CHAPTER 4

Conormal distributions at the boundary

As will become apparent, conormal distributions play a very basic rôle in the discussion of the inverses of elliptic operators. This is (assumed to be) familiar in the case of elliptic operators on a compact manifold without boundary and is even more true of the degenerate elliptic operators of interest here. Conormal distributions have, in a very real sense, the simplest types of singularities amongst all distributions with singularities located on a given submanifold. A very attractive account can be found in [**3**]. We concentrate on the (mild) novelty that the distributions have singularities only at the boundary and are supported, or defined, on only one side of the singular set. In Chapter 6 we shall consider more general conormal distributions associated to p-submanifolds.

4.1. The spaces $\mathcal{A}(X)$ and $\dot{\mathcal{A}}(X)$

As usual let X be a compact manifold with corners. We define the spaces of extendible, and of supported, conormal distributions of Sobolev order s by iterative regularity:

$$\begin{aligned}\mathcal{A}^{[s]}(X) &= \{u \in \mathcal{C}^{-\infty}(X); Pu \in \mathcal{H}^s(X)\ \forall\ P \in \mathrm{Diff}_b(X)\}\\ \dot{\mathcal{A}}^{[s]}(X) &= \{u \in \dot{\mathcal{C}}^{-\infty}(X); Pu \in \dot{\mathcal{H}}^s(X)\ \forall\ P \in \mathrm{Diff}_b(X)\}.\end{aligned} \tag{4.1.1}$$

The full spaces of conormal distributions at the boundary of X are then

$$\mathcal{A}(X) = \bigcup_s \mathcal{A}^{[s]}(X),\ \dot{\mathcal{A}}(X) = \bigcup_s \dot{\mathcal{A}}^{[s]}(X). \tag{4.1.2}$$

Both $\mathcal{H}^s(X)$ and $\dot{\mathcal{H}}^s(X)$ are Hilbertable spaces (i.e. the topology comes from a Hilbert space structure but no one inner product is distinguished.) The space $\mathrm{Diff}_b(X)$ has a countable spanning set, $\{P_i\}_{i=1}^{\infty}$ over $C^\infty(X)$. Thus $\mathcal{A}^{[s]}(X)$ and $\dot{\mathcal{A}}^{[s]}(X)$ have projective limit topologies which are metrizable:

$$d(u, v) = \sum_n 2^{-n} \frac{||P_n(u-v)||_s}{1 + ||P_n(u-v)||_s} \tag{4.1.3}$$

where $||\ ||_s$ denotes a norm on $\mathcal{H}^s(X)$ or $\dot{\mathcal{H}}^s(X)$ as appropriate. The full spaces, $\mathcal{A}(X)$ and $\dot{\mathcal{A}}(X)$, are then given the inductive limit topologies.

Applying the iterative regularity results (3.5.6) in the interior of X it follows that

$$\begin{aligned}u \in \mathcal{A}^{[s]}(X) &\Longrightarrow u \in \mathcal{C}^\infty(X^\circ)\\ u \in \dot{\mathcal{A}}^{[s]}(X) &\Longrightarrow u \restriction X^\circ \in \mathcal{C}^\infty(X^\circ).\end{aligned} \tag{4.1.4}$$

Thus all the singularities occur at the boundary. Of course the iterative regularity means more than this alone.

PROPOSITION 4.1.1. *With the vertical arrows surjective restriction maps, and all others inclusions, there is a commutative diagramme:*

$$\begin{tikzcd} & & \dot{\mathcal{A}}(X) \arrow[r] \arrow[dd] & \dot{\mathcal{C}}^{-\infty}(X) \arrow[dd] \\ \dot{\mathcal{C}}^{\infty}(X) \arrow[r] & \mathcal{C}^{\infty}(X) \arrow[ur] \arrow[dr] & & \\ & & \mathcal{A}(X) \arrow[r] & \mathcal{C}^{-\infty}(X). \end{tikzcd} \tag{4.1.5}$$

PROOF. We have already noted that the action of $\mathrm{Diff}_b(X)$ on $\mathcal{C}^\infty(X)$ is consistent with its inclusion in both $\dot{\mathcal{C}}^{-\infty}(X)$ and $\mathcal{C}^{-\infty}(X)$. Since

$$\mathcal{C}^\infty(X) \subset \mathcal{H}^s(X)\ \forall\ s,\ \mathcal{C}^\infty(X) \subset \dot{\mathcal{H}}^0(X) \tag{4.1.6}$$

we immediately conclude that $\mathcal{C}^\infty(X) \hookrightarrow \mathcal{A}(X)$, $\mathcal{C}^\infty(X) \hookrightarrow \dot{\mathcal{A}}^{[0]}(X)$.

Since $\dot{\mathcal{H}}^s(X) \longrightarrow \mathcal{H}^s(X)$ is a well-defined restriction map the first vertical arrow in (4.1.5) is also defined. The commutativity of the diagramme is then clear, so only the surjectivity of the map

$$\dot{\mathcal{A}}(X) \twoheadrightarrow \mathcal{A}(X) \tag{4.1.7}$$

remains to be established. This is proved below. □

LEMMA 4.1.1. *Suppose $F \in MX$ is a boundary face then there is an injection*

$$\dot{\mathcal{A}}(F;\Omega F) \longrightarrow \dot{\mathcal{A}}(X;\Omega X),\ v \longmapsto u,\ u(\phi) = v(\phi\restriction_F) \quad \forall\ \phi \in \mathcal{C}^\infty(X). \tag{4.1.8}$$

PROOF. The map (4.1.8) defines u as a Dirac distribution with support on F. Since the restriction of $\mathcal{C}^\infty$ functions from X to F is surjective we see that the map $v \longmapsto u$ is injective. It remains only to show that $u \in \dot{\mathcal{A}}(X;\Omega X)$. Let $v \in \dot{\mathcal{A}}(F;\Omega F)$, then $v \in \dot{\mathcal{A}}^{[s]}(F;\Omega F)$ for some s. Taking $s < -\dim F$ it follows that $u \in \dot{\mathcal{H}}^S(X;\Omega X)$, $S = s - \frac{1}{2}\operatorname{codim} F$, using (3.5.11). Now if $Q \in \mathrm{Diff}_b^m(X;\Omega X)$,

$$Qu(\phi) = u({}^tQ\phi) = v({}^tQ\phi\restriction_F) = v(Q_F\phi_F) = ({}^tQ_F)(\phi_F), \tag{4.1.9}$$

where $Q_F \in \mathrm{Diff}_b^m(F)$. Thus $u \in \dot{\mathcal{A}}^{[S]}(X;\Omega X)$. □

4.2. Characterization of $\mathcal{A}(X)$ by bounds

The space $\mathcal{A}(X)$ of extendible conormal distributions, which is to say conormal functions, at the boundary can also be considered as a space of (product-type) symbols if the boundary is placed at infinity. This just means that the L^2-based Sobolev spaces can be replaced by bounds (in L^∞) in the definition.

Let $L^\infty(X)$ be the space of bounded measurable functions on X. Thus $L^\infty(X) \hookrightarrow \mathcal{C}^{-\infty}(X)$. More generally if ρ is a total boundary defining function and $m \in \mathbb{R}$ we consider the space

$$\rho^m L^\infty(X) = \{u \in \mathcal{C}^{-\infty}(X);\ u = \rho^m v, v \in L^\infty(X)\}. \tag{4.2.1}$$

Notice that $\rho^m \cdot \phi \in \dot{\mathcal{C}}^\infty(X)$ if $\phi \in \dot{\mathcal{C}}^\infty(X)$ so multiplication by ρ^m is well-defined on $\mathcal{C}^{-\infty}(X)$.

PROPOSITION 4.2.1. *If X is a compact manifold with corners then*

$$\mathcal{A}(X) = \{u \in \mathcal{C}^{-\infty}(X);\ \text{for some } m = m(u),\ \mathcal{V}_b^k u \subset \rho^m L^\infty(X)\ \forall\ k \in \mathbb{N}\}. \tag{4.2.2}$$

PROOF. This is a form of the Sobolev embedding theorem. We first show that if $p \in \mathbb{N}$ and $p > \frac{n}{2} - s + 1$ then

$$\rho^p \mathcal{A}^{[s]}(X) \subset L^\infty(X). \tag{4.2.3}$$

Choosing $p > \frac{n}{2} - s + 1$ means there is an integer $k > \frac{n}{2}$ such that $p \geq k - s$, so it suffices to show that

$$\rho^p \mathcal{A}^{[s]}(X) \subset \mathcal{H}^k(X). \tag{4.2.4}$$

From the standard properties of Sobolev spaces we know that

$$u \in \mathcal{H}^s(X) \text{ with } Qu \in \mathcal{H}^s(X)\ \forall\ Q \in \operatorname{Diff}^p(X) \Longrightarrow u \in \mathcal{H}^k(X) \quad (p \geq k - s). \tag{4.2.5}$$

If $Q \in \operatorname{Diff}^p(X)$ then $Q\rho^p \in \operatorname{Diff}_b^p(X)$ so the hypothesis in (4.2.5) holds for $\rho^p u$ if $u \in \mathcal{A}^{[s]}(X)$:

$$u \in \mathcal{A}^{[s]}(X) \Longrightarrow Q(\rho^p u) \in \mathcal{H}^s(X)\ \forall\ Q \in \operatorname{Diff}^p(X). \tag{4.2.6}$$

This proves (4.2.4) and hence (4.2.3).

Now consider the commutation result

$$[V, \rho] = V\rho \in \rho\mathcal{C}^\infty(X) \text{ if } V \in \mathcal{V}_b. \tag{4.2.7}$$

Applying this repeatedly we conclude that if $Q \in \operatorname{Diff}_b^m(X)$ then

$$Q\rho^p u = \rho^p Q' u, Q' \in \operatorname{Diff}_b^m(X). \tag{4.2.8}$$

Thus

$$\operatorname{Diff}_b^m(X) \cdot \rho^p \mathcal{A}^{[s]}(X) \subset \rho^p \operatorname{Diff}_b^m(X) \cdot \mathcal{A}^{[s]}(X) \subset \rho^p \mathcal{A}^{[s]}(X) \subset L^\infty(X). \tag{4.2.9}$$

where (4.2.3) and the definition of $\mathcal{A}^{[s]}(X)$ have been used. From this we conclude that

$$\mathcal{A}^{[s]}(X) \subset \{u \in \mathcal{C}^{-\infty}(X);\ \text{for some } p\ (> \frac{n}{2} - s + 1)\ \mathcal{V}_b^k u \subset \rho^{-p} L^\infty(X)\ \forall\ k\}. \tag{4.2.10}$$

This shows that the left side of (4.2.2) is included in the right side. The reverse inclusion is similar, but simpler, using

$$\rho^m L^\infty(X) \subset \mathcal{H}^{-|m|N}(X) \quad N = \max \operatorname{codim} \partial X. \tag{4.2.11}$$

□

EXERCISE 4.2.23. Recall that the spaces of symbols on $\mathbb{R}^n$ are defined by

$$S^m(\mathbb{R}^n) = \{a \in \mathcal{C}^\infty(\mathbb{R}^n); |D_\xi^\alpha a| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}\ \forall \alpha\}. \tag{4.2.12}$$

Let $\mathbb{R}^n \xrightarrow{SP} \mathbb{S}_+^n$ be the stereographic compactification of $\mathbb{R}^n$ as a half sphere (i.e. a ball). Show that under this identification

$$S^\infty(\mathbb{R}^n) = \bigcup_m S^m(\mathbb{R}^n) \longleftrightarrow \mathcal{A}(\mathbb{S}_+^n). \tag{4.2.13}$$

In view of this alternate description of $\mathcal{A}(X)$ in terms of bounds set

$$\mathcal{A}^m(X) = \{u \in \mathcal{A}(X); \mathrm{Diff}_b(X)u \subset \rho^m L^\infty(X)\}. \tag{4.2.14}$$

Then we find:

$$\mathcal{A}(X) = \bigcup_m \mathcal{A}^m(X), \ \bigcap_m \mathcal{A}^m(X) = \dot{\mathcal{C}}^\infty(X). \tag{4.2.15}$$

This means that $\{\mathcal{A}^m(X)\}_{m\in\mathbb{R}}$ in *not* equivalent to $\{\mathcal{A}^{[s]}(X)\}_{s\in\mathbb{R}}$ as a filtration of $\mathcal{A}(X)$ since

$$\mathcal{A}(X) = \bigcup_s \mathcal{A}^{[s]}(X) \text{ but } \bigcap_s \mathcal{A}^{[s]}(X) = \mathcal{C}^\infty(X). \tag{4.2.16}$$

Exercise 4.2.24. Show that under the stereographic compactification in Exercise 4.2.23 $S^m(\mathbb{R}^n)$ is mapped onto $\mathcal{A}^{-m}(\mathbb{S}^n_+)$.

We shall also use a more precise 'multi-filtration' of $\mathcal{A}(X)$. If X is a manifold with corners let $\mathfrak{a}$ be a 'power set' that is a map

$$\mathfrak{a} : M_1X \longrightarrow \mathbb{R} \times \mathbb{N}_0 \tag{4.2.17}$$

which associates to each boundary hypersurface a real number and a nonnegative integer. Since we generally write $M_1X = \{H_1, \dots, H_N\}$ where each hypersurface $H_i = \{\rho_i = 0\}$ has a defining function, we can think of $\mathfrak{a}$ as a pair of N-vectors, $a \in \mathbb{R}^N$, $k \in \mathbb{N}_0^N$. Then we set

$$\begin{aligned} \mathcal{A}^{\mathfrak{a}}(X) &= \mathcal{A}^{(a_1,k_1),\dots,(a_N,k_N)}(X) \\ &= \left\{ u \in \mathcal{A}(X); \mathrm{Diff}_b(X)u \in \prod_{j=1}^N [\rho_j^{a_j}(|\log \rho_j| + 1)^{k_j}] L^\infty(X) \right\}. \end{aligned} \tag{4.2.18}$$

Thus $\mathcal{A}^{(a,k)}(X)$ corresponds to different weights at the different boundary hypersurfaces. If $k = (0, \dots, 0)$ we set $\mathcal{A}^{(a,0)}(X) = \mathcal{A}^a(X)$ and in the even more special case that $k = 0$ and $a = (m, m, \dots, m)$ we keep $\mathcal{A}^m(X)$ as an alternate notation. The obvious examples of such conormal distributions are the powers with logarithmic factors

$$\prod_{j=1}^N \rho_j^{a_j}(\log \rho_j)^{k_j} \in \mathcal{A}^{(a,k)}(X). \tag{4.2.19}$$

These conormal spaces have a useful interpolation property.

Lemma 4.2.1. *For any compact manifold with corners and any $u \in \mathcal{A}(X)$ the set A of power sets $\mathfrak{a}$ such that $u \in \mathcal{A}^{\mathfrak{a}}(X)$ has the properties*

$$\begin{gathered} \mathfrak{a} = (a,k) \in A, \ \mathfrak{b} = (b,l) \text{ with } b_j \le a_j \text{ and } l_j \ge k_j \text{ if } b_j = a_j \Longrightarrow \mathfrak{b} \in A \\ \mathfrak{a} = (a,k) \in A, \mathfrak{b} = (b,l) \in A, \mathfrak{c} = (c,p) \\ 0 \le s \le 1, \ c = sa + (1-s)b, p_j \ge sk_j + (1-s)l_j \Longrightarrow \mathfrak{c} \in A. \end{gathered} \tag{4.2.20}$$

Proof. The weighted L^∞ spaces in (4.2.18) have these properties and applying this to the iterated derivatives proves the result. □

We also define the following operations on power sets:

$$
\begin{aligned}
&\text{For } \mathfrak{a} = (a,k), \mathfrak{b} = (b,l), \\
&\mathfrak{a} + \mathfrak{b} = (c,p) \text{ if } c_j = a_j + b_j, p_j = k_j + l_j \\
&\mathfrak{a} \vee \mathfrak{b} = (c,p) \text{ if } c_j = \min(a_j,b_j), p_j = \max(k_j,l_j) \\
&\mathfrak{a} \le \mathfrak{b} \Longleftrightarrow a_j \ge b_j \text{ and } k_j \le l_j \text{ if } a_j = b_j.
\end{aligned}
\tag{4.2.21}
$$

Then

$$
\begin{aligned}
&\mathcal{A}^{\mathfrak{a}}(X) \subset \mathcal{A}^{\mathfrak{b}}(X) \Longleftrightarrow \mathfrak{a} \le \mathfrak{b}, \\
&\mathcal{A}^{\mathfrak{a}}(X) \cdot \mathcal{A}^{\mathfrak{b}}(X) \subset \mathcal{A}^{\mathfrak{a}+\mathfrak{b}}(X) \\
&\mathcal{A}^{\mathfrak{a}}(X) + \mathcal{A}^{\mathfrak{b}}(X) \subset \mathcal{A}^{\mathfrak{a}\vee\mathfrak{b}}(X).
\end{aligned}
\tag{4.2.22}
$$

In §4.13 we consider more particular, polyhomogeneous, conormal distributions which have expansions in terms of powers.

4.3. Symbol map

A fundamental tool in the study of $\mathcal{C}^\infty(X)$ is the Taylor series at each boundary hypersurface. The replacement for this in the conormal setting is the symbol mapping.

Suppose $\mathfrak{a}$ is a power set for a compact manifold with corners X. If $H \in M_1(X)$ let $-1(H) : M_1(X) \longrightarrow \mathbb{R} \times \mathbb{N}_0$ be the power set taking the value $(-1,0)$ at H and $(0,0)$ otherwise. The crudest form of the symbol mapping is just the projection to the quotient

$$
\mathcal{A}^{\mathfrak{a}}(X)\overset{\sigma}{'}_H \to\longrightarrow \mathcal{A}^{\{\mathfrak{a}\}_H}(X) = \mathcal{A}^{\mathfrak{a}}(X)/\mathcal{A}^{\mathfrak{a}-1(H)}(X).
\tag{4.3.1}
$$

One thing that makes this map rather useful is that it has very good invariance properties. Thus suppose that $f : X \longrightarrow X$ is a diffeomorphism which fixes $H \in M_1(X)$ to first order in the sense that $f(x) = x$ for all $x \in H$ and f^* is the identity on $N^*H \subset T^*_H X$. Then for each power set

$$
\mathcal{A}^{\mathfrak{a}}(X) \ni u \longmapsto f^*u - u \in \mathcal{A}^{\mathfrak{a}-1(H)}(X).
\tag{4.3.2}
$$

To see this it suffices to assume that u has support in a small neighbourhood of H and that $f = f_1$ is connected to the identity by a 1-parameter family fixing H to first order. This just corresponds to integration by a vector field V_t

$$
\frac{d}{dt} f_t^* \phi = f_t^* V_t \phi,\ t \in [0,1]
\tag{4.3.3}
$$

with $V_t \in \mathcal{V}_b(X)$. Since f_t fixes H to first order we must have $V_t = \rho_H W_t$ with $W_t \in \mathcal{V}_b(X)$. It follows that the difference (4.3.2) can be written as an integral

$$
f_1^* u - u = \int_0^1 f_t^*[\rho_H W_t u] dt \in \mathcal{A}^{\mathfrak{a}-1(H)}(X).
$$

Thus (4.3.2) holds.

To reinforce the invariance of the quotient it is convenient to note that there is a natural identification

$$
\mathcal{A}^{\{\mathfrak{a}\}_H}(X) \equiv \mathcal{A}^{\{\mathfrak{a}\}_H}(N^+H).
\tag{4.3.4}
$$

Here H, as the zero section, is considered as a boundary hypersurface of $N^+H \equiv H \times [0, \infty)$. The isomorphism (4.3.4) is given by any normal fibration of X around H, see §1.7. Since σ'_H in (4.3.1) is just projection we get the short exact sequence

$$0 \hookrightarrow \mathcal{A}^{\mathfrak{a}-1(H)}(X) \longrightarrow \mathcal{A}^{\mathfrak{a}}(X) \xrightarrow{\sigma_H} \mathcal{A}^{\{\mathfrak{a}\}_H}(N^+H) \longrightarrow 0. \tag{4.3.5}$$

This is the symbol sequence at H.

Suppose that $P \in \mathrm{Diff}^m_b(X)$ is a b-differential operator. We have shown in §2.11 that the normal operator of P at $H \in M_1(X)$, also called the indicial operator, can be identified with

$$I_H(P) \in \mathrm{Diff}^m_{b,H}(N^+H), \tag{4.3.6}$$

an $\mathbb{R}^+$-invariant operator on the inward pointing normal bundle to H.

Proposition 4.3.1. *If $P \in \mathrm{Diff}^m_b(X)$ then for any power set $\mathfrak{a}$ and any $H \in M_1(X)$ P acts on $\mathcal{A}^{\mathfrak{a}}(X)$ and*

$$\sigma_H(Pu) = I_H(P)\sigma_H(u) \quad \forall\ u \in \mathcal{A}^{\mathfrak{a}}(X). \tag{4.3.7}$$

Proof. It is only necessary to check this in the simple case where P an element of $\mathcal{V}_b(X)$ and then iterate. □

We also remark that the symbol behaves simply for products

$$\sigma_H(u \cdot v) = \sigma_H(u) \cdot \sigma_H(v). \tag{4.3.8}$$

4.4. Proof of (4.1.7)

Clearly (4.1.7) is local, so we can replace X by $\mathbb{R}^n_k$ and assume that $u \in \mathcal{A}(\mathbb{R}^n_k)$ has support in $\{|x| < 1\}$. By (4.2.2) $u \in \mathcal{A}^m(\mathbb{R}^n_k)$ for some m. Observe that integration in the first k variables increases the order, if it is negative:

$$\begin{gathered} Ju(x) = \phi(x) \int_1^{x_1}\int_1^{x_2}\cdots\int_1^{x_k} u(y_1, y_2, \dots, y_k, x'')dy \in \mathcal{A}^{m+\delta} \\ \text{if } m < 0,\ m - \delta \le 0,\ 0 \le \delta \le 1. \end{gathered} \tag{4.4.1}$$

Here $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n_k)$. By integrating at least $N > |m|$ times we ensure that $v = J^N u \in \mathcal{A}^0(\mathbb{R}^n_k)$ is bounded. Integrating twice more in each of these variables, now from the origin,

$$\begin{gathered} w(x) = (J')^2 v \in \mathcal{A}^{1+\epsilon}(\mathbb{R}^n_k), \\ J'v(x) = \phi(x) \int_0^{x_1}\cdots\int_0^{x_k} v(y_1, \dots, y_k, x'')dy \end{gathered} \tag{4.4.2}$$

gives a function which is continuously differentiable. Thus

$$Pu \in \dot{\mathcal{A}}^{[-k]}(\mathbb{R}^n_k) \text{ if } P \in \mathrm{Diff}^k(\mathbb{R}^n_k). \tag{4.4.3}$$

In particular it follows that $u' = [\partial_1\partial_2 \dots \partial_k]^{N+2} w \in \dot{\mathcal{A}}^{[-k(N+1)]}(\mathbb{R}^n_k)$. Noting that u' restricts to the interior to give u plus some error in $\mathcal{C}^\infty_c(\mathbb{R}^n_k)$ this proves the surjectivity in (4.1.7).

4.5. Asymptotic summation

Suppose $\{u_n\}$ is a sequence of conormal distributions. We want to consider the case that the u_n are becoming increasingly smooth. From the discussion above there are three cases

$$\begin{aligned} u_n &\in \mathcal{A}^{[s_n]}(X) \quad s_n \longrightarrow \infty \\ u_n &\in \mathcal{A}^{s_n}(X) \quad s_n \longrightarrow \infty \\ u_n &\in \dot{\mathcal{A}}^{[s_n]}(X) \quad s_n \longrightarrow \infty \end{aligned} \tag{4.5.1}$$

In all cases we wish to show that the sum of such a series is well-defined modulo the appropriate *residual space*

$$\mathcal{A}^{[\infty]}(X) = \mathcal{C}^\infty(X),\ \mathcal{A}^\infty(X) = \dot{\mathcal{A}}^{[\infty]}(X) = \dot{\mathcal{C}}^\infty(X). \tag{4.5.2}$$

Proposition 4.5.1. *If $\{u_n\}$ is a sequence as in (4.5.1) then the series $\sum\limits_n u_n$ has an asymptotic sum, $u \sim \sum\limits_n u_n$ in the sense that*

$$u - \sum_{k=1}^N u_k \in \mathcal{A}^{[R_N]} \quad (\text{resp } \mathcal{A}^{R_N}(X), \dot{\mathcal{A}}^{R_N}(X)) \tag{4.5.3}$$

where $R_N = \inf\limits_{k>N} s_k$.

Proof. The second two cases are actually the same, after a finite number of terms, but we simply give a proof uniform for all cases. Let ρ be a global boundary defining function. Choose some cutoff function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\phi(r) = 1$ if $r < \frac{1}{2}$, $\phi(r) = 0$ if $r \geq 1$. Then for some sequence $\infty \geq \epsilon_j \geq 0$, $\epsilon_j < \infty$ for $j > J$, with $\epsilon_j \longrightarrow 0$ as $j \longrightarrow \infty$. Consider

$$u = \sum_{k=1}^\infty \phi(\rho/\epsilon_k) u_k. \tag{4.5.4}$$

Here if $\epsilon_k = \infty$ then we interpret $\phi(\rho/\infty)$ as 1. Certainly u in (4.5.4) is well-defined. We wish to show that if the ϵ_j decrease sufficiently rapidly to 0 then u satisfies (4.5.3). This amounts to a countable set of conditions, requiring that $P_j(u - \sum\limits_{k=1}^N u_k)$ belong to the appropriate space for a set $\{P_j\}$ spanning $\mathrm{Diff}_b(X)$ and for each N. Each of these constraints is easily seen (cf. [Hö]) to take the following form:

$$\begin{aligned} &\text{for each } j, N\ \exists \text{ a sequence } \epsilon_{j,N,p} \longrightarrow 0 \text{ as } p \longrightarrow \infty \\ &\text{and } J = J(j,k) \text{ for which } \epsilon_p < \epsilon_{j,N,p}\ \forall\ p > J(j,k). \end{aligned} \tag{4.5.5}$$

By diagonalization all these constraints can be arranged to hold simultaneously. This proves the proposition. ☐

The proof of the following result is omitted, since it is completely analogous to that of Proposition 4.5.1.

Proposition 4.5.2. *Suppose $\mathfrak{a}_n$ is a sequence of power families for the boundary hypersurfaces of a compact manifold with corners X with*

$$\mathfrak{a}_n = ((s_n, l_n), (a_2, k_2), \dots, (a_N, k_N)),\ \text{and } s_n \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

then if $u_n \in \mathcal{A}^{\mathfrak{a}_n}(X)$ *there exists* $u \in \mathcal{A}^{\mathfrak{a}}(X)$, $\mathfrak{a} = (a,k)$, $a_1 = \max(s_n)$, $k_1 = \max\{l_n; s_n = a_1\}$ *such that*

$$\begin{gathered} u - \sum_{j=1}^{p} u_j \in \mathcal{A}^{\mathfrak{b}_p}(X) \quad \forall\ p, \\ \textit{if } \mathfrak{b}_p = ((b_p,0),(alpha_2,k_2),\dots,(a_N,k_N)),\ b_p < \sup_{j\geq p} s_j. \end{gathered} \tag{4.5.6}$$

The condition (4.5.6) *fixes u modulo*

$$\mathcal{A}^{\mathfrak{a}'}(X),\ \mathfrak{a}' = (\infty,(a_2,k_2),\dots,(a_N,k_N)). \tag{4.5.7}$$

4.6. Dual spaces

As a basic principal of Functional Analysis one should always examine dual spaces for signs of interest. To make sure that this does not introduce completely new objects we need:

LEMMA 4.6.1. $\dot{\mathcal{C}}^\infty(X) \hookrightarrow \dot{\mathcal{A}}(X)$ *and* $\mathcal{C}^\infty(X) \hookrightarrow \mathcal{A}(X)$ *are continuous dense inclusions.*

PROOF. We consider $\dot{\mathcal{A}}(X)$. Choose a total defining function $\rho \in \mathcal{C}^\infty(X)$ and a function $\mu \in \mathcal{C}^\infty_c(\mathbb{R})$. We shall show that

$$u \in \dot{\mathcal{A}}(X) \Longrightarrow \mu(n\rho)u \longrightarrow 0 \text{ in } \dot{\mathcal{A}}(X). \tag{4.6.1}$$

Choosing μ to satisfy $\mu(x) = 1$ for $|x| \leq 1$ this proves the desired density since then $(1-\mu(n\rho))u \in \mathcal{C}^\infty_c(X^\circ)$. If $u \in \dot{\mathcal{A}}^{[s]} \subset \dot{\mathcal{H}}^s(X)$ then $\mu(n\rho)u \longrightarrow 0$ in $\dot{\mathcal{H}}^S(X)$ where S depends only on s and not on u or μ. Now suppose that $V \in \mathcal{V}_b(X)$. Since $V\rho = a\rho$ with $a \in \mathcal{C}^\infty(X)$ we find

$$V[\mu(n\rho)u] = a\mu'(n\rho)u + \mu(n\rho)Vu. \tag{4.6.2}$$

Using the conormality of u and the same argument as before this sequence converges to 0 in $\dot{\mathcal{H}}^S(X)$. Clearly iteration of this argument shows that $P[\mu(n\rho)u] \longrightarrow 0$ in $\dot{\mathcal{H}}^S(X)$ for each $P \in \mathrm{Diff}_b(X)$. This proves the lemma for $\dot{\mathcal{A}}(X)$. The other case is similar. □

As a direct consequence of this lemma we can embed the dual space:

$$\mathcal{B}(X) = [\dot{\mathcal{A}}(X;\Omega X)]' \hookrightarrow \mathcal{C}^{-\infty}(X). \tag{4.6.3}$$

It then consists of those distributions for which the pairing $u(\phi)$ extends continuously from $\phi \in \dot{\mathcal{C}}^\infty(X;\Omega X)$ to $\phi \in \dot{\mathcal{A}}(X;\Omega X)$. In fact we have inclusions in both spaces of distributions

$$\begin{tikzcd} \mathcal{B}(X) \arrow[r, equal] \arrow[d, hook] & {[\dot{\mathcal{A}}(X,\Omega X)]'} \arrow[d, hook] \\ \dot{\mathcal{C}}^{-\infty}(X) \arrow[r] & \mathcal{C}^{-\infty}(X) \end{tikzcd} \tag{4.6.4}$$

with the diagramme commuting. The first inclusion here follows from the injection

$$\mathcal{C}^\infty(X;\Omega X) \hookrightarrow \dot{\mathcal{A}}^{[0]}(X;\Omega X).$$

This double inclusion (consistent with the restriction map from $\dot{\mathcal{C}}^{-\infty}(X)$ to $\mathcal{C}^{-\infty}(X)$) means that $\mathcal{B}(X)$ has some of the properties of $\mathcal{C}^\infty(X)$. Intuitively, $\mathcal{B}(X)$ consists of distributions which are 'smooth up to the boundary.' This statement is justified, at least in part, by the following two results, both of which follow from simple duality arguments.

PROPOSITION 4.6.1. *If* $P\colon \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X)$ *is an element* $P \in \operatorname{Diff}^m(X)$ *acting on extendible distributions then by restriction*

$$P\colon \mathcal{B}(X) \longrightarrow \mathcal{B}(X). \tag{4.6.5}$$

PROOF. Certainly ${}^tP$ acts on $\dot{\mathcal{A}}(X;\Omega X)$. □

PROPOSITION 4.6.2. *For any compact manifold with corners,* X, $\mathcal{C}^\infty(X)$ *is dense in* $\mathcal{B}(X)$ *and if* $F \in MX$ *is a boundary face then the restriction map to* F, *extends from* $\mathcal{C}^\infty(X)$ *to give*

$$\restriction_F\colon \mathcal{B}(X) \twoheadrightarrow \mathcal{B}(F). \tag{4.6.6}$$

PROOF. This restriction map is dual to the map (4.1.8). We leave the proof of surjectivity as an exercise. □

A standard method of proving the regularity of a distribution (for example the solution of some differential equation) is to prove *separately* the tangential regularity and the normal regularity. The following basic regularity result is a rather extreme form of this method.

THEOREM 4.6.1. *('ABC') If* X *is a compact manifold with corners then*

$$\mathcal{A}(X) \cap \mathcal{B}(X) = \mathcal{C}^\infty(X).$$

We shall prove this locally by using properties of the Mellin transform on $\mathbb{R}^n_k$, see §4.10.

4.7. Mellin transform

The Mellin transform is a close relative of the Fourier transform but related to $\mathbb{R}_+$-invariance rather than translation-invariance. Thus we consider first the basic homogeneous distributions on the half-line.

LEMMA 4.7.1. *The homogeneous distribution*

$$x^{iz} = \exp(iz\log x) \in \mathcal{C}^{-\infty}([0,\infty)) \tag{4.7.1}$$

is entire in $z \in \mathbb{C}$ *and its supported counterpart*

$$\gamma_z(x) = \frac{x^{iz-1}}{\Gamma(iz)} \in \dot{\mathcal{C}}^{-\infty}([0,\infty)) \tag{4.7.2}$$

extends from $\Im z << 0$ *to be entire.*

PROOF. We can use the functional equation

$$D_x\gamma_z(x) = \gamma_{z-1}(x) \tag{4.7.3}$$

to extend γ_z, once defined and holomorphic in a strip $\Im z < -k$ to the strip $\Im z < -k+1$. Thus γ_z is entire. Clearly then

$$x^{iz-1} = \Gamma(iz)\gamma_z(x)|_{(0,\infty)}. \tag{4.7.4}$$

From (4.7.3) it also follows that the values $\gamma_z(x)$ for $z \in i\mathbb{N}$ are multiples of successive derivatives of $\delta(x)$. Since $\Gamma(iz)$ has simple poles only at these points it follows that $x^{iz} \in \mathcal{C}^{-\infty}([0,\infty))$ is entire. □

On $\mathbb{R}^n_k$ we can consider products of such distributions:

$$x^{iz} = x_1^{iz_1} \dots x_k^{iz_k} \;\; z \in \mathbb{C}^k. \tag{4.7.5}$$

This is entire with values in $\mathcal{C}^{-\infty}(\mathbb{R}^k_n)$ but more importantly

$$x^{iz} \in \dot{\mathcal{C}}^p(\mathbb{R}^n_k) \text{ if } \Im z_i < -p \;\forall\; i = 1, \dots, k. \tag{4.7.6}$$

Here the function is simply independent of the last $n - k$ coordinates. From (4.7.6) it follows that the pairing

$$\langle x^{iz - \mathbb{1}} \phi(x''), u(x', x'') \rangle \tag{4.7.7}$$

is well defined if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^{n-k})$, $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n_k)$, when $\Im z_i < -K = -K(u)$. Here $\mathbb{1} = (1, \dots, 1) \in \mathbb{C}^k$. Thus we find:

Proposition 4.7.1. *The pairing* (4.7.7) *defines the Mellin transform which can be written formally*

$$u_M(z, x'') = \int (x')^{iz} u(x', x'') \frac{dx'}{x'}, \tag{4.7.8}$$

where $\frac{dx'}{x'}$ *is the measure* $|\frac{dx_1}{x_1} \dots \frac{dx_k}{x_k}|$. *For each* $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n_k)$ u_M *is defined as a map*

$$\{z \in \mathbb{C}^k; \Im z_i < -K, \; i = 1, \dots, k\} \longrightarrow \mathcal{C}^{-\infty}_c(\mathbb{R}^{n-k}) \tag{4.7.9}$$

which is holomorphic in a product of half-spaces (depending on u.)

The change of variable $t_i = -\log x_i$ formally reduces (4.7.8) to the Fourier transform:

$$u_M(z, x'') = \int e^{-iz\cdot t} u(e^{-t}, x'') dt. \tag{4.7.10}$$

Characterization of various spaces in terms of their Mellin transforms can therefore be reduced to standard (Paley-Wiener) theorems.

4.8. Paley-Wiener theorems

The basic Paley-Wiener theorem is a characterization of the Mellin transform of extendible distributions of compact support.

Theorem 4.8.1. *The Mellin transform,* (4.7.8)*, gives an isomorphism from the space* $\mathcal{C}^{-\infty}_c(\mathbb{R}^n_k)$ *onto the space of holomorphic maps* (4.7.9)*, for all k, which are uniformly bounded in the following sense:*

$$\exists \; R = R(u) \text{ s.t. } \operatorname{supp}(u_M(z, \cdot)) \subset \{x''; |x''| \le R\} \;\forall\; z. \tag{4.8.1}$$

and

$$\begin{cases} \exists \text{ an integer } p \text{ and constants } C, A \text{ and } N \text{ s.t.} \\ \qquad \|u_M(z, \cdot)\|_{(-p)} \le C \exp(A|\Im z|)(1 + |z|)^N \end{cases} \tag{4.8.2}$$

where $\| \;\|_{(-p)}$ *is the dual norm to the Sobolev norm on* $\mathcal{H}^p(\mathbb{R}^{n-k})$.

The proof of this result is by reduction to a standard Paley-Wiener theorem using (4.7.10). The constant K in the domain of u_M is directly related to the (Sobolev) regularity of u. Using this we can obtain a corresponding characterization of conormal distributions.

PROPOSITION 4.8.1. *The Mellin transform gives an isomorphism from*

$$\mathcal{A}_c(\mathbb{R}^n_k) = \{u \in \mathcal{A}(\mathbb{R}^n_k); u = 0 \text{ in } |x| > R, \text{ some } R\} \tag{4.8.3}$$

onto the space of holomorphic maps u_M *as in* (4.7.9) *which satisfy* (4.8.1)*, have values in* $\mathcal{C}^\infty(\mathbb{R}^{n-k})$ *and satisfy the following strengthened form of* (4.8.2)*:*

$$|D^\alpha_{x''} u_M(z, x'')| \le C_{N,\alpha} \exp(A|\Im z|)(1 + |z|)^{-N} \ \forall \ N, \alpha. \tag{4.8.4}$$

PROOF. It is only necessary to apply the estimates (4.8.2), using the uniformity in the dependence of the constants, to $(x' \cdot D_{x'})^\nu D^\alpha_{x''} u \in \mathcal{H}^s(X^n_k)$, s fixed, for all μ, α. Here $x' \cdot D_{x'} = (x_1 D_1, \dots, x_k D_k)$. □

Notice that (4.8.4) shows that the Mellin transform of a conormal distribution is rapidly decreasing at (real) infinity in the z variables. Next we wish to characterize the Mellin transform of the elements of $\mathcal{B}_c(\mathbb{R}^n_k) = \mathcal{B}(\mathbb{R}^n_k) \cap \mathcal{C}^{-\infty}_c(\mathbb{R}^n_k)$, where we only have polynomial bounds.

PROPOSITION 4.8.2. *The Mellin transform gives an isomorphism from* $\mathcal{B}_c(\mathbb{R}^n_k)$ *onto the space of meromorphic maps*

$$\mathbb{C}^k \backslash i\mathbb{N}^k \longrightarrow \mathcal{C}^{-\infty}_c(\mathbb{R}^{n-k}) \tag{4.8.5}$$

with uniform supports as in (4.8.1)*, satisfying* (4.8.2) *in some product of half spaces and such that*

$$u_\Gamma(z, \cdot) = \left[\prod_{l=1}^k \frac{1}{\Gamma(iz_l)}\right] u_M(z, \cdot) \text{ is entire with values in } \mathcal{C}^{-\infty}_c(\mathbb{R}^{n-k}) \tag{4.8.6}$$

and satisfies estimates in each strip $|\Im z| \le L$

$$|| \prod_{r=0}^{[L]} \prod_{j=1}^k (z_j - ir) u_M(z_j \cdot)||_{(-p)} \le C_L(1 + |z|)^p \tag{4.8.7}$$

for some $p = p(L)$.

PROOF. That u_Γ, given by (4.8.6), is entire follows from Lemma 4.7.1 since

$$\prod_{j=1}^k \frac{x_j^{iz_j - 1}}{\Gamma(z_j)} \phi = \prod_{j=1}^k \gamma_{z_j}(x_j)\phi \in \dot{\mathcal{A}}(\mathbb{R}^n_k) \tag{4.8.8}$$

is entire if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^{n-k})$. Thus

$$\langle u_\Gamma(z, \cdot), \phi\rangle = \langle u, \prod_{j=1}^k \gamma_{z_j}(x_j)\phi\rangle \tag{4.8.9}$$

must be entire if $u \in \mathcal{B} = (\dot{\mathcal{A}})'$. The estimates (4.8.4) follow from the continuity of u. The converse follows similarly, using the inversion formula for the Mellin (i.e. Fourier) transform. □

4.9. Dual space of $\mathcal{A}$

We shall denote the dual space to $\mathcal{A}$ by $\mathcal{Z}$:

$$\mathcal{Z}(X) = [\mathcal{A}(X;\Omega X)]'. \tag{4.9.1}$$

Since $\mathcal{C}^\infty(X;\Omega X) \hookrightarrow \mathcal{A}(X;\Omega X)$ is dense we certainly have an injection

$$\mathcal{Z}(X) \hookrightarrow \dot{\mathcal{C}}^{-\infty}(X). \tag{4.9.2}$$

In fact the restriction map $\dot{\mathcal{A}}(X;\Omega X) \longrightarrow \mathcal{A}(X;\Omega X)$ shows that $\mathcal{Z}(X) \hookrightarrow \mathcal{B}(X)$ includes in both spaces of distributions. It is straightforward to check that $\mathcal{Z}(X)$ is the subspace of $\mathcal{B}(X)$ consisting of the distribution with all boundary values vanishing (i.e. $Qu\restriction_F = 0$ for all $Q \in \mathrm{Diff}(X)$.)

Exercise 4.9.25. Generalize the Paley-Wiener Theorem, Proposition 4.8.2, to characterize the Mellin transforms of elements of $\mathcal{Z}_c(\mathbb{R}^n_k)$ as those entire functions

$$\mathbb{C}^k \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^{n-k}) \tag{4.9.3}$$

with uniformly compact support, satisfying (4.8.2) and in addition estimates

$$||u_M(z,\cdot)||_{(-p)} \le C_L(1+|z|)^p \tag{4.9.4}$$

in each strip $|\Im z| \le L$, with $p = p(L)$.

4.10. Proof of Theorem 4.6.1

This result is clearly local, so it suffices to show that

$$\mathcal{B}_c(\mathbb{R}^n_k) \cap \mathcal{A}_c(\mathbb{R}^n_k) = \mathcal{C}_c^\infty(\mathbb{R}^n_k). \tag{4.10.1}$$

If u is in the intersection of spaces on the left then both Proposition 4.8.1 and Lemma 4.8.2 apply. Thus the Mellin transform is meromorphic with poles only at $i\mathbb{N}^k$. The poles, in any product of lower half planes, can be removed by multiplication by a polynomial. For this holomorphic function the estimates (4.8.7) apply. Thus the function is of slow growth at real infinity, with values in some Sobolev spaces, but in $\Im z_i < -K$ it is rapidly decreasing at real infinity with values in $\mathcal{C}^\infty$.

Now we use a simple interpolation argument and (a trivial case of) Phragmen-Lindelöf bounds as in:

Lemma 4.10.1. *Suppose that $v(z,y)$ is a holomorphic function of z in some neighbourhood of the slab $0 \le \Im z_i \le 1$, $i = 1,\dots,k$, with values in $L^2(\mathbb{R}^{n-k})$ such that*

$$\sup_{0\le s_i\le 1} \int_{\mathbb{R}^n} |v(s+it,y)|^2 dt dy < \infty \tag{4.10.2}$$

$$v(s,y) \in \mathcal{S}(\mathbb{R}^n) \tag{4.10.3}$$

then for any differential operator, Q, in t, y with polynomial coefficients

$$Qv(s+it,y) \text{ is bounded on } [0,1-\epsilon]^k \times \mathbb{R}^{n-k} \quad \forall\, \epsilon > 0. \tag{4.10.4}$$

Proof. Take the Fourier transform of v in the variables t, with dual variables τ. Then

$$\hat{v}(s,\tau,y) = \int_{\mathbb{R}^k} e^{-it\cdot\tau} v(s+it,y)dt \tag{4.10.5}$$

satisfies the same conditions (4.10.2) and (4.10.3) and the differential equations

$$[\partial_{s_l} - \tau_l]\hat{v}(s,\tau,y) = 0. \tag{4.10.6}$$

These equations can be solved explicitly, giving

$$\hat{v}(s,\tau,y) = exp(s\cdot\tau)\hat{v}(0,\tau,y) \quad \forall\ 0 \le s_l \le 1. \tag{4.10.7}$$

By (4.10.3), $\hat{v}(0,\tau,y)$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and from (4.10.2)

$$e^{s\cdot\tau}\hat{v}(s,\tau,y) \in L^2(\mathbb{R}^n)$$

with a norm bounded independently of $s \in [0,1]^k$. If Q is any differential operator in τ, y with polynomial coefficients and $s_i \le \frac{1}{2}$

$$\begin{aligned} &\int_{\mathbb{R}^n} |Q\hat{v}(s,\tau,y)|^2 d\tau dy \\ &= \int_{\mathbb{R}^n} \overline{\hat{v}(2s,\tau,y)} G\hat{v}(0,\tau,y) d\tau dy \\ &\le \|\hat{v}(2s,\bullet)\|_{L^2}\|G\hat{v}(0,\bullet)\|_{L^2}. \end{aligned} \tag{4.10.8}$$

Here $G = \exp(-s\cdot\tau)Q^*Q\exp(s\cdot\tau)$ is a differential operator with polynomial coefficients in τ, y. Thus the right side is bounded and the inequality itself is easily justified by approximation. It follows that $Q\hat{v}(s,\bullet)$ is square-integrable for each $s \in [0,\frac{1}{2}]^k$, uniformly in s. These norms give the topology on $\mathcal{S}(\mathbb{R}^n)$ so the lemma is proved for $\epsilon \ge \frac{1}{2}$. However the same argument can be applied repeatedly by rescaling the interval $[\frac{1}{2},1]$ to $[0,1]$. This proves the lemma as stated. □

Returning to the consideration of u in the left side of (4.10.1) we see that for each $N \in \mathbb{N}$

$$w^{(N)}(z,y) = \left[\prod_{1\le l\le k}\prod_{1\le j\le N}(z_l - \sqrt{-1}j)\right] u_M(z,y) \tag{4.10.9}$$

is holomorphic in $\Im z_l < N$ with values in $\mathcal{H}^p(\mathbb{R}^{n-k})$ and satisfies estimates

$$\|w^{(N)}(z,\bullet)\}_p \le C(1+|z|)^p \text{ for } |\Im z_l| \le N. \tag{4.10.10}$$

For N large enough it also satisfies the sequence of estimates

$$\|w^{(N)}(z,\bullet)\|_q \le C_q(1+|z|)^{-q} \text{ when } \Im z_l = -N\ \forall\ i \quad \forall\ q \in \mathbb{N}. \tag{4.10.11}$$

Consider the function

$$v^{(N)}(z,y) = (z - iN - i)^{-P} L w^{(N)}(z,y) \tag{4.10.12}$$

where $P > p + k$ in (4.10.10) and L is an invertible elliptic operator of order $-P$. Clearly Theorem 4.10.1 applies to $v^{(N)}(Az,y)$ where A is a real linear transformation normalizing the slab $-N \le \Im z_l \le N$ to $[0,1]^k$. Since L is invertible it follows that $w^{(N)}$ is $\mathcal{C}^\infty$ and rapidly decreasing at real infinity in the slab $[-N,N]^k$.

From (4.10.9) we can therefore conclude that $u_M(z,y)$ is meromorphic with values in $\mathcal{C}_c^\infty(\mathbb{R}^n)$, at most simple poles at $i\mathbb{N}^k$ and rapid decrease at real infinity. Clearly this implies that $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, proving Exercise 4.6.1.

Exercise 4.10.26. Show that $\mathcal{C}^\infty(X) \cap \mathcal{Z}(X) = \dot{\mathcal{C}}^\infty(X)$.

4.11. Pull-back of $\mathcal{A}(Y)$

Let Y be a $\mathcal{C}^\infty$ manifold with corners. Now $\mathcal{A}(Y) \subset \mathcal{C}^\infty(Y^\circ)$ so for a $\mathcal{C}^\infty$ map $f : X \longrightarrow Y$

$$f(X^\circ) \subset Y^\circ \Longrightarrow f^*\mathcal{A}(Y) \subset \mathcal{C}^\infty(X^\circ). \tag{4.11.1}$$

Naturally we are interested in the boundary behaviour of these lifts of conormal functions. The hypothesis in (4.11.1) is just that $f^{-1}(\partial Y) \subset \partial X$ so is valid for a b-map unless $f^*\rho \equiv 0$ for some defining function for a boundary hypersurface. In fact

Proposition 4.11.1. *Under any b-map $f : X \longrightarrow Y$ with $f(X^\circ) \subset Y^\circ$*

$$f^* : \mathcal{A}(Y) \longrightarrow \mathcal{A}(X). \tag{4.11.2}$$

Proof. One quite revealing way to prove this is to use the (local) representation of a conormal distributions as an inverse Mellin transform and to lift this representation under f and see that this gives again a conormal distribution. We give a slightly more abstract proof because it gives a little more information on the regularity of the lift.

Push-forward under a $\mathcal{C}^\infty$ map, $f_* : T_xX \longrightarrow T_{f(x)}Y$ does not generally relate a smooth vector field on X to one on Y. However we can always interpret the push-forward of a smooth vector field as a section of f^*TY, the fibre of which is $T_{f(x)}$ over x. Similarly for a b-map we get a push-forward map

$$f_* : \mathcal{C}^\infty(X; {}^bTX) \longrightarrow \mathcal{C}^\infty(X; f^{*b}TY). \tag{4.11.3}$$

This means that locally on X we can write

$$f_*V = \sum_{i=1}^{\dim Y} a_iV_i,\ V_i \in \mathcal{C}^\infty(Y; {}^bTY), a_i \in \mathcal{C}^\infty(X). \tag{4.11.4}$$

This is important because of the duality between push-forward and pull-back:

$$Vf^*u = \sum_{i=1}^{\dim Y} a_i f^*[V_iu]. \tag{4.11.5}$$

Iterating this identity we see that for a b-map

$$u \in \mathcal{A}^0(Y) \Longrightarrow f^*u \in \mathcal{A}^0(X). \tag{4.11.6}$$

More generally if $\mathfrak{a}$ is a power family for Y then let

$$(f^\#\mathfrak{a})_l = \sum_{i=1}^{N(Y)} e(i,l)\alpha_i \tag{4.11.7}$$

where $e(i,l)$ are the boundary exponents, (1.12.2). Thus

$$f^*(\rho')^\alpha = a_\alpha\rho^\beta,\ \beta = f^\#\alpha,\ a_\alpha \in \mathcal{C}^\infty(X) \tag{4.11.8}$$

for any b-map. Combining this with (4.11.6) we have

$$f^*[\mathcal{A}^\alpha(Y)] \subset \mathcal{A}^\beta(X),\ \beta = f^\#\alpha \tag{4.11.9}$$

for any b-map. Then (4.11.2) follows from (4.11.9) and (4.2.15) – although of course (4.11.9) is more precise. □

Of course we can extend (4.11.2) and (4.11.9) to sections of a bundle. Thus, if $f : X \longrightarrow Y$ is an interior b-map then for any $\mathcal{C}^\infty$ vector bundle over Y

$$f^* : \mathcal{A}^\alpha(Y; H) \longrightarrow \mathcal{A}^\beta(X; f^*H),\ \beta = f^\# \alpha. \tag{4.11.10}$$

By duality a simple result on the push-forward follows:

Corollary 4.11.1. *If $f : X \longrightarrow Y$ is a b-map of compact manifolds with corners then*

$$f_* : \mathcal{Z}(X; \Omega \otimes f^*H) \longrightarrow \mathcal{Z}(Y; \Omega \otimes H) \tag{4.11.11}$$

for any $\mathcal{C}^\infty$ vector bundle H over Y.

Exercise 4.11.27. Prove Corollary 4.11.1 directly by using the characterization of $u \in \mathcal{Z}(X)$ in terms of the meromorphy of $\rho^{iz}u$.

4.12. Push-forward of $\dot{\mathcal{A}}(X;\Omega)$

As shown in §3.6 the push-forward under a $\mathcal{C}^\infty$ map of any supported distributional density on a compact manifold with corners is always well-defined. To know that the push-forward of a conormal distribution is conormal we need to restrict the map to be a b-submersion. Recall that two vector fields V on X and W on Y are f-related for a $\mathcal{C}^\infty$ map $F : X \longrightarrow Y$ if

$$f_*(V_x) = W_{f(x)} \quad \forall\ x \in X. \tag{4.12.1}$$

If f is a b-map and W and V are both tangent to the boundaries then (4.12.1) is equivalent to

$$^b f_*(V_x) = W_{f(x)} \text{ in } {}^bT_{f(x)}Y \ \forall\ x \in X. \tag{4.12.2}$$

The condition that f be a b-submersion arises because of

Lemma 4.12.1. *If $f : X \longrightarrow Y$ is a b-submersion then each $W \in \mathcal{V}_b(Y)$ is f-related to some $V \in \mathcal{V}_b(X)$.*

Proof. By definition of a b-submersion the map ${}^bf_* : {}^bT_xX \longrightarrow {}^bT_{f(x)}Y$ is surjective for every $x \in X$. Since the range has constant dimension the null space is a subbundle:

$$\operatorname{null}({}^bf_*) \subset {}^bTX. \tag{4.12.3}$$

If we choose a b-metric on X, i.e. a positive definite fibre metric on ${}^bTX$ and let $L = \operatorname{null}({}^bf_*)^\perp$ be the orthocomplement of the null space then

$$W_x \in L_x,\ {}^bf_*(W_x) = V_{f(x)} \ \forall\ x \in X \tag{4.12.4}$$

defines an element $W \in \mathcal{V}_b(X)$ for each $V \in \mathcal{V}_b(Y)$. Clearly W and V are f-related so the lemma is proved. $\square$

Proposition 4.12.1. *If $f : X \longrightarrow Y$ is a b-submersion of compact manifolds with corners then*

$$f_* : \dot{\mathcal{A}}(X; \Omega \otimes f^*H) \longrightarrow \dot{\mathcal{A}}(Y; \Omega \otimes H) \tag{4.12.5}$$

for any $\mathcal{C}^\infty$ vector bundle H over Y.

PROOF. Consider the case $H = \mathbb{R}$. The continuity properties of push-forward show that for each $s \in \mathbb{R}$ there exists $S \in \mathbb{R}$ such that

$$f_* : \dot{\mathcal{H}}^s(X;\Omega) \longrightarrow \dot{\mathcal{H}}^S(Y;\Omega). \tag{4.12.6}$$

It follows from Lemma 4.12.1 that

$$f_* : \dot{\mathcal{A}}^{[s]}(X;\Omega) \longrightarrow \dot{\mathcal{A}}^{[S]}(Y;\Omega). \tag{4.12.7}$$

Indeed, if $W \in \mathcal{V}_b(Y)$ then there exists some $V \in \mathcal{V}_b(X)$ which is f-related to it. Thus

$$W[f_*(u)] = f_*[Vu] \in \dot{\mathcal{H}}^S(Y;\Omega) \tag{4.12.8}$$

if $u \in \dot{\mathcal{A}}^{[s]}(X;\Omega)$. Iterating this argument (4.12.7) follows. This proves the proposition in case H is trivial. The general case follows by localizing, or by allowing the vector fields to act on the sections of the bundles through a connection. □

By duality we deduce the follow pull-back result:

COROLLARY 4.12.1. *If $f : X \longrightarrow Y$ is a b-submersion between compact manifolds then*

$$f^* : \mathcal{B}(Y;H) \longrightarrow \mathcal{B}(X;f^*H) \tag{4.12.9}$$

for any $\mathcal{C}^\infty$ vector bundle H over Y.

It is also straightforward to generalize Proposition 4.12.1 to partly extendible conormal distributions (we will give other generalizations in Chapter 6 below.)

PROPOSITION 4.12.2. *If $f : X \longrightarrow Y$ is a b-submersion and $A \subset M_1Y$ then if $B = M_1X \setminus f^{-1}(M_1Y \setminus A)$*

$$f_* : \mathcal{A}_B(X;\Omega \otimes f^*H) \longrightarrow \mathcal{A}_A(Y;\Omega \otimes H) \tag{4.12.10}$$

for any $\mathcal{C}^\infty$ vector bundle H over Y.

PROOF. Recall that the restriction map

$$\dot{\mathcal{A}}(X;G) \longrightarrow \mathcal{A}_A(X;G) \tag{4.12.11}$$

is surjective for any vector bundle G and $A \subset M_1X$. Using this, the fact that

$$f_* : \mathcal{C}_B^{-\infty}(X;\Omega \otimes f^*H) \longrightarrow \mathcal{C}_A^{-\infty}(Y;\Omega \otimes H) \tag{4.12.12}$$

and Proposition 4.12.1, (4.12.10) follows directly. □

EXERCISE 4.12.28. What is the dual statement, for pull-back, to (4.12.10)?

4.13. Polyhomogeneity

In discussing the Mellin transform in §4.7 the entire family of distributions $x^{iz} \in \mathcal{C}^\infty([0,\infty))$ was encountered. Polyhomogeneous conormal distributions are those having expansions in terms of these homogeneous functions and the quasi-homogeneous

$$(\log x)^k x^{iz} = D_z^k x^{iz}. \tag{4.13.1}$$

We fix those powers, iz, and their multiplicities, k, which can occur by index sets.

By an index set, E, we mean a countable discrete subset

$$E \subset \mathbb{C} \times \mathbb{N}_0 \tag{4.13.2}$$

with the properties

$$(z,k)\in E \Longrightarrow (z,l)\in E,\ 0\le l\le k \tag{4.13.3}$$

$$(z,k)\in E \Longrightarrow (z-il,k)\in E \quad \forall\, l\in\mathbb{N} \tag{4.13.4}$$

$$(z_j,k_j)\in E \text{ with } |(z_j,k_j)|\longrightarrow\infty \Longrightarrow \Im z_j\longrightarrow -\infty. \tag{4.13.5}$$

In view of the assumed discreteness this last condition can be restated as

$$\forall\ N \quad E\cap(\{z\in\mathbb{C};\Im z>-N\}\times\mathbb{N}_0) \text{ is finite.} \tag{4.13.6}$$

The simplest, non-empty, index sets are those of the form $\{(z-il,0);l\in\mathbb{N}_0\}$. Where it is not otherwise confusing we can denote this index set simply by z, since it is the smallest index set containing $(z,0)$.

More generally we consider an index family, $\mathcal{E}$, for a manifold with corners, X. This is just the assignment to each boundary hypersurface of X of an index set, i.e. $\mathcal{E}$ is a map from $M_1(X)$ to index sets. Setting $M_1(X)=\{H_1,\dots,H_N\}$ means we can put $\mathcal{E}=(E_1,\dots,E_N)$ where each E_j is an index set. We wish to define

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)\subset\mathcal{A}(X) \tag{4.13.7}$$

so that its elements have expansions at each boundary hypersurface with exponents from the corresponding index set. To do so we proceed inductively over the number of elements of $\mathcal{E}$ which are not 0.

Thus the simplest case is where $E_j=0$ for $j=1,\dots,N$ meaning the index sets are all $-i\mathbb{N}_0\times\{0\}$. If we denote this index family by 0 then we set

$$\mathcal{A}^0_{\text{phg}}=\mathcal{C}^\infty(X). \tag{4.13.8}$$

We remark, as part of the inductive statement, that this is invariant under diffeomorphisms mapping each boundary hypersurface to itself and that

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)\subset\mathcal{A}^{\alpha}(X) \text{ if } \alpha_j<-\sup_{(z,k)\in E_j}\Im z \tag{4.13.9}$$

and if $E_j=0$ then

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)=\mathcal{A}^{\mathcal{E}'(j)}_{\text{phg}}(2_jX)\restriction_X,\ \mathcal{E}'(j)=(E_1,\dots,E_{j-1},E_{j+1},\dots,E_N). \tag{4.13.10}$$

Proceeding inductively we can suppose $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ to be defined provided at most k of the E_j are not 0. Suppose $\mathcal{E}$ has $k+1$ terms not 0. We first restrict attention to the case that X itself has exactly $k+1$ boundary hypersurfaces. Then we fix α so that $-\alpha_j>\sup\{\Im z;(z,k)\in E_j\}$ and define

$$\begin{gathered}u\in\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)\Longleftrightarrow \text{ for each } j\ \exists\ \phi_j:E_j\longrightarrow\mathcal{A}^{\mathcal{E}(j)}_{\text{phg}}(X),\\ \mathcal{E}(j)=(E_1,\dots,E_{j-1},0,E_{j+1},\dots,E_N) \text{ for which,}\\ \text{with } \alpha(j)=(\alpha_1,\dots,\alpha_{j-1},\infty,\alpha_{j+1},\dots,\alpha_N),\\ u\sim\sum_{(z,k)\in E_j}\phi_j(z,k)(\log\rho_j)^k\rho_j^{iz}\quad\text{mod }\mathcal{A}^{\alpha(j)}(X).\end{gathered} \tag{4.13.11}$$

Here we use Proposition 4.5.2 to fix the meaning of the asymptotic expansion in (4.13.11). This condition is clearly invariant under diffeomorphisms mapping each boundary hypersurface to itself. In the general case where X has more than $k+1$ boundary hypersurfaces we simply replace it by 2_AX, where $A\subset M_1(X)$ is the subset with 0 as index set and put

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)=\mathcal{A}^{\mathcal{E}(A)}_{\text{phg}}(2_AX)\restriction_X, \tag{4.13.12}$$

where $\mathcal{E}(A)$ is the index family for $2_A X$ obtained by dropping the zero entries. The invariance under diffeomorphisms assumed inductively shows that this is independent of the choice of doubling and (4.13.9) and (4.13.10) follow. Thus we have defined $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ in general. Clearly if $E_j = z$ for some j then

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) = \rho_j^{iz}\mathcal{A}^{\mathcal{E}'}_{\text{phg}}(X),\ \mathcal{E}' = (E_1, \dots, E_{j-1}, 0, E_{j+1}, \dots, E_N). \tag{4.13.13}$$

We also remark that if $\mathcal{E}$ and $\mathcal{F}$ are two index families and we put

$$\begin{gathered} \mathcal{E} + \mathcal{F} = (E_1 + F_1, \dots E_N + F_N) \\ E + F = \{(z_1 + z_2, \max(k_1, k_2)); \exists\ (z_1, k_1) \in E, (z_2, k_2) \in F\} \end{gathered} \tag{4.13.14}$$

then

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \cdot \mathcal{A}^{\mathcal{F}}_{\text{phg}}(X) \subset \mathcal{A}^{\mathcal{E}+\mathcal{F}}_{\text{phg}}(X). \tag{4.13.15}$$

Similarly if

$$\mathcal{E} \cup \mathcal{F} = (E_1 \cup F_1, \dots, E_N \cup F_N) \tag{4.13.16}$$

then

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) + \mathcal{A}^{\mathcal{F}}_{\text{phg}}(X) \subset \mathcal{A}^{\mathcal{E}\cup\mathcal{F}}_{\text{phg}}(X). \tag{4.13.17}$$

If $\mathcal{E} = (E_1, \dots, E_N)$ is an index family for the boundary hypersurfaces of X then

$$\mathfrak{e} = (e, l),\ e_j = \sup\{\Im z; (z, k) \in E_j\},\ l_j = \sup\{k; (z, k) \in E_j, z = e_j\}, \tag{4.13.18}$$

defines a power set. Moreover from the iterative definition we have

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \subset \mathcal{A}^{\mathfrak{e}}(X) \tag{4.13.19}$$

and $\mathfrak{e}$ is the smallest power set (with respect to the partial order (4.2.21)) for which this is true. Again directly from the definition, for any power family $\mathfrak{b} = ((b_1, p_1), \dots, (b_N, p_N))$ and index family $\mathcal{E} = (E_1, \dots, E_N)$

$$\begin{gathered} \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \cap \mathcal{A}^{\mathfrak{b}}(X) = \mathcal{A}^{\mathcal{F}}_{\text{phg}}(X) \\ \mathcal{F} = (F_1, \dots, F_N),\ F_j = \{(z, k) \in E_j; \Im z \ge b_j \text{ and if } \Im z = b_j \text{ then } k \le p_j\}. \end{gathered} \tag{4.13.20}$$

The space of polyhomogeneous conormal distributions, $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$, with respect to a given index family $\mathcal{E}$ is topologized by the seminorms of $\mathcal{A}^{\mathfrak{e}}(X)$, with the power set given by (4.13.18) and also (inductively) the seminorms involved in all the expansions. More precisely for each boundary hypersurface H_j we can consider a normal fibration of X near H_j :

$$X \supset \Omega \cong H_j \times [0, 1)_{\rho_j}. \tag{4.13.21}$$

On Ω the coefficients in the expansion (4.13.11) are uniquely determined if we demand them to be independent of ρ_j, i.e. to be distributions on H_j. Then the seminorms are the polyhomogeneous seminorms on the coefficients $\phi_j(z, k)$, using the inductive hypothesis, and the seminorms in $\mathcal{A}^{\mathfrak{e}+Ni_j}(X)$ on the difference

$$u - \sum_{\substack{(z,k)\in E_j \\ \Im z \ge -N}} \phi_j(z, k)(\log \rho_j)^k \rho_j^{iz} \chi(\rho_j) \tag{4.13.22}$$

where $\chi \in \mathcal{C}^\infty_c([0, 1))$ is 1 near 0. It is straightforward to check that these seminorms define a topology which is independent of the choices. It does not even matter too much in which conormal space the seminorm on the difference (4.13.22) is taken, provided as $N \longrightarrow \infty$ the power set has jth entry tending to ∞ with the others bounded above and below.

LEMMA 4.13.1. *With these seminorms each $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ is a complete locally convex topological vector space.*

4.14. Mellin characterization of polyhomogeneity

The invariance under diffeomorphisms of the spaces of polyhomogeneous conormal distributions inherent in the definition can be put in the following form. If $F : X \longrightarrow X$ is a diffeomorphism then it induces a map on $M_1(X)$ and hence a transformation of index families

$$F^{\#}(\mathcal{E}) = (E_{i(1)}, \dots, E_{i(N)}) \text{ if } F(H_{i(k)}) = H_k,\ k = 1, \dots, N. \tag{4.14.1}$$

Then

$$F^* : \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \longrightarrow \mathcal{A}^{F^{\#}\mathcal{E}}_{\text{phg}}(X). \tag{4.14.2}$$

Moreover these spaces are local $\mathcal{C}^\infty$-modules, so to define them it is really only necessary to fix the spaces $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(\mathbb{R}^n_k)$. If $\mathcal{E}$ is an index family for $\mathbb{R}^n_k$ we write $\pi_1(\mathcal{E})$ for its projection onto $\mathbb{C}^k$.

PROPOSITION 4.14.1. *The (k-dimensional) Mellin transform maps $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(\mathbb{R}^n_k) \cap \mathcal{C}^{-\infty}_c(\mathbb{R}^n_k)$ into the space of meromorphic functions*

$$u_M : \mathbb{C}^k \setminus \pi_1(\mathcal{E}) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R}^{n-k}) \tag{4.14.3}$$

which have uniformly bounded supports and are such that for any $N \in \mathbb{R}$ if

$$q_N(s_1, \dots, s_k) = \prod_{j=1}^{k} \prod_{(z,k)\in E_j, \Im z \geq N} (s_j - z)^{k+1} \tag{4.14.4}$$

then $q_N u_M$ is holomorphic in $\Im s_j \leq N$ and for each $p \in \mathbb{N}$ satisfies estimates

$$\|q_N u_M(s, \cdot)\|_{(p)} \leq C_{p,N} \exp(A|\Im s|)(1 + |s|)^{-p},\ \Im s_j \leq N. \tag{4.14.5}$$

PROOF. These estimates just mean that the Mellin transform of a polyhomogeneous conormal function has at most a poles of order $k+1$ at $s_j = z$ if $(z, k) \in E_j$ (where a simple pole is of order 1), together with the rapid decrease at real infinity corresponding to conormal regularity and the exponential bound as $\Im s_j \longrightarrow -\infty$ arising from the assumption of compact supports.

Both the necessity and the sufficiency follow by inductive arguments over the number of boundary hypersurfaces on which the index set is 0. We concentrate on the necessity, the sufficiency can be proved by reversing the arguments. Thus suppose $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(\mathbb{R}^n_k)$ has support in $|x| \leq 1$. We suppose initially that the result has been established provided at least one of the index sets is 0. Consider the expansion of u in x_1 :

$$u \sim \sum_{(z,k)\in E_1} a(z, k)(x_1)^{iz}(\log x_1)^k \tag{4.14.6}$$

where can suppose that $a(z, k) \in \mathcal{A}^{\mathcal{E}(1)}_{\text{phg}}(\mathbb{R}^{n-1}_{k-1})$, the index family just being $E(1) = (E_2, \dots, E_k)$. Now in the asymptotic summation of such a conormal series (as in

§4.5):

$$u' = \sum_{(z,k)\in E_1} a(z,k)\phi(x_1/\epsilon(z,k))(x_1)^{iz}(\log x_1)^k \tag{4.14.7}$$

we can arrange that for any N the tail of the series converges with respect to the seminorms of $\mathcal{C}^k_c([0,1];\mathcal{A}^{\mathcal{E}(1)}_{\text{phg}}(\mathbb{R}^{n-1}_{k-1}))$. From this it follows readily, using the inductive hypothesis, that u' has Mellin transform satisfying the conditions of the Proposition. Moreover, taking the expansions of the coefficients at the other boundaries and using the convergence arranged we see that $u' \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(\mathbb{R}^n_k)$. The difference, $u - u'$ is then rapidly decreasing at $x_1 = 0$ and so polyhomogeneous with respect to the index family $(\emptyset, E_2, \dots, E_N)$. To this the inductive hypothesis can also be applied. If there are more than k boundary hypersurfaces then doubling across those with index set 0 reduces to this case, so the proposition is proved. □

As with the 'ABC' Theorem proved in §4.10 we can use interpolation arguments to give an alternative characterization of the Mellin transform of polyhomogeneous conormal functions.

Lemma 4.14.1. *If $u \in \mathcal{A}(X)$ then $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ if and only if the product with powers of boundary defining functions*

$$v(s) = \sum_{j=1}^{N} \rho_j^{is_j - 1} u \in \dot{\mathcal{C}}^{-\infty}(X) \tag{4.14.8}$$

extends from $\Im s_j < -C$, where is is initially defined, to be meromorphic on $\mathbb{C}^N$ and such that for each $M \in \mathbb{R}$ each product $q_M(s)v(s)$, with q_M given by (4.14.4), is holomorphic in $\Im s_j \le M$ with values in $\mathcal{H}^p(X)$ for some p and satisfies estimates (4.14.5).

Proof. The necessity of this condition follows easily from Proposition 4.14.1 and to prove the converse it suffices to check the polyhomogeneity in local coordinates, i.e. we can assume that $u \in \mathcal{A}(\mathbb{R}^n_k)$ has compact support. The assumption of holomorphy, for $q_M v$ with values in some Sobolev space on $\mathbb{R}^n_k$ allows us to integrate in $x_1, \dots, x_k$ and deduce that the Mellin transform has the same meromorphy, namely that $q_m(s)u_M(s)$ is also holomorphic and of slow growth at real infinity in $-M \le \Im s_j \le M$. Now, by the assumed conormality of u the Mellin transform, in $\Im s_j < -C$ takes values in $\mathcal{C}^\infty(\mathbb{R}^{n-k})$ with rapid decrease at real infinity. Thus, after suitable renormalization, Lemma 4.10.1 can be applied to show that $q_M(s)u_M(s)$ has values in $\mathcal{C}^\infty$ and is rapidly decreasing at real infinity. This recovers (4.14.5) and proves the lemma. □

We also note how the symbol mapping behaves for polyhomogeneous functions. namely if $\mathfrak{e}$ is the power set associated to $\mathcal{E}$ in (4.13.18) and we let $\mathcal{E}'(H)$ be the index family obtained from $\mathcal{E}$ be replacing the index set $E(H)$ for H by

$$E'(H) = \{(z,k) \in E(H); (-\Im z, k) \le \mathfrak{e} - 1\} \tag{4.14.9}$$

then the symbol sequence is just

$$\begin{gathered} 0 \longrightarrow \mathcal{A}^{\mathcal{E}'(H)}_{\text{phg}}(X) \hookrightarrow \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \longrightarrow \mathcal{A}^{\{\mathcal{E}(H)\}}_{\text{phg}}(N^+H) \longrightarrow 0 \\ \mathcal{A}^{\{\mathcal{E}(H)\}}_{\text{phg}}(N^+H) = \mathcal{A}^{\mathcal{E}}_{\text{phg}}(N^+H)/\mathcal{A}^{\mathcal{E}'(H)}_{\text{phg}}(N^+H). \end{gathered} \tag{4.14.10}$$

Polyhomogeneous conormal functions, with a given index family, are those having expansions at each boundary hypersurface with terms as specified by the corresponding index set. Of course the coefficients have to be polyhomogeneous too; this is why the definition above is somewhat intricate. In fact it is *not* necessary to assume *a priori* that the coefficients are polyhomogeneous. For each boundary hypersurface H_j consider a normal fibration of X near H_j as in (4.13.21). On Ω the coefficients in the expansion (4.13.11) are fixed if we demand them to be independent of ρ_j, i.e. to be distributions on H_j. Indeed let $V_j = \rho_j D_{\rho_j}$ be the $\mathbb{R}^+$-invariant vector field for the product decomposition (4.13.21). Then set

$$P_{M,j} = \prod_{(z,k)\in E_j;\Im z\ge -M} (V_j - z). \tag{4.14.11}$$

Thus if $k \ge 0$ is the maximum multiplicity at which z occurs in E_j and $\Im z \ge -M$ then $(D_j - z)^{k+1}$ occurs as a factor in $P_{M,j}$, and conversely. The asymptotic expansion (4.13.11) certainly implies that

(4.14.12)

$$P_{M,j}u \in \mathcal{A}^{\alpha(j,N)}(X) \quad \forall\ M, \quad \text{with } \alpha(j,M) = (\alpha_1,\dots,\alpha_{j-1},M,\alpha_{j+1},\dots,\alpha_N).$$

Proposition 4.14.2. *Let $\mathcal{E}$ be an index family on a compact manifold with corners X then if $u \in \mathcal{A}(X)$, $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ if and only if u satisfies* (4.14.12) *for a fixed power set $\mathfrak{e}$, all M and $j = 1,\dots,N$.*

Proof. The necessity of condition (4.14.12) for a polyhomogeneous conormal function has already been established, for any normal fibration. Conversely it is straightforward to check that the conditions (4.14.12), for all M, are independent of the choice of normal fibration, in so far as it determines V_j. Then the sufficiency of these conditions follows directly from the characterization in terms of the Mellin transform in Proposition 4.14.1. □

4.15. Operations on polyhomogeneous conormal functions

In §4.11 and §4.12 the pull-back and push-forward operations were discussed in relation to conormal distributions. The pull-back restricts in a straightforward way to the polyhomogeneous spaces. Namely if $f : X \longrightarrow Y$ is a b-map such that $f(X^\circ) \subset Y^\circ$ then we can define an operator on index families by

(4.15.1)

$$f^*\rho'_j = \prod_{l=1}^{N} \rho_l^{e(l,j)},\ j = 1,\dots N' \Longrightarrow f^{\#}\mathcal{E} = (F_1,\dots,F_N) \text{ where}$$

$$F_j = \{(z,k); z = \sum_{j=1}^{N'} e(j,l)z_j \text{ for some } (z_j,k_j) \in E_j \text{ with } k \le \sum_{\{j;e(l,j\neq 0\}} k_j\}.$$

As already noted the spaces $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ are always local $\mathcal{C}^\infty$-modules, so we can define (by taking the tensor product over $\mathcal{C}^\infty(X)$) the spaces of polyhomogeneous sections of any vector bundle, G, over X :

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X;G) = \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;G). \tag{4.15.2}$$

This is precisely the same as defining polyhomogeneous conormal sections to be those with polyhomogeneous conormal coefficients in trivializations.

PROPOSITION 4.15.1. *If $F : X \longrightarrow Y$ is an b-map of compact manifolds with corners such that $f(X^\circ) \subset Y^\circ$ then for any index family $\mathcal{E}$ for Y and any vector bundle G over Y pull-back gives a continuous linear map*

$$f^* : \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(Y; G) \longrightarrow \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X; f^*G),\ \mathcal{F} = f^\# \mathcal{E}. \tag{4.15.3}$$

PROOF. We leave the, essentially notational, complication introduced by the vector bundle to the reader and suppose $G = \mathbb{C}$ (or $\mathbb{R}$). The pull-back of index families in (4.15.1) is determined by noting that for $z = (z_1, \dots, z_N)$ and $k = (k_1, \dots, k_N) \in \mathbb{N}_0^N$

$$\begin{gathered} f^* \left[\rho_1^{iz_1} (\log \rho_1)^{k_1} \dots \rho_N^{iz_N} \log(\rho_N)^{k_N}\right] \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X), \\ \text{if for each } l,\ (\sum_{j=1}^N e(l,j) z_j,\ \sum_{\{j: e(l,j) \neq 0\}} k_j) \in F_l; \end{gathered} \tag{4.15.4}$$

although this does not completely capture the behaviour of the multiplicities. As usual we use induction over the number of non-zero index sets in $\mathcal{E}$. starting with the case $\mathcal{E} = 0$ given by (4.13.8). Thus we may suppose that $u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(Y)$ where if one non-zero index set, say E_1, of $\mathcal{E}$ is replaced by 0 the lifting properties are already established. If u is replaced by a sum, u' as in (4.14.7), of the asymptotic series at $\rho_1 = 0$ then each of the terms lifts, and the lifted series converges, i.e. $u' \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X)$. Since $u - u'$ lies in the polyhomogeneous space with index family $(\emptyset, E_1, \dots, E_N)$ the inductive hypothesis implies that it too lifts as desired. $\square$

The push-forward result is a little more subtle. It is *not* true that Proposition 4.12.1 extends directly to polyhomogeneous conormal functions. Rather we have to strengthen the assumptions and require that f be a b-fibration. In that case we can define a second operation on index families:

$$\begin{gathered} f_\# \mathcal{F} = (E_1, \dots, E_{N'}) \text{ where } (z, k) \in E_j \Longleftrightarrow \\ L(z) = \{l; e(l,j) \neq 0 \text{ and for some } k,\ (e(j,l)z, k_l) \in F_l\} \neq \emptyset,\ k \leq \#(L) + \sum_l k_l. \end{gathered} \tag{4.15.5}$$

PROPOSITION 4.15.2. *Let $f : X \longrightarrow Y$ be a b-fibration of compact manifolds with corners and suppose $\mathcal{F}$ is an index family for X and G is a vector bundle over Y then push-forward gives a continuous linear map*

$$f_* : \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X; \Omega \otimes f^*G) \longrightarrow \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(Y; \Omega \otimes G),\ E = f_\# F. \tag{4.15.6}$$

PROOF. Since we can work locally in Y we can assume that G is trivial and ignore it. We use Lemma 4.14.1 to show that the push-forward is polyhomogeneous, since from Proposition 4.12.1 we already know that it is conormal. Thus if $M_1(Y) = \{H'_1, \dots, H'_{N'}\}$ and ρ'_j, $j = 1, \dots, N'$ are corresponding defining functions we are interested in the meromorphy of

$$\prod_{j=1}^{N'} (\rho'_j)^{is_j - 1} f_*(u) = f_* \left[\prod_{l=1}^N \rho_l^{iZ_l - 1} u\right],\ Z_l = \begin{cases} e(l,j) z_j \text{ if } e(l,j) \neq 0 \\ 0 \text{ if } e(l,j) = 0 \quad \forall\ j. \end{cases} \tag{4.15.7}$$

Notice that this only makes sense because f is assumed to be a b-fibration and so, by Lemma 2.4.2, for each l the exponent $e(l,j)$ is non-zero for at most one j.

Applying 4.14.1 once, the distribution on the right in (4.15.7) is meromorphic in Z_l with poles and multiplicities specified by F_l, where only this l with $e(l,j) \neq 0$

for some j appear. Thus it follows (by the continuity of push-forward) that the distribution on the left is also meromorphic with poles at the points fixed by

$$z_j = Z_l/e(l,j),\ \ e(l,j) \neq 0 \tag{4.15.8}$$

The multiplicity of such poles in z_j is the sum of the multiplicities of the poles in Z_l which correspond to it. This is precisely the content of the definition (4.15.5) of the push-forward of index sets, proving the proposition. $\square$