = X_1 \times \{p_2\} \times X_3. \tag{5.8.14}$$

The blow up of Y_1 in X is given by the blow up of p_1 in X_1 and the blow-up of Y_2 in X is given by the blow up of p_2 in X_2. Thus the intermediate spaces are, locally,

$$\begin{gathered} \cong [X_1; \{p_1\}] \times X_2 \times X_3,\ \beta^* Y_2 \cong [X_1; \{p_1\}] \times \{p_2\} \times X_3 \\ [X; Y_2] \cong X_1 \times [X_2; \{p_2\}] \times X_3,\ \beta^* Y_1 \cong \{p_1\} \times [X_2; \{p_2\}] \times X_3. \end{gathered} \tag{5.8.15}$$

Thus the two doubly blown-up spaces are the same and

$$[X; Y_1, Y_2] \cong [X_1, \{p_1\}] \times [X_2, \{p_2\}] \times X_3 \tag{5.8.16}$$

gives a local product decomposition. □

5.9. Blowing up chains

In performing deeply iterated blow ups it is often convenient to consider chains of p-submanifolds, as in Definition 1.9.6. We shall always take the component manifolds to be closed in this section. Since each submanifold is a p-submanifold we can blow it up and consider the lifts of the remaining elements of the chain.

LEMMA 5.9.1. *Under the blow-up of any element Y_i of a chain of closed p-submanifolds in a manifold with corners X, the lifts of the elements of the chain contained in Y_i form a chain of p-submanifolds contained in the front face of* $\mathrm{ff}[X; Y_i]$ *and the lifts of the elements containing Y_i form a second chain which intersects the first transversally in the sense of Definition 1.9.7.*

Of course if the manifold blown up is either the smallest or the largest element of the chain then one or other of these chains is empty.

PROOF. That all of the elements of the chain lift to be p-submanifolds follows from Proposition 5.7.2. In fact the simultaneous product coordinates given by Lemma 1.9.1 allow Y_i to be written locally as

$$Y_i = \{x_l = 0, 1 \le l \le s,\ y_j = 0,\ 1 \le j \le m\} \tag{5.9.1}$$

with the other elements of the chain given by the vanishing of more of the coordinates, for the smaller elements, or less for the larger elements. The introduction of the polar coordinates

$$(x_1, \dots, x_s, y_1 \dots, y_m) = r(\omega, \eta)$$

of the blow up therefore expresses the other elements of the chain as

$$\begin{aligned} Y_{i'} = \{r = 0, x_l = 0,\ & l \in J_1(i') \subset \{s+1, \dots, k\}, \\ & y_j = 0,\ j \in J_2(i') \subset \{m+1, \dots, n-k\}\},\ i' < i \\ Y_{i'} = \{\omega_l = 0,\ & l \in I_1(i') \subset \{1, \dots, s\}, \\ & \eta_j = 0,\ j \in I_2(i') \subset \{1, \dots, m\}\},\ m > i \end{aligned} \tag{5.9.2}$$

where the $J_1(i')$, $J_2(i')$, $I_1(i')$ and $I_2(i')$ are all decreasing with increasing i'. This provides the local product decompositions to show that both the smaller and the larger elements form chains and that these two chains meet transversally. □

This argument extends immediately to the more general case of the blow up of an element of one chain of a set of transversally intersecting chains:

PROPOSITION 5.9.1. *If one element $Y_i^{(j)}$ of one chain, $Y^{(j)}$, of a finite collection of transversally intersecting chains of closed p-submanifolds in a manifold with corners X is blown up then the $Y^{(l)}$ for $l \neq j$ lift to chains of p-submanifolds of $[X; Y_i^{(j)}]$ which form a transversally intersecting system of chains together with the chains given by the lift of the remaining elements of $Y^{(j)}$.*

The lifting result Lemma 5.9.1 shows that if $\mathcal{Y} = \{Y_i; i = 1, \dots, r\}$ is a chain of closed p-submanifolds in a manifold with corners X then we can blow up all elements, starting from the smallest. We shall extend the notation, (5.8.1), for iterated blow up to this case by setting

$$[X; \mathcal{Y}] = [X; Y_1; Y_2; \cdots ; Y_r] = [[\cdots[X; Y_1]; \beta_1^* Y_2] \cdots ; \beta_{r-1}^* Y_r] \tag{5.9.3}$$

where β_i is the iterated blow-down map of the first i blow ups. More generally the commutativity of blow up for transversal intersections, as in Proposition 5.8.2, means that given a finite collection $\mathcal{Y}^{(j)}$, $j = 1, \dots, r$, of transversally intersecting chains of closed p-submanifolds the total blow up of all elements of all chains is well defined by blowing up the elements of each chain, in increasing order of dimension, irrespective of the order amongst the chains. Thus we can use the notation

$$[X; \mathcal{Y}^{(*)}] \tag{5.9.4}$$

unambiguously for this total blow up.

In fact even the order in which the elements of a chain are blown up is immaterial. That is if $Z_l = Y_{i(l)}^{(j(l))}$, $l = 1, \dots, N$, is any listing of the all the elements of all the chains in the finite collection $\mathcal{Y}^{(*)}$ of transversally intersecting chains then

$$[X; \mathcal{Y}^{(*)}] \equiv [X; Z_1; Z_2; \cdots ; Z_N] \tag{5.9.5}$$

with a natural isomorphism. Indeed, the lifting result Proposition 5.9.1 shows that the right side of (5.9.5) is defined for any such ordering, since all the other elements lift to the elements of transversally intersection chains. Since the left side is defined for one choice of ordering it suffices to prove that the same manifold results, with the natural identification, when two neighbouring elements among the Z_l, for $l{+}L$, $L{+}1$, are interchanged. It suffices to consider the case $L+1 = N$, since once this is shown the later blow ups are same in the two manifolds. Similarly X can be replaced by the space with all Z_l blown up for $l < L$. Then the identification follows from either Proposition 5.8.1 or Proposition 5.8.2 as Z_L and Z_{L+1} are either transversal or one is contained in the other. Thus the notation (5.9.4) is completely unambiguous.

We are also interested in the lifting properties of vector fields under such iterated blow up:

LEMMA 5.9.2. *Under the blow-down map $\beta \colon X' \longrightarrow X$, for the blow up of any subset of the elements of a finite collection of transversally intersecting chains, $\mathcal{Y}^{(j)}$, of closed p-submanifolds in the manifold with corners, X, the subspace of $\mathcal{V}_b(X)$ consisting of the vector fields tangent to each element of each chain lifts into, and spans over $\mathcal{C}^\infty(X')$, the subset of the $\mathcal{V}_b(X')$ consisting of the vector fields tangent to all the elements of the lifted transversal chains.*

PROOF. This is a local result and locally X can be decomposed into a product with one chain in each product. It therefore suffices to consider the case of the blow up of some elements of one chain, Y_p, $p = 1, \dots, r$, of closed p-submanifolds. Without loss of generality we can consider a point, p, in the smallest element, Y_1, of the chain and again proceeding by induction it suffices to prove that under the blow up of some element Y_q the vector fields tangent to all Y_p lift to span the vector fields tangent to the lifted chains.

By Lemma 1.9.1 local coordinates can be introduced in terms of which

$$\begin{gathered} Y_p = \{x_i = 0,\ i \ni I_p,\ y_j = 0,\ j \in J_p\}, \\ \{1, \dots, k\} \supset I_1 \supset I_2 \cdots \supset I_r,\ \{1, \dots, n-k\} \supset J_1 \supset J_2 \cdots \supset J_r. \end{gathered} \tag{5.9.6}$$

The elements of $\mathcal{V}_b(X)$ tangent to all the Y_p are spanned over $\mathcal{C}^\infty(X)$ by

$$\begin{gathered} x_i \frac{\partial}{\partial x_i},\ i = 1, \dots, k,\ y_l \frac{\partial}{\partial y_j},\ j \in I_1,\ l \in I_{M(j)},\ \frac{\partial}{\partial y_j},\ j \notin J_1 \\ \text{where } M(j) = \max\{p; j \in I_j\}. \end{gathered} \tag{5.9.7}$$

The front face of $X' = [X; Y_q]$ is covered by the projective coordinate systems discussed in §5.4. These are of two types, those where the front face is locally defined by x_s, $s \in I_q$ and those where it is defined by y_s, $s \in J_q$. The resulting coordinates are given by

$$\begin{gathered} u_s = x_s,\ u_i = \frac{x_i}{x_s},\ s \neq i \in I_q,\ z_j = \frac{y_j}{x_s},\ j \in J_q \\ u_{k+1} = y_s,\ u_i = \frac{x_i}{y_s},\ i \in I_q,\ z_j = \frac{y_j}{y_s},\ s \neq j \in J_q. \end{gathered} \tag{5.9.8}$$

together with the $u_i = x_i$ for $i \in \{1, \dots, k\} \setminus I_q$ and the $z_j = y_j$ for $j \in \{1, \dots, n-k\} \setminus J_q$, in the second case there is one more boundary defining function u_{k+1} and no z_s.

Lifting the elements of the chain other than Y_q gives the two, transversally intersecting chains, with elements, Y'_p, $p \neq q$, defined in these, respective, local coordinates by

$$\begin{gathered} Y'_p = \{u_s = 0,\ u_i = 0,\ i \in I_p \setminus I_q,\ z_j = 0,\ j \in J_p \setminus J_q\},\ p < q, \\ Y'_p = \{u_i = 0,\ i \in I_p,\ z_j = 0,\ j \in J_p\},\ p > q \\ Y'_p = \{u_i = 0,\ i \in (I_p \setminus I_q) \cup \{k+1\},\ z_j = 0,\ j \in I_p \setminus (J_q \cup \{s\})\},\ p < q \\ Y'_p = \{u_i = 0,\ i \in I_p \cup \{k+1\},\ z_j = 0,\ j \in I_p \setminus \{s\}\},\ p < q, \end{gathered} \tag{5.9.9}$$

and where, for a given coordinate system, a given Y'_p with $p > q$ can only appear if $s \notin I_p$ or $s \in J_p$ in the two cases.

Lifting the spanning vector fields in (5.9.7), for the first class of coordinate systems, gives

$$\begin{gathered} x_i \frac{\partial}{\partial x_i} \longmapsto u_i \frac{\partial}{\partial u_i},\ s \neq i \in \{1, \dots, k\} \\ x_s \frac{\partial}{\partial x_s} \longmapsto u_s \frac{\partial}{\partial u_s} - \sum_{\substack{1 \le i \le k \\ i \neq s}} u_i \frac{\partial}{\partial u_i} - \sum_{j \in I_s} z_j \frac{\partial}{\partial z_j} \\ y_l \frac{\partial}{\partial y_j} \longmapsto z_l \frac{\partial}{\partial z_j},\ j \in I_1 \\ \frac{\partial}{\partial z_j} \longmapsto \frac{\partial}{\partial z_j},\ j \notin I_1. \end{gathered} \tag{5.9.10}$$

Notice here that in case $j \in J_s$ necessarily $M(j) \ge s$ so the coefficient y_l has $l \in J_s$. It is now straightforward to check that these vector fields do span the Lie algebra of vector fields tangent to all the lifted submanifolds. The other case is similar. □

5.10. Lifting maps

If $G \subset {}^bTX$ is a subbundle we shall say that G is transversal to a finite collection, $\mathcal{Y}^{(*)}$, of transversal chains of p-submanifolds of X if at each point

(5.10.1)

$$G_p + {}^bT_p\mathcal{Y}^{(*)} = {}^bT_pX,$$

$${}^bT_p\mathcal{Y}^{(*)} = \left\{v \in {}^bT_pX; \exists\ V \in \mathcal{V}_b(X) \text{ tangent to all } \mathcal{Y}^{(i)} \text{ an with } V(p) = v\right\}$$

LEMMA 5.10.1. *The transversality condition* (5.10.1) *between the b-null bundle of a b-submersion and a finite collection of transversal p-submanifolds is equivalent to requiring that at each point* $p \in X$ *the null space of the (ordinary) differential of* $f \restriction \mathrm{Fa}(p)$ *be transversal to the sum*

$$\sum_{i,j} T_p\left(Y_j^{(i)} \cap \mathrm{Fa}(p)\right) \subset T_p\,\mathrm{Fa}(p). \tag{5.10.2}$$

The transversality condition is also equivalent to requiring that for each $V \in \mathcal{V}_b(X)$ *there exists* $W \in \mathcal{V}_b(X)$ *which is a section of* G *such that* $V - W$ *is tangent to each component manifold of* $\mathcal{Y}^{(*)}$.

PROOF. Fixing a point p let $i(j)$ be, for each j, the smallest value of i such that $p \in Y_i^{(j)}$. Then the transversality of the chains means that the space (5.10.2) is just the direct sum:

$$\bigoplus_j T_p\left(Y_{i(j)}^{(j)} \cap \mathrm{Fa}(p)\right). \tag{5.10.3}$$

Thus the null space of the b-differential must contain smooth vector fields which restrict to $\mathrm{Fa}(p)$ to span a space transversal to (5.10.3). The existence of these vector fields shows the transversality of the ordinary differential to (5.10.3). The converse also holds in view of the assumption that the G is a bundle.

The second reformulation of the transversality condition is local. A product decomposition can therefore be taken in which the transversal chains separate into the factors. Taking local coordinates in the factors as in (1.9.5) means that (5.10.3) is spanned by the $\partial/\partial y_j$ for some subset $j \in J \subset \{1, \dots, n-k\}$. Moreover the remaining elements of the coordinate basis of $\mathcal{V}_b(X)$, namely the $x_i\partial/\partial x_i$ and $\partial/\partial y_j$ for $i = 1, \dots, k$ and $j \notin J$, are all tangent to every component manifold of the $\mathcal{Y}^{(*)}$. The condition (5.10.1) means that there are local sections of G of the form

$$V_j = \frac{\partial}{\partial y_j} + V_j',\ V_j' \text{ tangent to } \mathcal{Y}^{(*)},\ j \in J. \tag{5.10.4}$$

Thus any element $V \in \mathcal{V}_b(X)$ can be written as

$$V = \sum_{j \in J} a_j(x)\frac{\partial}{\partial y_j} + V',\ V' \text{ tangent to } \mathcal{Y}^{(*)}. \tag{5.10.5}$$

The converse implication follows similarly. □

This transversality condition is useful mainly because of the following result:

PROPOSITION 5.10.1. *If $f\colon X \longrightarrow X'$ is a b-submersion with b-differential having null bundle transversal to a finite collection of chains, $\mathcal{Y}^{(*)}$, of closed p-submanifolds of X then the composite map $f' = f \circ \beta[X;\mathcal{Y}^{(*)}]$, $f' : [X;\mathcal{Y}^{(*)}] \longrightarrow X'$ is also a b-submersion.*

PROOF. A b-submersion is an interior b-map $f\colon X \longrightarrow X'$ under which every element of $\mathcal{V}_b(X')$ is f-related to an element of $\mathcal{V}_b(X)$ and this is a local condition. Thus suppose $V' \in \mathcal{V}_b(X')$ and $V \in \mathcal{V}_b(X)$ is f-related to V'. Then the transversality condition allows $W \in \mathcal{V}_b(X)$ to be chosen in the null space of the b-differential such that $V - W$ is tangent to each element of $\mathcal{Y}^{(*)}$. By Lemma 5.9.2 $V - W$ lifts under the blow-down map β for $\mathcal{Y}^{(*)}$ to a smooth tangent vector field V''. Then V'' is related to V' under $f \circ \beta$ which is therefore a b-submersion. □

COROLLARY 5.10.1. *If $f\colon X \longrightarrow X'$ is a b-fibration and $\mathcal{Y}^{(*)}$ is a finite collection of transversally intersecting chains of closed p-submanifolds in X satisfying the transversality condition of Proposition 5.10.1 and is such that no element of $\mathcal{Y}^{(*)}$ has image under f contained in a boundary face of X' of codimension 2 then f' is a b-fibration.*

PROOF. By Proposition5.10.1 f' is a b-submersion. The b-normality of f' therefore reduces to the condition that no boundary hypersurface of $[X;\mathcal{Y}^{(*)}]$ should be mapped by it into a boundary face of codimension 2 of X'. Each boundary hypersurface of $[X;\mathcal{Y}^{(*)}]$ is either the lift of a boundary hypersurface of X or else is the front face produced by the blow up of an element of $\mathcal{Y}$. In these two cases the closure of the image of the boundary hypersurface under f' is just the image, under f, of the corresponding boundary hypersurface or element of $\mathcal{Y}^{(*)}$. Thus the b-normality of f' follows from the b-normality of f and the assumption in the statement above. □

5.11. Blowing up normal families

In practice in the construction of stretched spaces it is often necessary to blow up a collection of p-submanifolds which have somewhat more complicated intersection properties than those just considered. We shall consider next the notion of a normal collection of closed p-submanifolds. A collection $\mathcal{F} = \{F_1, \dots, F_N\}$ of closed p-submanifolds is said to be normal at a point p if there are local coordinates in a neighbourhood Ω of p in terms of which

$$p \in F_l \Longrightarrow F_j \cap \Omega = \{x_i = 0,\ i \in I'_l,\ y_j = 0,\ j \in I''_l\}\,. \tag{5.11.1}$$

This just amounts to a common local product decomposition for all the elements. By a normal family of p-submanifolds we shall mean a normal collection of p-submanifolds with a given total order.

Clearly the union of any finite collection of transversal chains of closed p-submanifolds forms a normal collection. Furthermore any collection of boundary faces of a manifold with corners is normal. More generally:

LEMMA 5.11.1. *The union of any normal collection of p-submanifolds and any collection of boundary faces of a manifold with corners, with any order, forms a normal family.*

PROOF. Since the local coordinates in (5.11.1) are, as always, required to be admissible any boundary face has a product decomposition in the same coordinates. □

To show that the blow up of a normal family is well defined we need the following simple lifting result:

LEMMA 5.11.2. *Under the blow up of any element of a normal collection of closed p-submanifolds the remaining elements lift to a normal collection of p-submanifolds.*

PROOF. This follows directly from the normal form (5.11.1). Renumbering the coordinates if necessary the element to be blown up can be brought to the form

$$F = \{x_i = 1,\ i = 1, \dots, k',\ y_j = 0,\ j = 1, \dots, p'\}. \tag{5.11.2}$$

with the others still in the form (5.11.1). Away from the front face there is nothing to be proved and near any point of the front face some projective coordinate system can be used. Again renumbering the coordinates as necessary it can always be supposed that either x_1 or y_1 is a radial variable and the coordinates can be taken as

$$\begin{aligned} x_1,\ x_i' =&\frac{x_i}{x_1},\ i = 2, \dots, k',\ x_i,\ i = k' + 1, \dots, k, \\ &y_j' = \frac{y_j}{x_1},\ j = 1, \dots, p',\ y_j,\ j = p' + 1, \dots, n - k \text{ or} \\ y_1,\ x_i' =&\frac{x_i}{y_1},\ i = 1, \dots, k',\ x_i,\ i = k\text{''} + 1, \dots, k, \\ &y_j = \frac{y_j}{y_1},\ j = 2, \dots, p',\ y_j,\ j = p' + 1, \dots, n - k. \end{aligned}$$

In terms of these coordinates the lifted manifolds clearly have a common local product decomposition. □

This allows the blow up of a normal family to be defined by induction.

DEFINITION 5.11.17. If $\mathcal{F}$ is a normal family of closed p-submanifolds in a manifold with corners, X, then X blown up along $\mathcal{F}$, denoted $[X;\mathcal{F}]$ is defined inductively over the number of elements of the family by

$$[X;\mathcal{F}] = [[X;F_1];\mathcal{F}'] = [X;F_1;\mathcal{F}'] \tag{5.11.3}$$

where F_1 is the first element, with respect to the order, of $\mathcal{F}$ and $\mathcal{F}'$ is the normal family in $[X;F_1]$ given by the lifts of the elements of $\mathcal{F}$, other than F_1, with the induced order.

The normality of the intersection of the elements of $\mathcal{F}$ allows the lifting of vector fields to be described easily

PROPOSITION 5.11.1. *Under the blow up of a normal family, $\mathcal{F}$, of p-submanifolds of a manifold with corners X a vector field lifts to be smooth if and only if it is tangent to each element of the family; the subspace of $\mathcal{V}_b(X)$ of vector fields tangent to each element of the family lifts to span $\mathcal{V}_b\left([X;\mathcal{F}]\right)$ over $\mathcal{C}^\infty\left([X;\mathcal{F}]\right)$.*

PROOF. The local product decomposition, (5.11.1), shows that the Lie subalgebra of $\mathcal{V}_b(X)$ consisting of the vector fields tangent to all elements of $\mathcal{F}$ is spanned locally by

$$x_i\frac{\partial}{\partial x_i},\ i = 1, \dots, k,\ \frac{\partial}{\partial y_t},\ t \notin \bigcup_l I_l'',\ y_s\frac{\partial}{\partial y_t} \text{ where } t, s \in I_l''. \tag{5.11.4}$$

These are all homogeneous of degree 0 or 1 under the radial action corresponding to the blow up of S_1, in which $x_i \longmapsto sx_i$ if $i \in I_1'$ and $y_j \longmapsto sy_j$ if $j \in I_1''$ and the other variables are unchanged. Since the radial vector field for this action is tangent to all the elements of $\mathcal{F}$ that these vector fields lift and span follows as in Proposition 5.3.1. Conversely the failure of vector fields not tangent to the $\mathcal{F}$ to lift smoothly is easily established. $\square$

Again it is important to see the extent to which this construction gives a result which is independent of the order imposed on the family. In general the space produced by blow up depends on the order. However using the results above on commutation sufficient conditions to allow a change of order can be found. Notice that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are normal families which arise from the same collection of submanifolds, so only differ in the order imposed, there is at most one diffeomorphism, f, giving a commutative diagramme:

$$
\begin{tikzcd}
{[X;\mathcal{F}_1]} \arrow[r, leftrightarrow] \arrow[d, "\beta"] & {[X;\mathcal{F}_2]} \arrow[d, "\beta"] \\
X \arrow[r, leftrightarrow, "\mathrm{Id}"] & X.
\end{tikzcd}
\tag{5.11.5}
$$

Thus if there is such a diffeomorphism we say that the blown up spaces are naturally diffeomorphic.

Proposition 5.11.2. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are normal families in X with the same underlying collection of p-submanifolds and with orders coinciding except that two elements $F < G$ neighbouring in order in $\mathcal{F}_1$ are interchanged in $\mathcal{F}_2$ then $[X;\mathcal{F}_1]$ and $[X;\mathcal{F}_2]$ are naturally diffeomorphic provided that one of the following three conditions holds:*

$$F \pitchfork G \tag{5.11.6}$$

or

(5.11.7)

$$
\begin{gathered}
F \subset G \text{ or } G \subset F \text{ and either} \\
\mathcal{P} = \{P \in \mathcal{F}_1; P < F,\ P < G,\ P \supset (F \cap G),\ P \text{ does not contain } F \cup G\} \\
\text{is empty or has first element satisfying } P \subset F \cup G
\end{gathered}
$$

or else

$$
\begin{gathered}
\mathcal{P} = \{P \in \mathcal{F}_1; P < F,\ P < G,\ P \supset (F \cap G)\} \\
\text{is not empty and has first element containing neither } F \text{ nor } G.
\end{gathered}
\tag{5.11.8}
$$

In (5.11.7) $F \cap G$ is the smaller of F and G and $F \cup G$ is the larger. The order here is that in $\mathcal{F}_1$ but the condition is the same if the order is taken to be that of $\mathcal{F}_2$.

Proof. It can be assumed that G is the last element in $\mathcal{F}_1$, since, once the result is proved in this case, subsequent blow ups will be the same in the two spaces. The argument will proceed by induction on the number of earlier elements. Certainly if F is the first element then the condition in (5.11.7) is empty and (5.11.8) cannot occur, so the commutativity follows from Proposition 5.8.1 or Proposition 5.8.2.

To check the inductive step we need to show that if (5.11.6), (5.11.7) or (5.11.8) holds for a normal family $\mathcal{F}$ then it holds for the family obtained by lifting the remaining elements under the blow up of the first. Lemma 5.11.2 shows the lifted family to be normal. Certainly nothing changes away from the new front face. We can assume that the manifold to be blown up is given by (5.11.2) and the remaining elements are given by (5.11.1). If $F \pitchfork G$ then with $F = F_r$ and $G = F_{r+1}$ the corresponding index sets are disjoint, $I'_r \cap I'_{r+1} = \emptyset$, $I''_r \cap I''_{r+1} = \emptyset$. It follows that the lifts of these submanifolds under the blow up of F are again transversal. Thus is remains to consider the other two cases, (5.11.7) and (5.11.8).

Suppose F and G are as in (5.11.7) and for the sake of definiteness suppose that $F \subset G$. If the set $\mathcal{P}$ in (5.11.7) is empty then every element of $\mathcal{F}_1$ preceeding F and G either does not contain F or if it does it also contains G. The relation $F \subset G$ persists for the lifts under the blow up of such an element and the set $\mathcal{P}$ remains empty for the lifted family. Thus, still supposing that $F \subset G$, consider the case that $\mathcal{P}$ in (5.11.7) is not empty. If the first element of $\mathcal{F}_1$ is not in $\mathcal{P}$ then as before lifting under its blow-down map again gives (5.11.7). If this first element is in $\mathcal{P}$ then, by assumption in (5.11.7), it also satisfies $F \subset F_1 \subset G$. Under the blow up of F_1 the lifts of F and G become transversal, so that then (5.11.6) holds. Thus (5.11.6) and (5.11.7) are together preserved under the blow up of the first element.

Consider the third case, (5.11.8). Again if $F_1 \notin \mathcal{P}$ then lifting under its blow-down map the condition (5.11.8) persists. Othewise $F_1 \in \mathcal{P}$ is necessarily the first element and it is easly seen that the lifts of F and G are disjoint and hence transversal.

Thus the three conditions (5.11.6)–(5.11.8) are together preserved under lifting under the blow up of the first element of $\mathcal{F}_1$. This completes the proof of the inductive step. □

5.12. Lifting a blow up

Corollary 5.10.1 gives conditions under which a b-fibration remains so after the blow up of submanifolds of its domain. Next we consider the question of what needs to be blown up in the domain of a b-fibration to compensate for the blow-up of a sumbanifold of the range.

LEMMA 5.12.1. *The inverse image of a closed p-submanifold of the range of a b-fibration of compact manifolds with corners is a normal collection of closed p-submanifolds of the domain.*

PROOF. If $f\colon X \longrightarrow Y$ is the b-fibration and $S \subset Y$ is an interior p-submanifold then $f^{-1}(S)$ is a (possibly disconnected) interior p-submanifold of X, certainly this is a normal collection.

In the case of a boundary p-submanifold the local coordinate form near $p \in S$ is

$$S = \{x'_1 = \dots = x'_q = 0, y'_1 = \dots = y'_r = 0\} \tag{5.12.1}$$

where $j \le k$, with k' the local boundary codimension, and $r \le n - k'$. Coordinates, x_i, y_j can then be chosen near any point of $f^{-1}(p)$ such that

$$\begin{aligned} f^* x'_{i'} &= \prod_{p(i'-1) < i \le p(i')} (x_i)^{e(i',i)},\ i' = 1, \dots, k \\ f^* y'_j &= y_j,\ j = 1, \dots, r \end{aligned} \tag{5.12.2}$$

where $p(i')$ is a stricly increasing sequence with $p(0) = 0$ and the $e(i', i)$ are the non-vanishing boundary exponents. Thus, locally near $f^{-1}(p)$, the inverse image of

S is just the union of the local p-submanifolds

$$S'_\alpha = \{x_{\alpha(1)} = \cdots = x_{\alpha(k')} = 0, y_1 = \cdots = y_r = 0\} \tag{5.12.3}$$

where α runs over the maps $\alpha\colon \{1,\dots,k'\} \to \{1,\dots,p(k')\}$ with $p(i'-1) < \alpha(i') \le p(i')$. This common local product decomposition shows that the collection of lifts forms is normal. □

This lifting result for p-submanifolds suggest the following fundamental lifting result for b-fibrations.

PROPOSITION 5.12.1. *If $f\colon X \longrightarrow Y$ is a b-fibration and $S \in Y$ is a closed p-submanifold such that for each hypersurface $H \in \mathrm{Hu}(S)$ the boundary exponents of f on H are 0 or 1 :*

$$e_f(G,H) = 0 \text{ or } 1 \;\forall\; G \in M_1(X) \tag{5.12.4}$$

and $\mathcal{S}$ is normal family given by some ordering of all the lifts of S under f then f lifts from the interior to a b-fibration

$$f_S\colon [X;\mathcal{S}] \longrightarrow [Y;S] \tag{5.12.5}$$

PROOF. Consider first the case where S is a boundary face of Y. Let $I \subset M_1(Y)$ be the elements containing S, so that

$$S = \{\rho'_H = 0,\ H \in I\} \text{ near } p \in S. \tag{5.12.6}$$

Near $q \in f^{-1}(p)$ let

$$f^*\rho'_H = \prod_{\substack{G\in M_1(X),\ q\in G,\\ e(G,H)\neq 0}} \rho_G \tag{5.12.7}$$

be the pull-back of the defining functions; here we have used the assumption that the non-zero boundary exponents involving $H \in I$ are all equal to 1. Thus the elements of $\mathcal{S}$ containing q are the boundary faces S_α, parametrized by the maps $\alpha\colon I \longrightarrow M_1(X)$ with $e(\alpha(H),H) = 1$ for all $H \in I$ and $\rho_{\alpha(H)}(q) = 0$. Under the blow up of the first such surface, S_{α_1}, with blow-down map β_1, the front face of $X_1 = [X;S_{\alpha_1}]$ has defining function R_1 and the other boundary hypersurfaces can be identfied with the lifts of the boundary hypersurfaces of X. In terms of appropriate defining functions $R_G^{(1)}$, $G \in M_1(X)$ for the lifted boundary hypersurfaces

$$\beta^*_{\alpha_1}\rho_G = R_1 R^{(1)}_{\alpha_1(G)},\ G \in I,\ \beta^*_{\alpha_1}\rho_G = R^{(1)}_G,\ G \notin I, \tag{5.12.8}$$

The effect of the first such blow up is that f lifts to the map $f_1\colon X_1 \longrightarrow Y$ such that (5.12.7) is replaced by

$$f_1^*\rho'_H = R_1 \prod_{\substack{G\in M_1(X),\ q\in G,\\ e(G,H)\neq 0}} R^{(1)}_G. \tag{5.12.9}$$

Moreover near the front face of X_1 the lifts of the remaining S_α are given by the simultaneous vanishing of the corresponding $R^{(1)}_{\alpha(G)}$, $G \in I$. Notice that for α_1 this set is empty.

Proceeding by induction after the ith blow up a new fact R_i appear in the lifts of all the defining functions which vanish on S. Thus after all N boundary faces S_α

have been blown up (5.12.9) is replaced by

$$f^*_{(N)}\rho'_H = \prod_{i=1}^{N} R_i \prod_{\substack{G\in M_1(X),\ q\in G,\\ e(G,H)\neq 0}} R^{(N)}_G \tag{5.12.10}$$

As H runs over I the functions γ_H, given for each H by the second product in (5.12.10), can have no common zero. Indeed such a point would correspond to a common zero of the $R^{(N)}_{\alpha(H)}$ for some parametrizing map α. If this α is the jth in order then the functions at level j, $R^{(j)}_{\alpha(H)}$ can have no common zero, so there lifts under later blow ups can have no common zeros.

The blow up of S in the range corresponds to the introduction of the polar and angular functions

$$R = \left(\sum_{H\in I} \rho_H^2\right)^{\frac{1}{2}},\ \rho''_G = \rho'_G/R \tag{5.12.11}$$

as boundary defining functions. From (5.12.10) these functions lift to be smooth on the fully blown up space, $X_N = [X;\mathcal{S}]$. Thus f lifts to a smooth map $f' : [X;\mathcal{S}] \longrightarrow [Y;S]$ with

$$(f')^* R = a\prod_{i=1}^{N} R_i,\ (f')^*(\rho''_G) = a^{-1} \prod_{\substack{G\in M_1(X),\ q\in G,\\ e(G,H)\neq 0}} R^{(N)}_G, \tag{5.12.12}$$

where the fact that a is smooth and positive follows from the fact, noted above, that the second products in (5.12.10) have no common zero.

Together with the fact that the other boundary defining functions, for $G \notin I$, and the interior variables are not affected by these blow ups (5.12.12) also shows that the lifted map f' is a b-fibration. This completes the proof in case S is a boundary face. The general case is only notationally more complex. $\square$

Both Lemma 5.12.1 and Proposition 5.12.1 extend directly to the more general circumstances of a normal family in the range of a b-fibration.

PROPOSITION 5.12.2. *The lifts of a normal collection $\{S_i\}$ of p-submanifolds of the range of a b-fibration $f : X \longrightarrow Y$ is a normal collection of p-submanifolds $\{S_{j,i}\}$ of the domain. If $\mathcal{T}$ is a normal family given by an ordering of the collection in the range and $e(H,G) = 0$ or 1 for each $G \in M_1(X)$ and each $H \in M_1(Y)$ containing an element of $\mathcal{T}$ then f lifts to a b-fibration $f' : [X;\mathcal{S}] \longrightarrow [Y;\mathcal{T}]$ if $\mathcal{S}$ is a normal family given by an ordering of the S_{ij} in which each $S_{i'j'}$ preceeds S_{ij} if $S_{i'}$ preceeds S_i.*

PROOF. By Lemma 5.12.1 each of the p-submanifolds in Y lifts to a normal collection of p-submanifolds of X. To see that the totality of these manifolds forms a normal collection, i.e. to obtain the desired normal form in local coordinates, it is only necessary to apply the same argument using the local normal form for f.

The remainder of the proposition will be proved inductively over the numbe of elements of $\mathcal{T}$. Proposition5.12.1 gives the result if $\mathcal{T}$ has only one element. Blow up of the first element, S_1, reduces the number of elements by 1. Let f_1 be the resulting b-fibration, given by Proposition 5.12.1, and let β'_1 and β''_1 be the blow down maps in range and domain respectively. Then it is enough to show that for each of the other elements, S_i, of $\mathcal{T}$ the lift under f_1 of $(\beta_1')^* S_i$ is the normal collection given by the lifts $(\beta_1'')^* S_{ij}$ of the elements of its lift of S_i under f. This can be seen directly following the proof of Proposition 5.12.1 above, separately as $S_i \subset S_1$ or not. $\square$

A similar, simpler, discussion also serves to extend Corollary 5.10.1 to the case of the blow up of a normal family:

PROPOSITION 5.12.3. *If $f : X \longrightarrow Y$ is a b-fibration and $\mathcal{S}$ is a normal family of closed p-submanifolds in X to which f is b-transversal in the sense that*

$$\big(\operatorname{null}(f_*) \cap T_p \operatorname{Fa}(p)\big) + \left(\bigcap_{p \in S_i} T_p \left(S_i \cap \operatorname{Fa}(p)\right)\right) = T_p \operatorname{Fa}(x), \ \forall\ p \in X \tag{5.12.13}$$

and such that no element of $\mathcal{S}$ has image under f contained in a boundary face of Y of codimension 2 then $f' = f \circ \beta_{\mathcal{S}}$ is a b-fibration.

5.13. Total boundary blow up

One basic example of a normal collection of submanifolds is $M'(X)$, the set of all boundary faces of X. Consider an order on $M'(X)$ which is consistent with inclusion, i.e. if $B_1 \subset B_2$ then $B_1 \le B_2$. Such an order always exists, as can be seen by listing the boundary faces so that the codimension is non-decreasing. The manifold obtained by blowing up this normal family will be denoted X_{tb}, with total blow-down map $\beta_{\text{tb}} : X_{\text{tb}} \longrightarrow X$. We will call X_{tb} the total boundary blow up of X. As the notation indicates the resulting manifold is independent of the order chosen. This can be deduced from the following somewhat more general result:

PROPOSITION 5.13.1. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are normal families given by orders on the same collection of boundary faces of a manifold with corners X where the collection is closed under non-transversal intersection and both orders satisy*

$$B_1 \subset B_2 \Longrightarrow B_1 \le B_2 \tag{5.13.1}$$

then $[X; \mathcal{T}_1]$ and $[X; \mathcal{T}_2]$ are naturally diffeomorphic.

PROOF. If B_r and B_{r+1} are neighbouring elements with respect to one of the orders and $B_r \pitchfork B_{r+1}$ then the order obtained by reversing them still satifies (5.13.1) and Proposition 5.11.2 shows that it gives the same blow up. With repsect to either the order fixing $\mathcal{T}_1$ or that fixing $\mathcal{T}_2$ the first element must be minimal, i.e. cannot be contained in any other. Let the order for $\mathcal{T}_1$ be given by $B_1, \dots, B_N$ and consider the position of the first element with respect to the order of $\mathcal{T}_2$, suppose it is B_l. The fact that the underlying collection is closed under non-transversal intersection means that each element B_i for $i = 1, \dots, l-1$ must be transversal to B_l, since otherwise it could not be minimal. Thus, using the commutation described above, the second normal family can be replaced by one with the same blown up space but with first element B_1.

This allows the result to be proved by induction over the number of elements of the underlying collection. Certainly it is true if the collection has only one element. To prove the inductive step the discussion above shows that the two families can be supposed to have the same first element. Blowing up this element and lifting to the blown up manifold clearly gives two families satisfing the same hypotheses and with one less element. $\square$

Clearly the boundary hypersurfaces of X_{tb} are in 1-1 correspondence with the non-trivial boundary faces of X

$$M_1(X_{\mathrm{tb}}) \ni B \longleftrightarrow \beta_{\mathrm{tb}}(B) \in M'(X). \tag{5.13.2}$$

LEMMA 5.13.1. *If X is a manifold with corners then the set of non-trivial boundary faces, $M'(X_{\mathrm{tb}})$, of the total boundary blow up of X is in* 1-1 *correspondence with the subset of the power set of $M(X)$ comprised of the chains.*

PROOF. A boundary face of X_{tb} is a component of a non-empty intersection of boundary hypersurfaces. Since $M_1(X_{\mathrm{tb}}) \equiv M'(X)$ this gives a map into the power set:

$$M'(X_{\mathrm{tb}}) \longrightarrow \mathcal{P}\left(M(X)\right). \tag{5.13.3}$$

If B_1 and B_2 are boundary faces of X then $B_1 \cap B_2$ is a finite union of boundary faces of X, each a component. If $B_1 \cap B_2$ has smaller dimension than either B_1 or B_2 then on blowing up all its components B_1 and B_2 become separated. Thus if the hypersurfaces of X_{tb} corresponding to B_1 and B_2 are to intersect then $\dim(B_1 \cap B_2)$ must equal the smaller of $\dim B_1$ and $\dim B_2$. For boundary faces this means that either $B_1 \subset B_2$ or $B_2 \subset B_1$. Applying this argument repeatedly shows that if a collection of boundary hypersurfaces of X_{tb} have non-empty intersection then they must define a chain of boundary faces of X. Conversely consider the boundary hypersurfaces formed by the blow up of the elements of a chain of boundary faces of X. Near its smallest element, B, the chain has a product decomposition, as in (1.9.5), as bundles over B. It follows that, near the hypersurfaces defined by the blow up of B, the blow up of each element of the chain is a bundle over B which has fibre given by the blow up of one of the elements of the appropriate chain of boundary faces of some $\mathbb{R}^n_k$. In particular the intersection of the hypersurfaces of X_{tb} corresponding to a chain of boundary faces of X is not only non-empty it is also connected. Thus the map (5.13.3) is an identification of $M'(X_{\mathrm{tb}})$ and the set of chains of boundary faces of X. □

Since the blow up of each boundary face corresponds to the introduction of polar coordinates in the corresponding defining function it is relatively easy to find a set of generators of $\mathcal{C}^\infty(X_{\mathrm{tb}})$.

LEMMA 5.13.2. *For any manifold with corners X, let ρ_H, for $H \in M_1(X)$ be a complete set of boundary defining functions. If $A \subset M_1(X)$ set*

$$\rho_A = \left(\sum_{H \in A} \rho_H^2\right)^{\frac{1}{2}} \tag{5.13.4}$$

then the functions ρ_A and the quotients

$$\omega_{A,B} = \frac{\rho_A}{\rho_B},\ A \subset B \subset M_1(X), \tag{5.13.5}$$

where if $A = \{H\}$ has just one element we write $\omega_{A,B} = \omega_{H,B}$, generate $\mathcal{C}^\infty(X_{\mathrm{tb}})$ over $\mathcal{C}^\infty(X)$.

PROOF. Let X_j be the manifold defined inductively by blowing up all boundary faces of dimension j or less. Thus X_0 is obtained from X by blowing up all point boundary faces, X_1 is obtained by blowing up the lifts to X_0 of all one-dimensional boundary faces of X, and so on. Thus $X_{\mathrm{tb}} = X_{n-1}$ if $n = \dim X$. Certainly it suffices to show inductively that $\mathcal{C}^\infty(X_k)$ is generated by the lifts of the functions

$$\begin{gathered} \rho_A,\ \omega_{H,A} \text{ and } \omega_{A,B} \text{ for } A \subset B \\ \text{where } A, B \text{ have at least } n-k \text{ elements.} \end{gathered} \tag{5.13.6}$$

and that this set of functions contains a complete set of boundary defining functions. In case $k = 0$ (5.13.6) just corresponds to the introduction of polar coordinates around the boundary faces of dimension 0, if there are any. If there are no such faces then all the ρ_A, where A has at least n elements, are smooth and strictly positive, so (5.13.6) just gives $\mathcal{C}^\infty(X)$ again, as it should.

Consider the transition from X_0 to X_1. The boundary faces in X_0 to be blown up are the lifts of the one dimensional boundary faces of X. In terms of (5.13.6), these faces are defined by the vanishing of any $n-1$ of the quotients $\omega_{H,B}$ where B is fixed (and has n elements) and $H \in B$. If $A \subset B$ is such a subset the corresponding radial variable is

$$\left(\sum_{H\in A} \omega_{H,B}^2\right)^{\frac{1}{2}} = \frac{\rho_A}{\rho_B} = \omega_{A,B}. \tag{5.13.7}$$

The angular variables corresponding to this blow up are just the

$$\frac{\omega_{H,B}}{\omega_{A,B}} = \omega_{H,A}. \tag{5.13.8}$$

This gives (5.13.6) for $k = 1$. In fact the general inductive step is essentially the same. In X_{k-1} the boundary surfaces of dimension k to be blown up to get X_k are the lifts of the elements of $M_k(X)$ and are defined by the vanishing of the $\omega_{H,B}$ for $H \in A \subset B$ where A has $n-k$ elements and B has $n-k+1$. Then (5.13.7) and (5.13.8) again give the inductive step. □

One consequence of the commutation of blow ups is the following lifting result for projections:

Proposition 5.13.2. *If X and Y are manifolds with corners and $Z = X \times Y$ then there is a uniquely defined map $\pi_{\mathrm{tb}}\colon Z_{\mathrm{tb}} \longrightarrow Y_{\mathrm{tb}}$ such that the diagram*

$$\begin{tikzcd} Z_{\mathrm{tb}} \arrow[r, "\pi_{\mathrm{tb}}"] \arrow[d, "\beta_{\mathrm{tb}}"] & Y_{\mathrm{tb}} \arrow[d, "\beta_{\mathrm{tb}}"] \\ Z \arrow[r, "\pi"] & Y \end{tikzcd} \tag{5.13.9}$$

commutes, where $\pi\colon Z = X \times Y \longrightarrow Y$ is projection onto Y, and moreover π_{tb} is a b-fibration.

Proof. The boundary faces of Z are precisely the products $B \times F$ of a boundary face of B and a boundary face of Y, including X and Y. Select an order on $M'(Y)$ under which dimension is non-decreasing, so giving a normal family defining Y_{tb}. Select a similar order for $M(X)$. On $M'(Z)$ consider the order in which dimension is non-decreasing and within a fixed dimension the order is lexicographic, the $B \times F$ of the given dimension are listed in groups as B is ordered and within each group are ordered by F. This certainly gives a normal family, $\mathcal{F}$, defining Z_{tb} by blow up. We will apply Proposition 5.11.2 to change the order, without affecting the resulting space, by moving all the products $X \times F$ to the front. Thus consider the normal family, $\mathcal{G}$, also consisting of the elements of $M'(Z)$ but ordered with the $X \times F$

appearing first, in the order of F in $M'(Y)$, and then followed by all the remaining elements in the order in which they appear in $\mathcal{F}$. Notice that the $X \times F$ appear in this order in $\mathcal{F}$, interspersed with the other faces.

Thus it suffices to consider an intermediate order, $\mathcal{G}_s$, in which $X \times F_i$, for $i = 1, \dots, s$ come first, followed by the remaining elements in the order in which they appear in $\mathcal{F}$, and show that it defines the same blow up as $\mathcal{G}_{s-1}$. This in turn can be accomplished by commutations in which $X \times F_s$ is successively moved past each of the boundary faces of Z occuring between it and $X \times F_{s-1}$ which is the element in position $s-1$ in $\mathcal{G}_{s-1}$. Each of the intervening boundary faces is of the form $B \times F$ for some $B \in M'(X)$. If $F \neq Y$ then these submanifolds are not tranversal. Thus it suffices to check that either (5.11.7) or (5.11.8) holds. If (5.11.7) is to hold then $B \times F \subset X \times F_s$, which can occur only if $F \subset F_s$. The elements in $\mathcal{P}$ in (5.11.7) can only be of the form $B' \times F'$ with $B' \subset B$ and $F' \subset F$, which is not possible since it must follow $B \times F$ in order, or $X \times F'$ with $F' \supset F$. In the latter case $X \times F' \in \mathcal{P}$ iff F' does not contain F_s. Thus it is possible that $\mathcal{P}$ is non-empty but if so its first element is the first element of $M'(Y)$ containing F, this is F itself, so contained in F_s which shows that (5.11.7) holds in this case.

Thus it suffices to check that (5.11.8) holds if $X \times F_s$ does not contain $B \times F$, i.e. F_s does not contain F but $F \cap F_s \neq \emptyset$ (since otherwise (5.11.6) holds.) The set $\mathcal{P}$ in (5.11.8) is non-empty and in fact has first element $X \times (F \cap F_s)$ if $F \cap F_s$ is strictly smaller than F_s, containing neither $X \times F_s$ nor $B \times F$ and if $F_s \subset F$ then it must be a strict subset and the first element is $B \times F_s$ which again contains neither.

This shows that Z_{tb} can be defined by the blow up of the normal family $\mathcal{G}$. The first part of $\mathcal{G}$ just gives the blow up of the boundary of Y so showing that there is indeed a smooth map π_{tb} in (5.13.9). It remains only to show that it is a b-fibration. Under the blow-down map for the first part of $\mathcal{G}$, consisting of the elements of $X \times M'(Y)$, the remaining elements lift to form a normal family to which the projection $X \times Y_{\mathrm{tb}} \longrightarrow Y_{\mathrm{tb}}$ is transversal. Furthermore the image of each of these boundary faces is a boundary hypersurface of Y_{tb}, with the lift of $B \times F$ always mapping onto the boundary hypersurface produced by the blow up of F, so Proposition 5.12.3 applies and shows that the lift, which gives π_{tb}, is a b-fibration. $\square$

One useful example of a generally non-smooth function which lifts to be smooth on the total boundary blow up is

$$f = \frac{1}{a + \sum\limits_{H \in A} \frac{1}{\rho_H}} \tag{5.13.10}$$
$$\text{where } A \subset M_1(X) \text{ and } 0 \leq a \in \mathcal{C}^\infty(X).$$

Lemma 5.13.3. *If $\rho_H \in \mathcal{C}^\infty(X)$ for $H \in M_1(X)$ are boundary defining functions $A \subset M_1(X)$ and $0 \leq a \in \mathcal{C}^\infty(X)$ then the function in* (5.13.10) *lifts to be smooth on X_{tb} :*

$$\begin{gathered} \beta_{\mathrm{tb}}^* f = a_f \prod_{B \subset \widetilde{A}} \rho'_B, \\ \widetilde{A} = \bigcap_{H \in A} H, \ 0 < a_f \in \mathcal{C}^\infty(X_{\mathrm{tb}}), \end{gathered} \tag{5.13.11}$$

where the ρ'_B, parametrized by the boundary faces of X, are defining functions for the boundary hypersurfaces of X_{tb}.

PROOF. It suffices to check the smoothness of $\beta^*_{\text{tb}}f$ in a small neighbourhood of the preimage in X_{tb} of each point where all the ρ_H for $H \in A$ vanish, since the non-vanishing ρ_H can be absorbed into a. Thus X can be replaced by $\mathbb{R}^n_k$ and the ρ_H for $H \in A$ by the corresponding x_j for $j \le p$.

In case $p = 1$, f is of the form $x_1/(1 + x_1 a)$ and is therefore already $\mathcal{C}^\infty$. For general p consider the product decomposition $\mathbb{R}^n_k = \mathbb{R}^p_p \times \mathbb{R}^{n-p}_{k-p}$. The total blow up can be factored through the total blow up of the first factor. Under the blow up of $\{0\}$ in $\mathbb{R}^p_p$ the x_j are replaced by $r\omega_j$. Thus f lifts to

$$r \times \frac{1}{ra + \sum\limits_{j=1}^{p} \frac{1}{\omega_j}} \tag{5.13.12}$$

Now $\mathbb{R}^p_p$ is replaced by $[0, \infty) \times Y$ where Y is a manifold with corners of dimension $p-1$ and the ω_j are defining functions for the bounddary hypersurfaces of Y. Thus a simple inductive argument shows that f lifts to $\left(\mathbb{R}^p_p\right)_{\text{tb}} \times \mathbb{R}^{n-p}_{k-p}$ and vanishes simply at all boundary hypersurfaces of the first factor. The further lift to X_{tb} is a b-map so the lift under β_{tb} vanishes simply at all the boundary hypersurfaces of X_{tb} which correspond to boundary faces in X on which all the ρ_H for $H \in A$. This is precisely the content of (5.13.11). □

COROLLARY 5.13.1. *If $A' \subset A \subset M_1(X)$, $a_H \in \mathcal{C}^\infty(X)$ for $H \in A'$, $a' \in \mathcal{C}^\infty(X)$ and $0 \le a \in \mathcal{C}^\infty(X)$ then the function*

$$\frac{a' + \sum\limits_{H \in A'} \frac{a'_H}{\rho_H}}{a + \sum\limits_{H \in A} \frac{1}{\rho_H}} \tag{5.13.13}$$

lifts into $\mathcal{C}^\infty(X_{\text{tb}})$.

Since we shall consider the total boundary blow up of products of a given manifold with itself we shall introduce some further notation:

DEFINITION 5.13.18. If X is a manifold with corners we shall define (the otherwise ambiguous) notation

$$X^2_{\text{tb}} = \left(X^2\right)_{\text{tb}},\ X^3_{\text{tb}} = \left(X^3\right)_{\text{tb}}. \tag{5.13.14}$$

Let $\pi^2_h \colon X^2 \longrightarrow X$ be, for $h = l, r$, the projections respectively onto the left and right factors of X. Similarly let $\pi^3_o \colon X^3 \longrightarrow X^2$ be, for $o = f, s, c$ the three projections where π^3_f drops the first factor, π^3_s drops the third factor and π^3_c drops the central factor; the subscripts here correspond to the operators in composition, f is for first, s is for *s*econd and c is for *c*omposite. Applying Proposition 5.13.2 to each of these projections gives:

COROLLARY 5.13.2. *For any manifold with corners, X, there are two b-fibrations*

$$\pi^2_{\text{tb},h} \colon X^2_{\text{tb}} \longrightarrow X_{\text{tb}}, \tag{5.13.15}$$

for $h = l, r$ and three b-fibrations

$$\pi^3_{\text{tb},o} \colon X^3_{\text{tb}} \longrightarrow X^2_{\text{tb}} \tag{5.13.16}$$

for $o = f, s, c$ such that the following diagrammes commute:

$$
\begin{tikzcd}
X_{\mathrm{tb}} \arrow[r, "\pi^2_{\mathrm{tb},l}"] \arrow[d, "\beta_{\mathrm{tb}}"] & X^2_{\mathrm{tb}} \arrow[r, "\pi^2_{\mathrm{tb},r}"] \arrow[d, "\beta_{\mathrm{tb}}"] & X_{\mathrm{tb}} \arrow[d, "\beta_{\mathrm{tb}}"] \\
X \arrow[r, "\pi^2_l"] & X^2 \arrow[r, "\pi^2_r"] & X
\end{tikzcd}
\tag{5.13.17}
$$

$$
\begin{tikzcd}
X^2_{\mathrm{tb}} \arrow[d, "\beta_{\mathrm{tb}}"] & X^3_{\mathrm{tb}} \arrow[l, "\pi^3_{\mathrm{tb},l}"'] \arrow[r, "\pi^3_{\mathrm{tb},r}"] \arrow[d, "\beta_{\mathrm{tb}}"] & X^2_{\mathrm{tb}} \arrow[d, "\beta_{\mathrm{tb}}"] \\
X^2 & X^3 \arrow[l, "\pi^3_f"'] \arrow[r, "\pi^3_s"] & X^2.
\end{tikzcd}
\tag{5.13.18}
$$

5.14. Logaritmic blow up

By logarithmic blow up of a boundary face, with defining function ρ, we mean the introduction of

$$t = \mathrm{ilg}\,\rho = \frac{1}{\log \frac{1}{\rho}} \tag{5.14.1}$$

as a new boundary defining function near $\rho = 0$. This defines a new manifold:

Lemma 5.14.1. *If $\rho < 1$ is a defining function for a boundary hypersurface $B \in M_1(X)$ of a manifold with corners then t, in (5.14.1), and $\mathcal{C}^\infty(X)$ generate a $\mathcal{C}^\infty$ structure on X as a manifold with corners, $[X, B]_{\log}$, which is independent of the choice of ρ and such that the identity map $[X, B]_{\log} \longrightarrow X$ is $\mathcal{C}^\infty$.*

Proof. Clearly if $\rho < 1$ then t is $\mathcal{C}^\infty$ away from $B = \{\rho = 0\}$ so the $\mathcal{C}^\infty$ structure is unchanged away from B. Taking a product decomposition of X near B, $X \sim B \times [0, \epsilon)_\rho$ in which ρ is the normal variable to the boundary and noting that $\rho = \exp(-1/t)$ is $\mathcal{C}^\infty$ shows that t and $\mathcal{C}^\infty(X)$ do fix a new $\mathcal{C}^\infty$ structure on X as a manifold with corners, $[X, B]_{\log}$. To see that this structure is independent of ρ observe that if $\rho' < 1$ is another defining function for B then $\rho' = a\rho$, with $0 < a \in \mathcal{C}^\infty(X)$ and

$$t' = \mathrm{ilg}\,\rho' = \mathrm{ilg}\,a\rho = \frac{1}{\log \frac{1}{\rho} + \log \frac{1}{a}} = \frac{t}{1 - t\log a} = \frac{\mathrm{ilg}\,\rho \cdot \mathrm{ilg}\,a}{\mathrm{ilg}\,\rho + \mathrm{ilg}\,a}. \tag{5.14.2}$$

This shows that t' is $\mathcal{C}^\infty$, since $1 - t\log a > 0$. □

In view of the existence of local product decompostions such logarthmic blow up can be performed at each boundary hypersurface and the different blow ups commute. Thus the 'total logarithmic blow up' of X :

$$\begin{gathered} X_{\log} = X_k,\ X_j = [X_{j-1}, B_j]_{\log},\ j = 1, \dots, k, \\ X_0 = X,\ M_1(X) = \{B_1, \dots, B_k\} \end{gathered} \tag{5.14.3}$$

is well defined, with overall blow down map $\beta_{\log} \colon X_{\log} \longrightarrow X$. Under this blow down map polynomially decaying functions on X become rapidly decaying.

There is an important lifting result relating total blow up of the boundary in the ordinary sense and logarithmic blow up:

PROPOSITION 5.14.1. *For any manifold with corners the natural identification of the interiors lifts under iterated blow up to give a b-map:*

$$\left((X_{\mathrm{tb}})_{\log}\right)_{\mathrm{tb}} \longrightarrow (X_{\log})_{\mathrm{tb}}\,. \tag{5.14.4}$$

PROOF. Let ρ_H, for $H \in M_1(X)$ be a complete set of boundary defining functions. We can certainly assume that $\rho_H < \frac{1}{2}$ everywhere, so the functions $t_H = \operatorname{ilg}\rho_H$ are a complete set of boundary defining functions for $X_{\log}$. By Lemma 5.13.2, applied to $X_{\log}$, we see that it suffices to show that the functions

$$\begin{gathered} r_A = \left(\sum_{H\in A} (\operatorname{ilg}\rho_H)^2\right)^{\frac{1}{2}}, \\ w_{A,B} = \frac{r_A}{r_B},\ A \subset B \end{gathered} \tag{5.14.5}$$

lift to be $\mathcal{C}^\infty$ on the space on the left in (5.14.4).

Lemma 5.13.2 shows that the functions in (5.13.4) and (5.13.5), together with $\mathcal{C}^\infty(X)$, generate $\mathcal{C}^\infty(X_{\mathrm{tb}})$. Moreover this set of functions contains a complete set of boundary defining functions, the remaining elements are either non-vanishing or products of defining functions. If $g = \rho_1 \cdots \rho_l$ is a product of defining functions then

$$\operatorname{ilg} g = \frac{1}{\log\frac{1}{g}} = \frac{1}{\frac{1}{\operatorname{ilg}\rho_1} + \cdots + \frac{1}{\operatorname{ilg}\rho_l}} \tag{5.14.6}$$

is $\mathcal{C}^\infty$ by Lemma 5.13.3, and a product of defining functions. Thus the $\operatorname{ilg} c\rho_A$ and $\operatorname{ilg} c\omega_{A,B}$, for $A \subset B \subset M_1(X)$ and $c > 0$ a sufficiently small constant, similarly contain a complete set of boundary defining functions for $(X_{\mathrm{tb}})_{\log}$ and together with $\mathcal{C}^\infty(X)$ generate the $\mathcal{C}^\infty$ structure on this manifold.

Consider the lift of the r_A, in (5.14.5), to $(X_{\mathrm{tb}})_{\log}$. Applying Lemma 5.13.2 this lifts to be of the form

$$\left(\sum_{H\in A}\left(\frac{1}{a_H + \sum\limits_{B\subset H}\frac{1}{\operatorname{ilg}\rho'_B}}\right)^2\right)^{\frac{1}{2}}. \tag{5.14.7}$$

Here we have chosen the $\rho'_B < \frac{1}{2}$ first and then chosen the ρ_H so small that the a_H are positive. From Lemma 5.13.1 each of the terms in the outer sum lifts to the space on the left in (5.14.4) to be $\mathcal{C}^\infty$ and of the form

$$a'_H \prod_{\mathcal{B}\in\mathcal{C}(H)} \rho''_{\mathcal{B}},\ a'_H > 0 \tag{5.14.8}$$

where $\mathcal{C}(H)$ is the collection of chains in $M_1(X)$ with the property that each element contains H and where the $\rho''_{\mathcal{B}}$ are boundary defining functions for $((X_{\mathrm{tb}})_{\log})_{\mathrm{tb}}$. The factor common to each of these products, as H ranges over A is just the product

$$\prod_{\mathcal{B}\in\mathcal{C}(A)} \rho''_{\mathcal{B}} \tag{5.14.9}$$

where $\mathcal{C}(A)$ is the collection of chains which have minimal element (as a subset of $M'(X)$) contained in the intersection of the $H \in A$. Furthermore

$$\sum_{H\in A}\left(\prod_{(\mathcal{C}(A)\setminus\mathcal{C}(H))}\rho''_{\mathcal{B}}\right)^2 > 0 \tag{5.14.10}$$

for any subset $A \subset M_1(X)$. Thus r_A lifts to be a smooth positive multiple of the product (5.14.9). The hypersurfaces of $\left((X_{\mathrm{tb}})_{\log}\right)_{\mathrm{tb}}$ corresponding to the chains in $\mathcal{C}(A)$ are those mapped into each of the elements of A in X. Thus if $A \subset B \subset M_1(X)$ then $\mathcal{C}(A) \supset \mathcal{C}(B)$. Thus the quotient $\omega_{A,B} = r_A/r_B$ lifts to into $\mathcal{C}^\infty\left(\left((X_{\mathrm{tb}})_{\log}\right)_{\mathrm{tb}}\right)$ provided $A \subset B$. □

5.15. Quasihomogneneous blow up

In the remaining sections of this chapter many of the results above are extended to the case of the non-radial blow up of a submanifold corrresponding to a quasi-homogeneity structure. First we show how such a blown up manifold is associated to the quasi-homogeneity.

Consider first a quasi-homogeneity structure, $\mathcal{F}$, at tge origin of a real vector space V. As shown in §1.15 this corresponds to a nested sequence of subspaces of the dual, with associated non-negative integers:

$$V^* = S_1 \supset S_2 \supset S_3 \supset \dots S_q = \{0\} = S_{q+1} = \dots. \tag{5.15.1}$$

where the homogeneity of a linear function on V with differential in S_k is (at least) k. Choosing an appropriate basis of V, and corresponding linear coordinates, x_i, $i = 1, \dots, n$, we can arrange that each S_k is spanned by the linear functions dx_i for $i \le N(k)$:

$$S_k = \left\{\sum_{j=1}^{N(k)} a_j dx_j\right\}, \tag{5.15.2}$$

where $N(k)$ is a non-increasing function of k, with $N(q) = 0$.

On $\mathbb{R}^n$ consider the $\mathbb{R}^+$ action given by integration of the vector field

$$R_h = \sum_{i=1}^n h(i)x_i\frac{\partial}{\partial x_i},\ j = h(i) \Longleftrightarrow dx_i \in S_j \text{ but } dx_i \notin S_{j+1}. \tag{5.15.3}$$

Consider also the related non-round sphere:

$$\mathbb{S}_h^{n-1} = \{x \in \mathbb{R}^n; r_h = 1\},\ r_h(x) = \left(\sum_{i=1}^n x_i^{2H(i)}\right)^{1/2L} \\ \text{where } L = (h(n))! \text{ and } H(i) = (h(n))!/h(i). \tag{5.15.4}$$

Notice here that $h(n)$ is the largest of the integers $h(i)$. Clearly each x_i is homogeneous of degree $h(i)$ under the $\mathbb{R}^+$ generated by R_h. Thus $x_i^{2H(i)}$ is homogeneous of degree $2H(i)h(i) = 2L$. It follows that r_h is a smooth positive function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree 1. Thus there is an isomorphism

$$\mathbb{R}^n \setminus \{0\} \ni x \longmapsto (r_h(x), \omega) \in (0, \infty) \times \mathbb{S}_h^{n-1} \tag{5.15.5}$$

where the second factor is given by projection along the itegral curves of R_h, so

$$\omega_i = \frac{x_i}{r(x)^{h(i)}},\ i = 1, \dots, n. \tag{5.15.6}$$

In particular the inverse of the map (5.15.5) extends to be smooth down to $r = 0$. It is this map which is the model quasi-homogeneous blow down map, for the quasi-homogeneity structure at $\{0\}$ determined by the S_k :

$$\beta\colon [\mathbb{R}^n; \{0\}, \mathcal{F}] = [0, \infty) \times \mathbb{S}_h^{n-1} \ni (r, \omega) \longmapsto (r^{h(1)}\omega_1, \dots, r^{h(n)}\omega_n) \in \mathbb{R}^n. \tag{5.15.7}$$

We proceed to show, as the notation indicates, that this blow up proecdure does really only depend on the quasi-homogeneity structure. First notice that the vector field R_h is not, in general, determined by the subspaces S_j at least as a vector field on V. However it is determined as a vector field on the graded vector space associated to the filtration of V :

$$V_S = \bigoplus_{j=1} 1^q \left(S_j/S_{j+1}\right)^* . \tag{5.15.8}$$

Thus we can identify the non-round sphere $\mathbb{S}_h^{n-1}$ with the quotient $\left(V_S \setminus \{0\}\right)/\mathbb{R}^+$. By doing so we have identified the product on the right in (5.15.7) with a fixed set, which does not depend on the basis. This means that if we define V with $\{0\}$ blown up with quasi-homogeneity arising from (5.15.1) to be

$$[V; \{0\}, \mathcal{F}] = \left(V_S \setminus \{0\}\right)/\mathbb{R}^+ \sqcup \left(V \setminus \{0\}\right) \tag{5.15.9}$$

then (5.15.7) gives this the structure of a manifold with boundary. Notice that we replace the filtration in (5.15.1) by the corresponding quasihomogeneity structure. However it remains to be shown that the $\mathcal{C}^\infty$ structure does not depend on the choice of basis. A change of the basis which reduces the S_j to normal form can be construed as a linear transformation, A, of the original vector space V such that the dual transformation A^t maps each S_k into itself. Thus the naturality of the construction follows from:

Lemma 5.15.1. *If A is a linear transformation of a vector space V and A^t maps each element in a filtration* (5.15.1) *into itself then A extends from $V \setminus \{0\}$ to a diffeomorphism of $[V; \{0\}, \mathcal{F}]$.*

Proof. The assumption that A preserves the filtration (5.15.1) means that when expressed with respect to any basis which reduces the S_j to the form (5.15.2) it takes a block-triangular form. Expressed in terms of the transpose this becomes the condition:

(5.15.10)

$$(A^t)(dx_j) = \sum_{j=N(k+1)+1}^{N(k)} A_{jl}^{(k)} dx_l + \sum_{j \le N(k+1)} A_{jl} dx_l,\ N(k+1) < l \le N(k),$$

where the block-diagonal matrices $A^{(k)}$ must all be invertible.

Suppose first that the off-block-diagonal terms vanish, $A_{jl} = 0$ for $j \le N(k+1)$, $l > N(k)$ for each k. Then the homogeneity of all terms is preserved and

$$
(5.15.11) \qquad \begin{aligned} r_h(Ax) &= \left(\sum_{i=1}^{n} (Ax)_i^{2H(i)}\right)^{1/2L} = a(\omega) r_h(x) \\ a_A(\omega) &= \left(\sum_{i=1}^{n} (A\omega)_i^{2H(i)}\right)^{1/2L} \in \mathcal{C}^\infty(\mathbb{S}_h^{n-1} \end{aligned}
$$

If the decomposition into terms of fixed homogeneity is written $x = (x^{(1)}, \dots, x^{(q)})$ where $x^{(k)} = (x_{N(k+1)+1}, \dots, x_{N(k)})$ then the lifted diffeomorphism is just

(5.15.12)
$$
\tilde{A}(r_h, \omega^{(1)}, \dots, \omega^{(q)}) = (a_A(\omega) r_h, a_A^{-h(1)}(\omega) A^{(1)} \omega^{(1)}, \dots, a_A^{-h(q)}(\omega) A^{(q)} \omega^{(q)}).
$$

Since the block-diagonal terms of a block-triangular matirx compose under composition this analysis of the block-diagonal case allows a general linear tranformation preserving the filtration (5.15.1) to be reduced to one with all block-diagonal entries identity matrices. Then instead of (5.15.11) one finds that the new radial variable is of the form $r_h(1 + r_H f(r_H, \omega))^{1/2L}$. It is therefore $\mathcal{C}^\infty$, since it is certainly $\mathcal{C}^\infty$ in $r_h > 0$. The angular variables behave similarly, so the general linear map preserving the filtration does indeed lift to be smooth. □

We can proceed to generalize this to the case of a quasi-homogeneity structure around a p-submanifold much as in the radial case discussed above. As a first step we consider the lift of vector fields. The homogeneity order of a differential operator, and in particular of a vector field, is defined in §1.15.

Lemma 5.15.2. *The smooth vector fields vanishing at* 0 *and of non-negative order with respect to a quasi-homogeneity structure at* $\{0\}$ *in a vector space* V *are spanned over* $\mathcal{C}^\infty(V)$ *by the polynomial vector fields of the form* $x^\alpha \partial/\partial x_j$ *where the total homogeneity of coefficient* $h(j) \le h(\alpha) = \alpha_1 h(1) + \cdots + \alpha_n h(n) \le h(j) + q$.

Proof. The first inequality on the indices certainly ensures that they have non-negative homogeneity order. Conversly the condition that a vector field

$$
W = \sum_{i=1}^{n} a_j(x) \frac{\partial}{\partial x_j}
$$

should have non-negative homogeneity order implies that $Vx_j = a_j$ should only have terms of homogeneity at least $h(j)$ in its Taylor series. Terms of homogeneity $q + h(j)$ need not be included to get a spanning set since they can be decomposed into products of linear factor and terms of homogeneity at least $h(j)$. □

As a simple consequence we conclude that

Proposition 5.15.1. *The smooth vector fields vanishing at* 0 *and of non-negative homogeneity order on a vector space* V*, with quasihomogeneity structure given by* (5.15.1)*, lift to be smooth on* $[V; \{0\}, \mathcal{F}]$ *and span, over* $\mathcal{C}^\infty([V; \{0\}, \mathcal{F}])$*, the Lie algebra* $\mathcal{V}_b([V; \{0\}, \mathcal{F}])$.

PROOF. In view of Lemma 5.15.2 to show the smoothness of the lifts it suffices to show that the spanning set lift to be smooth. Since these all have non-negative homogeneity they each lift to be of the form

$$r^m(ar\partial_r + W) \tag{5.15.13}$$

for some integer m, function a on $\mathbb{S}_h^{n-1}$ and vector field W on $\mathbb{S}_h^{n-1}$. The smoothness of a and W follow from the smoothness of the lift away from 0. Furthermore the radial vector field R_h lifts to $r\partial_r$, so the fact that these lifts span $\mathcal{V}_b([V;\{0\},\mathcal{F}])$ follows from the fact that they span outside the origin. □

As in the radial case integration of these vector fields allows the lifts of diffeomorphisms to be analyzed easily:

PROPOSITION 5.15.2. *Any local diffeomorphism fixing* 0 *and preserving a given quasi-homogeneity structure* $\mathcal{F}$ *on a vector space* V *lifts to a diffeomorphism near the boundary of* $[V;\{0\},\mathcal{F}]$.

PROOF. Lemma 5.15.1 shows that the linearization at 0 of the diffeomorphism lifts to a diffeomorphism of the quasi-homogeneous blow up. Thus it suffices to consider the case of a diffeomorphism with Jacobian at 0 the identity. In Taylor series such a diffeomorphism is given by the integration of a vector field which necessarily has non-negative homogeneity. Since Proposition 5.15.1 shows that the vector field lifts to be smooth (and tangent to the boundary) there is a diffeomorphism equal to the given one to arbitrarily high order in Taylor series at 0 which also lift to be smooth. Thus it can be assumed that the diffeomorphism is equal to the identity to very high order in Taylor series. If the order is high enough this not only implies the (assumed) non-negativity of the homogeneity order but means that the diffeomorphism is homotopic, through such diffeomorphisms, to the identity. Now the proof of Lemma 5.1.2 applies. The parameter dependent vector fields which integrates to the given family of diffeomorphisms must be of positive homogeneity order and hence lifts and allows the family of diffeomorphisms to be lifted. □

This coordinate invariance, i.e. naturality, of the quasi-homogeneous blow up can now be extended to the general case of a closed p-submanifold with given quasi-homogeneity structure as in the radial case above. First consider the case of a vector bundle, V, with quasi-homogeneity structure, $\mathcal{F}$, at the zero section. The quasi-homogeneous blow up is defined as the blow up of 0 in each fibre. The naturality of the $\mathcal{C}^\infty$ structure follows as in the radial case. Now to treat the general case of the quasi-homogeneous blow up of a closed p-submanifold it is enough to use the existence of a quasi-homogeneous normal fibration.