CHAPTER 3

Distributions on manifolds with corners

On a manifold without boundary the space of distributions is the dual of the space of compactly supported $\mathcal{C}^\infty$ densities. The situation is more complicated for a manifold with corners. There are three 'obvious' spaces of distributions. First there is the space, $\mathcal{C}^{-\infty}(X^\circ)$ of distributions on the interior of the manifold. Next there is the space of extendible distributions, denoted $\mathcal{C}^{-\infty}(X)$. This consists of the dual of the space of compactly supported densities which vanish to all orders at all boundary components. Finally there is the space of supported distributions, which is the dual of the space of all compactly supported densities smooth up to the boundary. These various spaces and the relationships between them will now be discussed. To remove at least one of the sources of variety and possible confusion we shall deal throughout with compact manifolds with corners. The general case is only notationally more complicated since one then has to deal separately (and dually) with spaces with compact and general supports.

3.1. Densities

Recall that TX is a $\mathcal{C}^\infty$ vector bundle over X, a compact manifold with corners, of fibre dimension $n = \dim X$. As noted in §1.10 it follows that all 'associated bundles' are smooth. In particular the bundles $\Omega^\alpha X$, $\alpha \in \mathbb{R}$, of α-densities are $\mathcal{C}^\infty$ line bundles over X. We proceed to recall the definition and basic properties of these bundles.

If V is a vector space then $\Lambda^n V$, $n = \dim V$, is the totally antisymmetric part of the n-fold tensor product $V \otimes \cdots \otimes V$; it is 1-dimensional. The space of α-densities, $\Omega^\alpha V$,[1] is the 1-dimensional linear space

$$\Omega^\alpha V = \{f : \Lambda^n V\backslash 0 \longrightarrow \mathbb{R}; f(sv) = |s|^\alpha f(v)\ \forall\ v \in \Lambda^n V\backslash 0\}. \tag{3.1.1}$$

The bundle of α-densities, over a manifold with corners, has as fibres

$$\Omega^\alpha_p X = \Omega^\alpha(T_pX) \quad \alpha \in \mathbb{R}. \tag{3.1.2}$$

Locally if $\partial_1, \dots, \partial_n$ is a basis for T_pX (from local coordinates) then $\partial_1 \wedge \cdots \wedge \partial_n$ is a basis of $\Lambda^n T_pX$ so $|dx_1 \wedge \cdots \wedge dx_n|^\alpha$ is a basis of $\Omega^\alpha_p X$.

$$|dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|^\alpha (t\partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_n) = |t|^\alpha. \tag{3.1.3}$$

In particular if $\mu \in \mathcal{C}^\infty(X; \Lambda^n X)$ is a $\mathcal{C}^\infty$ n-form then $|\mu|$ is a (non-negative) density, although in general not a smooth one. If μ is non-vanishing then $|\mu|^\alpha \in \mathcal{C}^\infty(X; \Omega^\alpha)$; of course μ and $-\mu$ give the same density section. If the manifold is orientable then a choice of orientation gives an identification of the n-form and density bundles, conversely an isomorphism between these bundles consistent with the relationship

[1] N.B. For a manifold the convention is that $\Lambda^k_p X = \Lambda^k(T^*_pX)$ is the antisymmetric part of the k-fold tensor product of the cotangent, not the tangent, space. It's really an historical accident that the star finished up in the wrong place.

between μ and $|\mu|$ fixes an orientation. Densities have natural properties under tensor product,

$$\Omega^\alpha X \otimes \Omega^\beta X \cong \Omega^{\alpha+\beta} X, \ \Omega^0 X \cong \mathbb{R}. \tag{3.1.4}$$

Thus, for example, the dual bundle of $\Omega^\alpha X$ is $\Omega^{-\alpha} X$.

Each of the densities bundles is trivial but, except for $\Omega^0 X$, not canonically so. That is, there is a smooth section $\nu \in \mathcal{C}^\infty(X; \Omega X)$ which is everywhere positive and then ν^α is a global $\mathcal{C}^\infty$ section of $\Omega^\alpha X$:

$$f \in \mathcal{C}^\infty(X; \Omega^\alpha X) \Longleftrightarrow f = g\nu^\alpha, \ g \in \mathcal{C}^\infty(X). \tag{3.1.5}$$

The topology (of uniform convergence of all derivatives on compact sets) from local coordinates is consistent with this isomorphism so that each $\mathcal{C}^\infty(X; \Omega^\alpha)$ is a complete locally convex topological vector space; since X is assumed compact it is actually a Fréchet space.

The most important property of $\Omega X = \Omega^1 X$ is that its smooth sections can be invariantly integrated:

$$\mathcal{C}^\infty(X; \Omega X) \xrightarrow{\int} \mathbb{C}. \tag{3.1.6}$$

The map (3.1.6) is defined for densities with support in a coordinate patch, Ω, as follows. For $\mathbb{R}^n$ any smooth density is of the form $\alpha = \phi(x)|dx|$ where $\phi \in \mathcal{C}^\infty(\Omega')$, Ω' being the image of Ω under the coordinate system, and $|dx|$ the function on $\Lambda^n \mathbb{R}^n$ determined by

$$|dx| \cdot (t\partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_n) = |t|. \tag{3.1.7}$$

Under a coordinate transformation $y = f(x)$ the density transforms to $\alpha = \psi(y)|dy|$ with

$$\phi(x) = \psi(f(x))|\det \frac{\partial f_i}{\partial x_j}|. \tag{3.1.8}$$

Thus, if $\phi \in \mathcal{C}_c^\infty(\Omega')$ then

$$\int_{\mathbb{R}^n} \alpha = \int_{\mathbb{R}^n} \phi(x)dx = \int_{\mathbb{R}^n} \psi(y)dy. \tag{3.1.9}$$

The coordinate independence means that (3.1.6) is well defined by use of a partition of unity to reduce supports to a covering by coordinate patches.

The subspace $\mathcal{C}_c^\infty(X^\circ; \Omega X)$ of densities with compact support in the interior of X is a complete locally convex topological vector space with topology given by the inductive limit over compact sets $K_j \subset\subset X^\circ$ exhausting X°. Its dual is the space of distributions on X° :

$$\mathcal{C}^{-\infty}(X^\circ) = [\mathcal{C}_c^\infty(X^\circ; \Omega X)]'. \tag{3.1.10}$$

We generally do not use the space $\mathcal{C}^\infty(X^\circ)$. Its elements can behave extremely badly as the boundary of X is approached. For example they may be exponentially increasing or be of unbounded negative order. The minimal regularity that we shall require is that distributions be of 'slow growth' at the boundary. This is realized below in the (intrinsic) notion of an extendible distribution. First we consider what is an apparently quite different space; it is of course closely related.

3.2. Supported distributions

The space of *supported distributions* on X is the dual space

$$\dot{\mathcal{C}}^{-\infty}(X) = [\mathcal{C}^\infty(X;\Omega X)]'. \tag{3.2.1}$$

The density factor is included, as always, precisely so that there is a continuous linear injection

$$\mathcal{C}^\infty(X) \hookrightarrow \dot{\mathcal{C}}^{-\infty}(X) \tag{3.2.2}$$

determined by

$$u(\phi) = \int u\phi \quad u \in \mathcal{C}^\infty(X) \quad \forall\ \phi \in \mathcal{C}^\infty(X;\Omega X) \tag{3.2.3}$$

using (3.1.6).

Let $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(X)$ be the subspace of smooth functions which vanish, with all derivatives, at the boundary of X. If $\tilde{X} \supset X$ is an open extension of X then we can clearly identify

$$\dot{\mathcal{C}}^\infty(X) = \{u \in \mathcal{C}^\infty(\tilde{X}); \operatorname{supp}(u) \subset X\}. \tag{3.2.4}$$

As a general principle this is the meaning of the dot; that the space in question is that subspace of some similarly defined space on an extension $\tilde{X}$, $\partial\tilde{X} = \emptyset$, with supports in X.

Lemma 3.2.1. *The injection* (3.2.2) *has dense range, in the weak topology, as do the smaller spaces*

$$\mathcal{C}^\infty_c(X^\circ) \hookrightarrow \dot{\mathcal{C}}^\infty(X) \hookrightarrow \mathcal{C}^\infty(X) \hookrightarrow \dot{\mathcal{C}}^{-\infty}(X). \tag{3.2.5}$$

This follows as a fairly simple consequence of the following result which also justifies, at least in part, the notation:

Proposition 3.2.1. *If $\tilde{X} \hookleftarrow X$ extends X and is compact with $\partial\tilde{X} = \emptyset$, then there is a natural identification*

$$\dot{\mathcal{C}}^{-\infty}(X) = \{u \in \dot{\mathcal{C}}^{-\infty}(\tilde{X}); \operatorname{supp}(u) \subset X\}. \tag{3.2.6}$$

Here $\operatorname{supp}(u)$ is the complement of the largest open set $\Omega' \subset \tilde{X}$ such that $u(\phi) = 0$ for all $\phi \in \mathcal{C}^\infty(\tilde{X};\Omega\tilde{X})$ with $\operatorname{supp}(\phi) \subset \Omega'$.

Proof. There is a restriction map

$$R : \mathcal{C}^\infty(\tilde{X};\Omega\tilde{X}) \longrightarrow \mathcal{C}^\infty(X;\Omega X) \tag{3.2.7}$$

which is surjective. Thus if $u \in \dot{\mathcal{C}}^{-\infty}(X)$ then define $\tilde{u} \in \dot{\mathcal{C}}^{-\infty}(\tilde{X})$ by

$$\tilde{u}(\phi) = u(R\phi) \Longrightarrow \dot{\mathcal{C}}^{-\infty}(X) \hookrightarrow \dot{\mathcal{C}}^{-\infty}(\tilde{X}). \tag{3.2.8}$$

Conversely, suppose $v \in \dot{\mathcal{C}}^{-\infty}(\tilde{X})$ has support in X. Then define u by

$$u(\phi) = v(E\phi) \tag{3.2.9}$$

where $E : \mathcal{C}^\infty(X;\Omega X) \longrightarrow \mathcal{C}^\infty(\tilde{X};\Omega\tilde{X})$ is a Seeley extension operator. Certainly $u \in \dot{\mathcal{C}}^{-\infty}(X)$. Moreover it is straightforward to check that $\mathcal{C}^\infty_c(\tilde{X}\backslash X;\Omega\tilde{X})$, the space of $\mathcal{C}^\infty$ densities with compact support in this open set, is dense in

$$\{\psi \in \mathcal{C}^\infty(\tilde{X};\Omega\tilde{X}); \operatorname{supp}(\psi) \subset \overline{\tilde{X}\backslash X}\}. \tag{3.2.10}$$

Since $v(\psi) = 0$ when $\psi \in \mathcal{C}^\infty_c(\tilde{X}\backslash X)$, by the assumption on supports, $v(ER\psi) = v(\psi)$ for all $\psi \in \mathcal{C}^\infty(\tilde{X};\Omega\tilde{X})$. Thus $v(\psi) = u(R\psi)$, i.e. $v = \tilde{u}$. This proves (3.2.6) □

As an example of a supported distribution, consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$. Then the Dirac delta 'function' at the origin

$$u = \delta(x)\delta(y) \in \dot{\mathcal{C}}^{-\infty}(X).$$

It is defined by $\langle u, \phi(x)dxdy\rangle = \phi(0)$.

In general if G is a $\mathcal{C}^\infty$ vector bundle over X, let G^* be its dual bundle. The space of *supported distributional sections* of G is:

$$\dot{\mathcal{C}}^{-\infty}(X;G) = [\mathcal{C}^\infty(X;G^* \otimes \Omega X)]'. \tag{3.2.11}$$

Since $\Omega^* X \otimes \Omega X \cong \mathbb{R}$ this means, for example, that

$$\dot{\mathcal{C}}^{-\infty}(X;\Omega X) = [\mathcal{C}^\infty(X)]'. \tag{3.2.12}$$

Integration of such supported distributional densities, over X, is then well-defined:

$$\int u = \langle u, 1\rangle, \quad u \in \dot{\mathcal{C}}^{-\infty}(X;\Omega X). \tag{3.2.13}$$

3.3. Extendible distributions

We have defined the space $\dot{\mathcal{C}}^\infty(X)$, which is clearly a closed subspace of $\mathcal{C}^\infty(X)$. Similarly we can define $\dot{\mathcal{C}}^\infty(X;G)$ for any $\mathcal{C}^\infty$ vector bundle G. The space of *extendible distributions* on X is then, by definition:

$$\mathcal{C}^{-\infty}(X) = [\dot{\mathcal{C}}^\infty(X;\Omega X)]'. \tag{3.3.1}$$

Notice that if $\partial X = \emptyset$ then $\dot{\mathcal{C}}^\infty(X;\Omega X) \equiv \mathcal{C}^\infty(X;\Omega X)$, so

$$\dot{\mathcal{C}}^{-\infty}(X) \equiv \mathcal{C}^{-\infty}(X) \quad \text{if } \partial X = \emptyset. \tag{3.3.2}$$

In this case we use the simpler notation $\mathcal{C}^{-\infty}(X)$ which is consistent with the usual notation.

The function $f(x) = 1/x$ is an element of $\mathcal{C}^{-\infty}([0,1])$. Indeed, if $\phi \in \dot{\mathcal{C}}^\infty([0,1];\Omega)$ then $\phi(x)/x \in \dot{\mathcal{C}}^\infty([0,1];\Omega)$ so

$$\langle 1/x, \phi\rangle = \int \phi(x)/x. \tag{3.3.3}$$

There is no natural sense in which $1/x$ is a supported distribution on $[0,1]$ since there is no completely natural regularization of the (generally non-convergent) integral

$$\int_{[0,1]} \frac{f(x)}{x} dx \quad f \in \mathcal{C}^\infty([0,1]). \tag{3.3.4}$$

Similarly $\mathcal{C}^{-\infty}(X)$ should not be confused with the larger space $\mathcal{C}^{-\infty}(X^\circ)$.

LEMMA 3.3.1. *For any (compact) manifold with corners there is a natural injection*

$$\mathcal{C}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(X^\circ) \tag{3.3.5}$$

which is surjective if and only if $\partial X = \emptyset$.

PROOF. The injection (3.3.5) arises from the density of $\mathcal{C}_c^\infty(X^\circ)$ in $\dot{\mathcal{C}}^\infty(X)$. The non-surjectivity follows from the simple observation that if ρ is a total boundary defining function then $\exp(1/\rho)$ is never in $\mathcal{C}^{-\infty}(X)$, unless of course $\partial X = \emptyset$. ☐

More importantly the 'extendibility' of the elements of $\mathcal{C}^{-\infty}(X)$ is made evident by the following result.

LEMMA 3.3.2. *If $\tilde{X}$ is compact without boundary and extends X then the restriction map*

$$R : \mathcal{C}^{-\infty}(\tilde{X}) \longrightarrow \mathcal{C}^{-\infty}(X), \tag{3.3.6}$$

given by $Ru(\phi) = u(\phi)$ for all $\phi \in \dot{\mathcal{C}}^{\infty}(X;\Omega X) \hookrightarrow \mathcal{C}^{\infty}(\tilde{X};\Omega\tilde{X})$, is surjective.

In fact there is an intrinsic version of this which, following Proposition 3.2.1, is even stronger.

PROPOSITION 3.3.1. *The restriction $R : \dot{\mathcal{C}}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X)$ is surjective with null space consisting of the supported distributions with support contained in ∂X.*

PROOF. The characterization of the null space follows from the density, already noted,

$$\mathcal{C}^{\infty}_c(X^\circ;\Omega X) \hookrightarrow \dot{\mathcal{C}}^{\infty}(X;\Omega X), \tag{3.3.7}$$

since from this it follows that

$$u(\phi) = 0 \ \forall \ \phi \in \mathcal{C}^{\infty}_c(X^\circ;\Omega X) \Longleftrightarrow u(\phi) = 0 \ \forall \ \phi \in \dot{\mathcal{C}}^{\infty}(X;\Omega X). \tag{3.3.8}$$

Thus it remains to check the surjectivity of R. The first step in extending $u \in \mathcal{C}^{-\infty}(X)$ to $\tilde{u} \in \dot{\mathcal{C}}^{-\infty}(X)$ is to prove Lemma 3.3.2 i.e. to extend it to $v \in \mathcal{C}^{-\infty}(\tilde{X})$. Since $\dot{\mathcal{C}}^{\infty}(X;\Omega X) \subset \mathcal{C}^{\infty}(\tilde{X};\Omega\tilde{X})$ is closed this follows by a direct application of the Hahn-Banach theorem. Thus we need to show that if $v \in \mathcal{C}^{-\infty}(\tilde{X})$ there exists $v' \in \dot{\mathcal{C}}^{-\infty}(X)$ with $Rv = v'$, R being restriction to X.

This latter problem is local. By use of a partition of unity we are reduced to the case $X = \mathbb{R}^n_k$ with $\operatorname{supp}(v) \subset \{|x| < 1\}, \tilde{X} = \mathbb{R}^n$. Then we use the Schwartz structure theorem to write

$$v = \sum_{\text{finite}} D^\alpha v_\alpha, \quad v_\alpha \in \mathcal{C}^0(\mathbb{R}^n),\ \operatorname{supp}(v_\alpha) \subset \{|x| < 1\}. \tag{3.3.9}$$

Define

$$v_1(\phi) = \sum_{\phi}(-1)^{|\alpha|} \int\limits_{\mathbb{R}^n_k} v_\alpha D^\alpha \phi dx \quad \forall \ \phi|dx| \in \mathcal{C}^{\infty}(\mathbb{R}^n_k, \Omega\mathbb{R}^n_k). \tag{3.3.10}$$

Certainly $v_1 \in \dot{\mathcal{C}}^{-\infty}(\mathbb{R}^n_k)$ has support in $\{|x| < 1\}$. Moreover if $\phi \in \dot{\mathcal{C}}^{\infty}_c(\mathbb{R}^n_k)$ then $v(\phi) = v_1(\phi)$, proving the proposition. □

EXERCISE 3.3.18. Let X be a compact manifold with boundary. Show that the space $\left\{u \in \dot{\mathcal{C}}^{-\infty}(X); \operatorname{supp}(u) \subset \partial X\right\}$ can be identified (but not naturally) with the sum of countably many copies of $\mathcal{C}^{-\infty}(\partial X)$, i.e.

$$\left\{f = (f_i)_{i=0}^{\infty}; f_i \in \mathcal{C}^{-\infty}(\partial X) \ \forall \ i = 0, \dots, \infty,\ f_i = 0 \text{ for } i > I(f)\right\}.$$

3.4. Action of differential operators

The action of a $\mathcal{C}^{\infty}$ vector field on $\mathcal{C}^{\infty}(X)$ follows directly from the definition of the tangent bundle in §2.3:

$$Vf(p) = df \cdot V(p) \quad V \in \mathcal{V}_E(X),\ f \in \mathcal{C}^{\infty}(X). \tag{3.4.1}$$

This action, together with the ring structure of $\mathcal{C}^\infty(X)$, extends to give the basic action of a differential operator:

$$\operatorname{Diff}^*(X) \ni P : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X). \tag{3.4.2}$$

By restriction P also acts on the corresponding supported space.

$$P : \dot{\mathcal{C}}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X). \tag{3.4.3}$$

The dual, or transpose operators, to (3.4.2) and (3.4.3), are therefore

$$\begin{cases} {}^tP : \dot{\mathcal{C}}^{-\infty}(X;\Omega X) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X;\Omega X) \\ {}^tP : \mathcal{C}^{-\infty}(X;\Omega X) \longrightarrow \mathcal{C}^{-\infty}(X;\Omega X). \end{cases} \tag{3.4.4}$$

Integration by parts in the formula

$$\int_X P\phi \cdot \psi = \int_X \phi \cdot {}^tP\psi \quad \phi \in \mathcal{C}^\infty(X), \psi \in \dot{\mathcal{C}}^\infty(X;\Omega X) \tag{3.4.5}$$

shows that restriction to $\dot{\mathcal{C}}^\infty(X;\Omega X)$ in the first map in (3.4.4), or to $\mathcal{C}^\infty(X;\Omega X)$ in the second, gives a differential operator

$$\begin{cases} {}^tP : \dot{\mathcal{C}}^{\infty}(X;\Omega X) \longrightarrow \dot{\mathcal{C}}^{\infty}(X;\Omega X) \\ {}^tP : \mathcal{C}^{\infty}(X;\Omega X) \longrightarrow \mathcal{C}^{\infty}(X;\Omega X). \end{cases} \tag{3.4.6}$$

However some care needs to be taken with these formulæ, especially as regards the pair of inclusions

$$\begin{tikzcd} & \dot{\mathcal{C}}^{-\infty}(X) \arrow[dd] \\ \mathcal{C}^\infty(X) \arrow[ur] \arrow[dr] & \\ & \mathcal{C}^{-\infty}(X) \end{tikzcd} \tag{3.4.7}$$

From (3.4.2) and (3.4.6) P acts on each space. However if $()_c$ (cutoff at the boundary) denotes the injection $\mathcal{C}^\infty(X) \hookrightarrow \dot{\mathcal{C}}^{-\infty}(X)$ then, in general, $(Pu)_c \neq P(u)_c$, $u \in \mathcal{C}^\infty(X)$. However

$$P(u)_c - (Pu)_c \in \dot{\mathcal{C}}^{-\infty}(X) \text{ has support in } \partial X. \tag{3.4.8}$$

This distribution arises through differentiation across the 'jumps' of u at the boundary. An expression for the difference (3.4.8) in terms of the boundary values of u will be obtained below ('formule des sauts').

Exercise 3.4.19. Show that if $P \in \operatorname{Diff}^*(X)$ then the diagram, with vertical arrows given by restriction:

$$\begin{tikzcd} \dot{\mathcal{C}}^{-\infty}(X) \arrow[r, "P"] \arrow[d] & \dot{\mathcal{C}}^{-\infty}(X) \arrow[d] \\ \mathcal{C}^{-\infty}(X) \arrow[r, "P"'] & \mathcal{C}^{-\infty}(X) \end{tikzcd} \tag{3.4.9}$$

commutes.

EXERCISE 3.4.20. Show that if $P \in \mathrm{Diff}^m(X)$ then

$$P(u)_c = (Pu)_c \; \forall u \in \mathcal{C}^\infty(X) \Longleftrightarrow P \in \mathrm{Diff}^m_b(X). \tag{3.4.10}$$

The following is a simple example to show the effect of differentiation across the boundary. If $P = D_x$ with $X = [0,1]$ then

$$D_x(u)_c = (D_x u)_c - iu(0)\delta(x) + iu(1)\delta(x-1) \quad \forall\, u \in \mathcal{C}^\infty([0,1]). \tag{3.4.11}$$

3.5. Sobolev spaces

To avoid a discussion of coordinate invariance we shall assume familiarity with the Sobolev spaces $\mathcal{H}^s(\tilde X)$ for all $s \in \mathbb{R}$, where $\tilde X$ is a compact manifold without boundary. Then if X is a compact manifold with corners we can suppose $X \hookrightarrow \tilde X$ is extended across the boundary to a compact boundaryless manifold. Then define, with $\tilde u$ as in (3.2.8),

$$\begin{cases} \dot{\mathcal{H}}^s(X) &= \{u \in \dot{\mathcal{C}}^{-\infty}(X); \tilde u \in \mathcal{H}^s(\tilde X)\} \\ \mathcal{H}^s(X) &= \{u \in \mathcal{C}^{-\infty}(X); \exists\; v \in \mathcal{H}^s(\tilde X) \text{ s.t. } u = v|_X\}. \end{cases} \tag{3.5.1}$$

That these spaces are independent of the extension chosen, and invariant under diffeomorphisms of X, follows from the fact that any diffeomorphism of X extends to a neighbourhood of X in any two extensions.

The Schwartz structure theorem takes the form

$$\begin{cases} \dot{\mathcal{C}}^{-\infty}(X) = \bigcup\limits_s \dot{\mathcal{H}}^s(X) \\ \mathcal{C}^{-\infty}(X) = \bigcup\limits_s \mathcal{H}^s(X). \end{cases} \tag{3.5.2}$$

Similarly the Sobolev embedding theorem implies that

$$\begin{cases} \dot{\mathcal{C}}^{\infty}(X) = \bigcap\limits_s \dot{\mathcal{H}}^s(X) \\ \mathcal{C}^{\infty}(X) = \bigcap\limits_s \mathcal{H}^s(X). \end{cases} \tag{3.5.3}$$

It is straightforward to check many of the usual properties of Sobolev spaces. Thus, in terms of the appropriate actions (3.4.6)

$$\begin{cases} P : \dot{\mathcal{H}}^s(X) \longrightarrow \dot{\mathcal{H}}^{s-m}(X) \;\forall\; P \in \mathrm{Diff}^m(X),\; s \in \mathbb{R} \\ P : \mathcal{H}^s(X) \longrightarrow \mathcal{H}^{s-m}(X) \;\forall\; P \in \mathrm{Diff}^m(X),\; s \in \mathbb{R}. \end{cases} \tag{3.5.4}$$

Then for any $s \in \mathbb{R}$ and any $m \in \mathbb{N}$

$$\begin{cases} u \in \mathcal{H}^s \Longleftrightarrow u = \sum\limits_{j=1}^N P_j u_j,\; P_j \in \mathrm{Diff}^m(X),\; u_j \in \mathcal{H}^{m+s}(X) \\ u \in \dot{\mathcal{H}}^s \Longleftrightarrow u = \sum\limits_{j=1}^N P_j u_j,\; P_j \in \mathrm{Diff}^m(X),\; u_j \in \dot{\mathcal{H}}^{m+s}(X). \end{cases} \tag{3.5.5}$$

Conversely for any $s \in \mathbb{R}$ and any $m \in \mathbb{N}$

$$\begin{cases} u \in \mathcal{C}^{-\infty}(X),\; Pu \in \mathcal{H}^s(X) \;\forall\; P \in \mathrm{Diff}^m(X) \Longrightarrow u \in \mathcal{H}^{s+m}(X) \\ u \in \dot{\mathcal{C}}^{-\infty}(X),\; Pu \in \dot{\mathcal{H}}^s(X) \;\forall\; P \in \mathrm{Diff}^m(X) \Longrightarrow u \in \dot{\mathcal{H}}^{s+m}(X). \end{cases} \tag{3.5.6}$$

There is no problem in defining the Sobolev spaces of section of any $\mathcal{C}^\infty$ vector bundle G over X.

PROPOSITION 3.5.1. *For every $s \in \mathbb{R}$ there is a natural continuous pairing*

$$\mathcal{H}^s(X) \times \dot{\mathcal{H}}^{-s}(X;\Omega X) \longrightarrow \mathbb{C} \tag{3.5.7}$$

given by extension of the pairing of $\mathcal{C}^\infty(X) \hookrightarrow \mathcal{H}^s(X)$, which is always dense, and $\dot{\mathcal{H}}^{-s}(X;\Omega X) \subset \dot{\mathcal{C}}^{-\infty}(X;\Omega X)$. This pairing gives a natural duality

$$\left[\mathcal{H}^{-s}(X)\right]' \cong \dot{\mathcal{H}}^s(X;\Omega X). \tag{3.5.8}$$

The restriction map of Proposition 3.3.1 gives

$$\dot{\mathcal{H}}^s(X) \longrightarrow \mathcal{H}^s(X) \tag{3.5.9}$$

which is surjective if $s \le \frac{1}{2}$ and injective if $s \ge -\frac{1}{2}$.

PROOF. We refer this result back to the case of a compact manifold without boundary, where it is standard. Thus, let $X \hookrightarrow \tilde{X}$ be an embedding of X in a compact manifold without boundary, $\tilde{X}$. The definitions of the spaces of supported and extendible Sobolev spaces show that

$$\begin{gathered}\dot{\mathcal{H}}^s(X) \cong \{u \in \mathcal{H}^s(\tilde{X}); \operatorname{supp}(u) \subset X\},\\ \mathcal{H}^s(X;\Omega X) \cong \mathcal{H}^s(\tilde{X};\Omega X)/\{u \in \mathcal{H}^s(\tilde{X};\Omega X); \operatorname{supp}(u) \subset \tilde{X} \setminus X^\circ\}.\end{gathered} \tag{3.5.10}$$

The density of $\mathcal{C}^\infty_c(X^\circ)$ in $\dot{\mathcal{H}}^s(X)$ and of $\mathcal{C}^\infty_c(\tilde{X} \setminus X^\circ;\Omega X)$ in

$$\{u \in \mathcal{H}^s(\tilde{X};\Omega X); \operatorname{supp}(u) \subset \tilde{X} \setminus X^\circ\}$$

therefore gives the non-degeneracy of (3.5.7) and hence the duality (3.5.8).

An element u in the null space of (3.5.9) can be identified with $u \in \mathcal{H}^s(\tilde{X})$ with $\operatorname{supp}(u) \subset \partial X$. It is a standard consequence of a result of Schwartz that any element of $\mathcal{H}^s(\mathbb{R}^n)$ $s \ge -\frac{1}{2}$ with support in a subspace of positive codimension is necessarily zero. Localizing near an interior point of a hypersurface face of $\partial X \subset \tilde{X}$ shows that u has support in $\partial^2 X$. Proceeding to localize near an interior point of successively higher codimension boundary faces it follows that $u = 0$. This proves the injectivity of (3.5.9) for $s \ge -\frac{1}{2}$.

The dual of the restriction map is the restriction map

$$\dot{\mathcal{H}}^{-s}(X;\Omega X) \longrightarrow \mathcal{H}^{-s}(X;\Omega X).$$

Since the density factor can be canceled to give (3.5.9) again, the surjectivity for $s \le \frac{1}{2}$ follows by duality. This proves the proposition. □

In particular $\dot{\mathcal{H}}^s(X) \cong \mathcal{H}^s(X)$ for $-\frac{1}{2} \le s \le \frac{1}{2}$.

We also recall certain results on boundary values of functions in Sobolev spaces.

PROPOSITION 3.5.2. *If $F \in M_l X$ is a boundary face of a compact manifold with corners then there is a surjective restriction map*

$$\mathcal{H}^s(X) \longrightarrow \mathcal{H}^{s-\frac{1}{2}l}(F) \quad \forall\ s > \frac{1}{2}l. \tag{3.5.11}$$

PROOF. Since $v \in \mathcal{H}^s(X)$ can be represented by $\tilde{v} \in \mathcal{H}^s(\tilde{X})$ if $\tilde{X}$ is any compact manifold without boundary in which X is embedded as the closure of an open set, this result is a consequence of the standard restriction theorem for Sobolev functions to submanifolds. □

3.6. Operations

The most basic operation is the pull-back of smooth functions, if $F : X \longrightarrow Y$ is $\mathcal{C}^\infty$ then

$$F^* : \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X),\ F^*\phi = \phi \cdot F\ \forall\ \phi \in \mathcal{C}^\infty(Y) \tag{3.6.1}$$

Indeed this is equivalent to the smoothness of F. Dually we get the basic operation of push-forward of distibutional densities – recall that we assume manifolds to be compact unless otherwise stated –

$$F_* : \dot{\mathcal{C}}^{-\infty}(X;\Omega X) \longrightarrow \dot{\mathcal{C}}^{-\infty}(Y;\Omega Y) \tag{3.6.2}$$

defined by

$$F_*(u)(\phi) = u(F^*\phi) = u(\phi \cdot F) \quad \forall\ \phi \in \mathcal{C}^\infty(Y). \tag{3.6.3}$$

A fundamental pair of (closely related) questions is the extent to which F^* in (3.6.1) can be extended by continuity to larger spaces and the regularity properties of (3.6.2), i.e. the extent to which it can be restricted to smaller spaces (in both domain and range of course!) We shall discuss a simple result of this type in the next section. First however we discuss a geometric generalization of (3.6.1) and (3.6.2).

If G is a $\mathcal{C}^\infty$ vector bundle over Y then the pull-back over X, F^*G, is defined as follows. The total space of the bundle is

$$F^*G = \left\{(p,\xi); p \in X,\ \xi \in G_{F(p)}\right\}. \tag{3.6.4}$$

Thus the fibre over $p \in X$ is isomorphic to the fibre, $G_{F(p)}$, of G over the image of p. The projection to X just takes (p,ξ) to p. If

$$\pi^{-1}(\Omega) \overset{\Phi}{\rightarrow\!\!\longleftrightarrow} \Omega \times \mathbb{R}^n$$

is a local trivialization of G over $\Omega \ni F(p)$ then for $\Omega' \ni p$ a small neighbourhood

$$\pi^{-1}(\Omega') \ni (p,\xi) \longmapsto (p, \Phi(F(p),\xi) \in \Omega \times \mathbb{R}^n \tag{3.6.5}$$

is a local trivialization of F^*G. It is straightforward to check that the transition functions are $\mathcal{C}^\infty$ so F^*G is always a $\mathcal{C}^\infty$ vector bundle over X.

The pull-back operation can be generalized to give

$$F^* : \mathcal{C}^\infty(Y;G) \longrightarrow \mathcal{C}^\infty(X;F^*G),\ F^*(u)(p) = (p,\xi(F(p)))\ \forall\ (p,\xi) \in F^*G. \tag{3.6.6}$$

This reduces to a product with (3.6.1) in case $G = Y \times \mathbb{R}^n$. As we shall see below in some cases the bundle F^*G can be identified with some other, more intrinsic, bundle over X.

Of course by duality from (3.6.6) we always have a corresponding push-forward operation on distributional sections of the appropriate bundle:

$$F_* : \dot{\mathcal{C}}^{-\infty}(X;\Omega \otimes F^*G) \longrightarrow \dot{\mathcal{C}}^{-\infty}(Y;\Omega \otimes G). \tag{3.6.7}$$

3.7. Fibrations

The case in which (3.6.1) extends to the space of all supported distributions is that of a fibration. If $F : X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map the vector fields $V \in \mathcal{V}_E(X)$ and $W \in \mathcal{V}_E(Y)$ are said to be F-related, written

$$F_*V = W \text{ if } F_*V_p = W_{F(p)} \quad \forall\ p \in X. \tag{3.7.1}$$

For a general $\mathcal{C}^\infty$ map F this determines neither V nor W however:

PROPOSITION 3.7.1. *If $F : X \longrightarrow Y$ is a fibration then there is a continuous (lifting) map*

$$L : \mathcal{V}_E(Y) \longrightarrow \mathcal{V}_E(X) \tag{3.7.2}$$

such that $F_*(L(V)) = V$, i.e. $L(V)$ and V are F-related, and $L(V)$ is tangent to each component of $\partial X \setminus F^{-1}(\partial Y)$

PROOF. Locally in Y this follows directly from the local product decomposition of X and F. The global lifting can be constructed from these local maps by use of a partition of unity on Y. □

The important (and obvious) property of F-related vector fields is

$$F_*V = W \Longrightarrow F^*(W\phi) = VF^*\phi \quad \forall\ \phi \in \mathcal{C}^\infty(Y). \tag{3.7.3}$$

Indeed this follows at each point $p \in X$ from the definitions:

$$F^*(W\phi)(p) = W\phi(F(p)) = W_{F(p)} \cdot d\phi(F(p)) = V_p \cdot dF^*\phi.$$

Using this identity we can discuss the lift of distributions.

PROPOSITION 3.7.2. *If $F : X \longrightarrow Y$ is a fibration of compact manifolds then*

$$\begin{aligned} F_* &: \dot{\mathcal{C}}^\infty(X;\Omega X) \longrightarrow \dot{\mathcal{C}}^\infty(Y;\Omega Y)\\ F_* &: \mathcal{C}^\infty(X;\Omega X) \longrightarrow \mathcal{C}^\infty(Y;\Omega Y)\end{aligned} \tag{3.7.4}$$

and hence the pull-back of smooth functions extends by continuity to

$$\begin{aligned} F^* &: \mathcal{C}^{-\infty}(Y) \longrightarrow \mathcal{C}^{-\infty}(X)\\ F^* &: \dot{\mathcal{C}}^{-\infty}(Y) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X).\end{aligned} \tag{3.7.5}$$

PROOF. By the continuity of the pull-back operation on $\mathcal{C}^\infty$, there is some S, depending only on F, such that

$$F_*(\dot{\mathcal{H}}^0(X;\Omega X)) \subset \dot{\mathcal{H}}^S(Y;\Omega Y). \tag{3.7.6}$$

Suppose that $W \in \mathcal{V}_E(Y)$. Then W^t, the formal transpose, acts on (supported) densities:

$$W^t : \dot{\mathcal{C}}^{-\infty}(Y;\Omega Y) \longrightarrow \dot{\mathcal{C}}^{-\infty}(Y;\Omega Y). \tag{3.7.7}$$

From Proposition 3.7.1 we can lift W to a $\mathcal{C}^\infty$ vector field on X. Then

$$W^t[F_*\psi] = F_*[V^t\psi] \in \dot{\mathcal{H}}^S(Y;\Omega Y) \quad \forall\ \psi \in \dot{\mathcal{C}}^\infty(X;\Omega X). \tag{3.7.8}$$

Obviously this identity can be iterated, so that for any $\mathcal{C}^\infty$ differential operator on Y, $P^t[F_*\psi]$ has the fixed Sobolev regularity S. This implies that $F_*\psi$ is $\mathcal{C}^\infty$, proving the first part of (3.7.4). The second part of (3.7.4) follows by the same argument since the difference between the action of W on $\mathcal{C}^\infty(X;\Omega X)$ as a supported and an extendible distribution pushes forward to have support in ∂Y. By duality (3.7.5) follows, proving the proposition. □

EXERCISE 3.7.21. Show that if $F : X \longrightarrow Y$ is a fibration of compact manifolds and G is a vector bundle over Y then the lifting map (4.7.5) extends by continuity to

$$\begin{aligned} F^* &: \mathcal{C}^{-\infty}(Y;G) \longrightarrow \mathcal{C}^{-\infty}(X;F^*G)\\ F^* &: \dot{\mathcal{C}}^{-\infty}(Y;G) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X;F^*G).\end{aligned} \tag{3.7.9}$$

3.8. Intermediate spaces

We have defined the spaces of supported and extendible distributions and on a manifold with boundary these are the only two basic spaces. However on a manifold with corners there are intermediate spaces, supported across some boundary hypersurfaces and extendible across the others. These spaces are particularly useful in the discussion of linear operators.

Let X be a compact manifold with corners, with M_1X the set of boundary hypersurfaces. If $A \subset M_1X$ set

$$\rho_A = \prod_{H \in A} \rho_H \tag{3.8.1}$$

where ρ_H is a defining function for H. Then set

$$\dot{\mathcal{C}}^\infty_A(X) = \bigcap_{k \in \mathbb{N}} \rho_A^k \mathcal{C}^\infty(X). \tag{3.8.2}$$

This is simply the subspace of $\mathcal{C}^\infty(X)$ consisting of the function which vanish to infinite order at all boundary hypersurfaces in A. We also use the 'dual' notation

$$\mathcal{C}^\infty_B(X) = \dot{\mathcal{C}}^\infty_A(X) \text{ if } A = M_1X \setminus B \text{ for } B \subset M_1X. \tag{3.8.3}$$

Thus

$$\begin{aligned} \dot{\mathcal{C}}^\infty_{M_1X}(X) &= \mathcal{C}^\infty_\emptyset(X) = \dot{\mathcal{C}}^\infty(X), \\ \dot{\mathcal{C}}^\infty_\emptyset(X) &= \mathcal{C}^\infty_{M_1X}(X) = \mathcal{C}^\infty(X). \end{aligned} \tag{3.8.4}$$

These spaces are also well-defined for sections of any $\mathcal{C}^\infty$ vector bundle. For instance

$$\begin{aligned} \dot{\mathcal{C}}^\infty_A(X;G) &= \dot{\mathcal{C}}^\infty_A(X) \cdot \mathcal{C}^\infty(X;G) \\ &= \{u \in \mathcal{C}^\infty(X;G); u = \sum_{\text{finite}} \phi_i u_i, \phi_i \in \dot{\mathcal{C}}^\infty_A(X), u_i \in \mathcal{C}^\infty(X;G)\}. \end{aligned} \tag{3.8.5}$$

The topology on $\dot{\mathcal{C}}^\infty_A(X;G)$ is that induced from $\mathcal{C}^\infty(X;G)$, of which it is a closed subspace.

This leads us to define the corresponding intermediate spaces of distributions by:

$$\dot{\mathcal{C}}^{-\infty}_A(X;G) = [\mathcal{C}^\infty_A(X;\Omega X \otimes G^*]' . \tag{3.8.6}$$

As noted in §1.11 we can double any compact manifold with corners across any collection of boundary hypersurfaces. Let 2_AX be the compact manifold obtained by doubling across the hypersurfaces in $A \subset M_1X$. The proof of Proposition 3.3.1 extends easily to show that

Proposition 3.8.1. *For any $A \subset M_1X$, X a compact manifold with corners,*

$$\dot{\mathcal{C}}^{-\infty}_A(X) \cong \big\{u \in \mathcal{C}^{-\infty}(2_AX); \operatorname{supp}(u) \subset X\big\} \tag{3.8.7}$$

and the restriction map

$$\dot{\mathcal{C}}^{-\infty}_A(X) \longrightarrow \dot{\mathcal{C}}^{-\infty}_{A'}(X),\ A' \subset A \subset M_1X \tag{3.8.8}$$

is always surjective.

Consider a b-map between $\mathcal{C}^\infty$ manifolds with corners, $F : X \longrightarrow Y$. From the definition, (1.12.2), if $H' \in M_1Y$ is any boundary hypersurface with $F^{-1}(H') \neq X$ then

$$F^{-1}(H') = \bigcup_{H \in A} H, \ A \subset M_1X. \tag{3.8.9}$$

Thus F defines a relation between M_1Y and M_1X which we write in the form

$$\begin{gathered} F^{-1}(B) = A \text{ if } B \subset M_1Y \text{ and } A \subset M_1X \\ \text{satisfy } F^{-1}\left(\bigcup_{H' \in B} H'\right) = \bigcup_{H \in A} H. \end{gathered} \tag{3.8.10}$$

Lemma 3.8.1. *If $F : X \longrightarrow Y$ is a b-map and $B \subset M_1Y$ then*

$$F^* : \dot{\mathcal{C}}^\infty_B(Y) \longrightarrow \dot{\mathcal{C}}^\infty_A(X) \text{ if } A = F^{-1}(B) \subset M_1X. \tag{3.8.11}$$

Proof. This follows from the definition, (3.8.10), and (1.12.2). Thus if

$$\rho'_B = \prod_{H' \in B} \rho'_H$$

is a product of defining functions for the hypersurfaces in B then from (1.12.2)

$$F^*\rho'_B = \alpha\rho_A, \ \alpha \in \mathcal{C}^\infty(X) \tag{3.8.12}$$

where α may vanish at various of the hypersurfaces in A. Thus if $\phi \in \mathcal{C}^\infty(Y)$ then

$$F^*[(\rho'_B)^k\phi] \in \rho_A^k\mathcal{C}^\infty(X)$$

which proves (3.8.11). □

Since the map (3.8.11) is clearly continuous we find by duality that

$$\begin{gathered} F_* : \dot{\mathcal{C}}^{-\infty}_A(X; \Omega X) \longrightarrow \dot{\mathcal{C}}^{-\infty}_B(Y; \Omega Y) \\ \text{if } F^{-1}(M_1Y \setminus B) \cup A = M_1X. \end{gathered} \tag{3.8.13}$$

More generally for any vector bundle G, over Y,

$$F_* : \dot{\mathcal{C}}^{-\infty}_A(X; F^*G \otimes \Omega X) \longrightarrow \dot{\mathcal{C}}^{-\infty}_B(Y; G \otimes \Omega Y) \tag{3.8.14}$$

under the same assumptions.

Note from (3.8.11) that multiplication gives

$$\dot{\mathcal{C}}^\infty_A(X) \times \mathcal{C}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty_A(X) \text{ for any } A \subset M_1X. \tag{3.8.15}$$

It follows that if $u \in \mathcal{C}^{-\infty}(X)$ and $\phi \in \dot{\mathcal{C}}^\infty_A(X)$ then the pairing with the product

$$\langle \phi u, \psi \rangle = \langle u, \phi\psi \rangle$$

is well-defined for any $\psi \in \mathcal{C}^\infty_A(X; \Omega X)$. Thus (3.8.15) extends by continuity to a bilinear, multiplication, map

$$\dot{\mathcal{C}}^\infty_A(X) \times \mathcal{C}^{-\infty}(X) \longrightarrow \dot{\mathcal{C}}^{-\infty}_A(X). \tag{3.8.16}$$

3.9. Schwartz kernel theorem

Schwartz' kernel theorem identifies continuous linear operators with distributions. The type of operator in which we are interested is

$$Q : \dot{\mathcal{C}}^\infty(Y) \longrightarrow \mathcal{C}^{-\infty}(X), \tag{3.9.1}$$

for two compact manifolds with corners; in general it also acts between sections of vector bundles.

Let $\pi_L : X \times Y \longrightarrow X$ and $\pi_R : X \times Y \longrightarrow Y$ be the two projections from the product. Suppose that $k \in \mathcal{C}^{-\infty}(X \times Y; \pi_R^*\Omega)$ is an extendible 'right density.' Then if $\phi \in \dot{\mathcal{C}}^\infty(Y)$, $\pi_R^*\phi \in \dot{\mathcal{C}}^\infty_{\partial Y}(X \times Y)$ where, with some slight abuse of notation,

$$\partial Y = \{X \times H; H \in M_1 Y\} \subset M_1(X \times Y). \tag{3.9.2}$$

Then from (3.8.16)

$$\pi_R^*\phi \cdot k \in \dot{\mathcal{C}}^{-\infty}_{\partial Y}(X \times Y; \pi_R^*\Omega). \tag{3.9.3}$$

Recall that for a product manifold the product of a left and a right density is canonically a density:

$$\pi_L\Omega \otimes \pi_R^*\Omega \equiv \Omega. \tag{3.9.4}$$

If we apply the identification (3.9.4) to (3.8.14) with $G = \Omega^{-1}X$ we get a push-forward map

$$(\pi_L)_* : \dot{\mathcal{C}}^{-\infty}_A(X \times Y; \pi_R^*\Omega) \longrightarrow \dot{\mathcal{C}}^{-\infty}_B(X) \tag{3.9.5}$$

provided $\pi_L^{-1}(\partial X \setminus B) \cup A = M_1(X \times Y)$. For $A = \partial Y$ this gives

$$(\pi_L)_* : \dot{\mathcal{C}}^{-\infty}_{\partial Y}(X \times Y; \pi_R^*\Omega) \longrightarrow \mathcal{C}^{-\infty}(X). \tag{3.9.6}$$

Then from (3.9.3) we see that

$$(\pi_L)_*[\pi_R^*\phi \cdot k] \in \mathcal{C}^{-\infty}(X). \tag{3.9.7}$$

THEOREM 3.9.1. *(Schwartz kernel) For any compact manifolds with corners X and Y the map*

$$\begin{gathered}\mathcal{C}^{-\infty}(X \times Y; \pi_R^*\Omega) \ni k \longmapsto Q_k \\ Q_k : \dot{\mathcal{C}}^\infty(Y) \longrightarrow \mathcal{C}^{-\infty}(X),\ Q_k(\phi) = (\pi_L)_*[\pi_R^*\phi \cdot k]\end{gathered} \tag{3.9.8}$$

is a 1-1 *map onto all the continuous linear operators* (3.9.1).

PROOF. We simply reduce this to the case of compact manifolds without boundary (see [**3**].) First, given $k \in \mathcal{C}^{-\infty}(X \times Y; \pi_R^*\Omega)$ the continuity of Q_k is clear so the map $k \longmapsto Q_k$ is well-defined. The representations (3.5.3), of $\dot{\mathcal{C}}^\infty(Y)$ as a projective limit of Sobolev spaces, and (3.5.2) of $\mathcal{C}^{-\infty}(X)$, as an inductive limit of Sobolev spaces, combined with a standard argument using the Banach-Steinhaus theorem, shows that any continuous linear operator (3.9.1) actually extends to a continuous linear map

$$Q : \dot{\mathcal{H}}^s(Y) \longrightarrow \mathcal{H}^S(X) \tag{3.9.9}$$

for some $s, S \in \mathbb{R}$. Then we can write

$$\begin{gathered}\dot{\mathcal{H}}^s(Y) = \{\phi \in \mathcal{H}^s(2Y); \operatorname{supp}(\phi) \subset Y\} \\ \mathcal{H}^S(X) = \{u \in \mathcal{H}^S(2X); \operatorname{supp}(u) \cap X^\circ = \emptyset\}^\perp\end{gathered} \tag{3.9.10}$$

with respect to Hilbert space structures. Thus Q extends, as zero, to

$$Q' : \mathcal{H}^s(2Y) \longmapsto \mathcal{H}^S(2X). \tag{3.9.11}$$

We apply the Schwartz kernel theorem for compact manifolds without boundary to deduce the existence of

$$\begin{gathered} k' \in \mathcal{C}^{-\infty}(2X \times 2Y; \pi_R^*\Omega) \text{ s.t.} \\ Q'\phi = (\pi_L)_*[\pi_R^*\phi \cdot k']. \end{gathered} \tag{3.9.12}$$

It is straightforward to check that

$$k = k'_{|X\times Y} \Longrightarrow Q = Q_k. \tag{3.9.13}$$

The injectivity of the map $k \longmapsto Q_k$ is easily shown, so the theorem is proved. $\square$

Of course this result has various extensions which can be readily deduced from it. If G and H are vector bundles over X and Y then $Q : \dot{\mathcal{C}}^\infty(Y; H) \longrightarrow \mathcal{C}^{-\infty}(X; G)$ is identified with its kernel $k \in \mathcal{C}^{-\infty}(X \times Y; \pi_L^*H \otimes \pi_R^*G^* \otimes \pi_R^*\Omega)$.

Exercise 3.9.22. Show that if $A \subset M_1Y$ and $B \subset M_1X$ then continuous linear operators

$$Q : \dot{\mathcal{C}}^\infty_A(Y) \longrightarrow \mathcal{C}^{-\infty}_B(X) \tag{3.9.14}$$

can be identified with their Schwartz kernels

$$\begin{gathered} k \in \mathcal{C}^{-\infty}_D(X \times y; \pi_R^*\Omega), \\ D = \{H' \times Y; H' \in B\} \cup \{X \times H; H \in A\}. \end{gathered} \tag{3.9.15}$$

There are other general types of linear operators. For instance we could consider, in place of (3.9.1), continuous linear maps

$$A : \mathcal{C}^\infty(Y) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X). \tag{3.9.16}$$

For these the kernel theorem becomes

$$A_k \longleftrightarrow k_Q \in \dot{\mathcal{C}}^{-\infty}(X \times Y; \pi_R^*\Omega). \tag{3.9.17}$$

This correspondence is defined by noting that if $k \in \dot{\mathcal{C}}^{-\infty}(X \times Y; \pi_R^*\Omega)$ and $\psi \in \mathcal{C}^\infty(Y)$ then

$$\pi_R^*\psi \cdot k \in \dot{\mathcal{C}}^{-\infty}(X \times Y; \pi_R^*\Omega). \tag{3.9.18}$$

By (3.7.9) this pushes forward to $Q\psi \in \dot{\mathcal{C}}^{-\infty}(Y)$.