CHAPTER 2

Boundary-fibration structures

The notion of a boundary-fibration structure on a manifold with corners is fundamental in the sequel. Once chosen it fixes the class of geometric entities, such as metrics, to be admitted. A boundary-fibration structure specifies a particular 'degeneracy' at the boundary of the manifold and a corresponding class of (elliptic) differential operators. It also indicates the geometry to be expected in the inversion of these operators. The inclusion of the 'fibration' part allows us to examine transition problems where, as some parameter approaches a limiting value, the behaviour changes type. Various examples are discussed in §2.6 and §2.7.

The notion of a boundary-fibration structure as defined here is based on the existence of appropriate local homogeneities for a Lie algebra of vector fields which must also satisfy conditions of constancy of ranks.

2.1. Preliminary examples

To prepare the way for the general discussion below we first consider two examples, essentially those from the beginning of Chapter 1. These illustrate not only the way that manifolds with corners (in these cases with boundary) arise but also that they typically have Lie algebras of vector fields associated to them in the process.

First consider $\mathbb{R}^n$ with its linear structure, so we are interested in the constant coefficient vector fields $D_j = -i\partial/\partial x_j$, $j = 1, \dots, n$. We shall consider again the compactification of $\mathbb{R}^n$ through stereographic projection:

$$\text{SP}(x) = \omega = \frac{(x,1)}{(1+|x|^2)^{\frac{1}{2}}} \in \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}. \tag{2.1.1}$$

In fact this is clearly an isomorphism onto the interior of the upper half sphere

$$\text{SP}\colon \mathbb{R}^n \longleftrightarrow \mathbb{S}^{n\,\circ}_+, \ \mathbb{S}^n_+ = \{(\omega_1, \dots, \omega_{n+1}) \in \mathbb{S}^n; \omega_{n+1} \ge 0\}. \tag{2.1.2}$$

We want to consider the image under SP of the constant coefficient vector fields

$$\text{SP}_*(D_j) = V_j. \tag{2.1.3}$$

LEMMA 2.1.1. *The vector fields V_j are $\mathcal{C}^\infty$ on $\mathbb{S}^n_+$ and are of the form*

$$V_j = \rho W_j, \ W_j \in \mathcal{V}_E(\mathbb{S}^n_+) = \mathcal{C}^\infty(\mathbb{S}^n_+; T\mathbb{S}^n_+) \tag{2.1.4}$$

with the W_j tangent to $\partial\mathbb{S}^n_+ \cong \mathbb{S}^{n-1}$ and where $\rho = \omega_{n+1}$ is a boundary defining function. Moreover any $\mathcal{C}^\infty$ vector field on $\mathbb{S}^n_+$ which is tangent to the boundary is of the form

$$W = \sum_{j=1}^n a_j W_j, \ a_j \in \mathcal{C}^\infty(\mathbb{S}^n_+). \tag{2.1.5}$$

PROOF. Only the behavior at the boundary, (2.1.4) and (2.1.5) is in any doubt. Clearly it suffices to prove (2.1.4) locally near any boundary point, so consider $\omega = (1, 0, \dots, 0)$. Near this point $\omega_{n+1}, \omega_2, \dots, \omega_n$ can be used as coordinates on $\mathbb{S}^n_+$. From (2.1.1)

$$\omega_{n+1} = \frac{1}{(1+|x|^2)^{\frac{1}{2}}},\ (\omega_2, \cdots, \omega_n) = \frac{(x_2, \cdots, x_n)}{(1+|x|^2)^{\frac{1}{2}}}. \tag{2.1.6}$$

Since $\omega_1 = 1$ at the base point p, $x_1^2 > x_2^2 + \cdots + x_n^2$ in $\mathrm{SP}^{-1}(\Omega)$ for a small neighbourhood, Ω, of p. Thus (2.1.6) can be replaced by the simpler *projective* coordinates

$$y_1 = \frac{1}{x_1},\ (y_2, \cdots, y_n) = (\frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}) \tag{2.1.7}$$

near p. In terms of these coordinates

$$\begin{gathered} D_{x_1} = -y_1^2 D_{y_1} - y_1(y_2, \cdots, y_n) \cdot (D_{y_2}, \cdots, D_{y_n}), \\ D_{x_i} = y_1 D_{y_i}\ i > 1. \end{gathered} \tag{2.1.8}$$

This shows that the $\mathrm{SP}_*(D_j)$ are of the form (2.1.4), since $\rho = y_1$ is a local boundary defining function. The representation (2.1.5) follows easily too, see the discussion of $\mathcal{V}_b(X)$ in §2.2 below. □

The second example is *blowing up the origin* in $\mathbb{R}^n$. The linear vector fields $x_i D_j$ all vanish at 0, and form a Lie algebra. We replace $\mathbb{R}^n$ by a blown-up version, simply by introducing polar coordinates:

$$[\mathbb{R}^n, \{0\}] = X = [0, \infty) \times \mathbb{S}^{n-1}. \tag{2.1.9}$$

The notation here is part of a general notation for blow up discussed in Chapter 5. The natural 'blow-down map' is

$$\beta_0 \colon X \ni (r, \omega) \longmapsto r\omega \in \mathbb{R}^n \tag{2.1.10}$$

which is $\mathcal{C}^\infty$ and is an isomorphism from X° onto $\mathbb{R}^n \backslash \{0\}$.

LEMMA 2.1.2. *The linear vector fields $x_i D_j$ lift to be $\mathcal{C}^\infty$ on X and tangent to the boundary and any $\mathcal{C}^\infty$ vector field on X tangent to the boundary can be written*

$$V = \sum_{i,j=1}^{n} a_{ij} V_{ij},\ a_{ij} \in \mathcal{C}^\infty(X),\ (\beta_0)_*(V_{ij}) = x_i D_j. \tag{2.1.11}$$

PROOF. Away from ∂X there is nothing to prove; the V_{ij} are $\mathcal{C}^\infty$ and span all vector fields. Moreover each $x_i D_j$ is invariant under the homotheity $x \longrightarrow ax, a > 0$. Thus each V_{ij} is invariant under $r \longrightarrow ar$. Clearly this means

$$V_{ij} = b_{ij}(\omega) r D_r + W_{ij} \tag{2.1.12}$$

where W_{ij} is a vector field on $\mathbb{S}^{n-1}$. Since V_{ij} is $\mathcal{C}^\infty$ in $r > 0$ the coefficients b_{ij} must be smooth, as must W_{ij} as a vector field on the sphere. Thus V_{ij} is $\mathcal{C}^\infty$ on $[\mathbb{R}^n, \{0\}]$. Clearly V_{ij} is tangent to the boundary, so it only remains to show that the V_{ij} span all such tangent vector fields. By noting that

$$(\beta_0)_*(r D_r) = x \cdot D_x \tag{2.1.13}$$

we only need show that the W_{ij} span all $\mathcal{C}^\infty$ vector fields on the sphere. This in turn is obvious since the V_{ij} span all vector fields in $r > 0$. □

These two examples illustrate the ubiquity of the space of vector fields tangent to the boundary, so we proceed to consider its structure.

2.2. Compressed tangent bundle

For a manifold with corners, X, we have denoted the space of all smooth vector fields $\mathcal{V}_E(X)$. As already noted this is a Lie algebra with respect to the standard commutator bracket. In general elements of $\mathcal{V}_E(X)$ are not well-behaved on a manifold with corners. Integration of $V \in \mathcal{V}_E(X)$ does not generally give a diffeomorphism of X, since V can be inward (or outward) pointing at the boundary. Consider instead the smaller space

$$\begin{aligned} \mathcal{V}_b(X) &= \{V \in \mathcal{V}_E(X); V \text{ is tangent to all boundary components}\} \\ &= \{V \in \mathcal{V}_E(X); V\rho_H = a_H\rho_H,\ a_H \in \mathcal{C}^\infty(X),\ \forall\ H \in M_1(X)\} \end{aligned} \tag{2.2.1}$$

where, as always, the ρ_H are defining functions for the boundary hypersurfaces of X.

LEMMA 2.2.1. *In local coordinates at $p \in \partial_k X$, $V \in \mathcal{V}_b(X)$ is of the form*

$$V = \sum_{j=1}^{k} a_j(x)x_jD_j + \sum_{j>k} a_j(x)D_j,\ \ a_j \in \mathcal{C}^\infty. \tag{2.2.2}$$

PROOF. Writing

$$V = \sum_{j=1}^{k} b_j(x)D_j + \sum_{j>k} a_j(x)D_j$$

it follows from the second part of (2.2.1) that, for $j \le k$, $b_j = 0$ on $x_j = 0$ hence $b_j = x_ja_j$ for some smooth a_j. □

COROLLARY 2.2.1. *There is a $\mathcal{C}^\infty$ vector bundle over X, denoted ${}^bTX$, with fibre at $p \in X$ given by*

$${}^bT_pX = \mathcal{V}_b(X)\,/(\mathcal{I}(p)\cdot\mathcal{V}_b(X)) \tag{2.2.3}$$

where $\mathcal{I}(p) \subset \mathcal{C}^\infty(X)$ is the subspace of functions vanishing at p.

PROOF. From (2.2.2) the elements x_iD_i, $i \le k$, D_j, $j > k$ give a basis for $\mathcal{V}_b(X)$ locally. The definition (2.2.3) of the fibre simply picks out the coefficients of the basis elements at the point as linear coordinates in ${}^bT_pX$. To show that ${}^bTX$ is $\mathcal{C}^\infty$ it suffices to show that the coefficients in (2.2.2) transform via a $\mathcal{C}^\infty$ matrix under coordinate transformation.

Let ϕ be the coordinate transformation so $x'_i = \phi_i(x)$ are the new coordinates. By assumption the first k coordinates in each system define, locally, the boundary hypersurfaces. By relabelling the x'_i we can arrange that x_i and x'_i, $i = 1, \dots, k$ define the same boundary hypersurface. Thus

$$x'_i = \phi_i(x) = x_i\alpha_i(x),\ \ i = 1, \dots, k,\ \ \alpha_i(x) > 0. \tag{2.2.4}$$

The Jacobian matrix of the transformation is

$$J_{ij}(x) = \partial_i\phi_j(x)$$

and the matrix transforming the coefficients of (2.2.2) is therefore just

$$
{}^bJ_{ij}(x) = \begin{cases} \frac{x_i}{\phi_j(x)}\partial_i\phi_j(x) & i,j\le k \\ x_i\partial_i\phi_j(x) & i\le k, j>k \\ \frac{1}{\phi_j(x)}\partial_i\phi_j(x) & i>k, j\le k \\ \partial_i\phi_j(x) & i,j>k. \end{cases} \tag{2.2.5}
$$

From (2.2.4) it follows that the entries of this matrix are $\mathcal{C}^\infty$. □

The space $\mathcal{V}_b(X)$ is the simplest, and the most fundamental, example of the general structure that we shall consider below. One rather particular aspect is its naturality. It is fair to say that $\mathcal{V}_b(X)$, and ${}^bTX$, are 'as good as' $\mathcal{V}_E(X)$ and TX, depending on the context. One way to see the naturality of $\mathcal{V}_b(X)$ is to note:

PROPOSITION 2.2.1. *On a compact manifold with corners integration of a real vector field in $\mathcal{V}_b(X)$ defines a 1-parameter family of diffeomorphisms of X and conversely, i.e. $\mathcal{V}_b(X)$ is the space of infinitesimal diffeomorphisms of X.*

PROOF. Certainly any element $V\in\mathcal{V}_b(X)$ is inward pointing, so if X is compact Corollary 1.13.1 shows that integration of such a vector field gives a one-parameter family of diffeomorphisms, $\exp(tV)$, at least for $t\ge 0$ and small. However $-V$ is also inward pointing and the construction can be iterated to conclude that $\exp(tV)$ is a diffeomorphism for all t. Conversely $\exp(tV)$, defined for some extension of X only preserves X, and so defines a diffeomorphism $\exp(tV)$ for $|t|$ small if both V and $-V$ are inward pointing. Clearly this implies that $V\in\mathcal{V}_b(X)$. □

We call ${}^bTX$ the 'compressed' or 'b' tangent bundle of X. Naturally there is a close relationship between the ordinary, or extension, tangent bundle and the compressed bundle:

LEMMA 2.2.2. *There is a natural $\mathcal{C}^\infty$ vector bundle map*

$$
e_b\colon {}^bTX \longrightarrow TX, \tag{2.2.6}
$$

determined by the condition that

$$
e_b^*\mathcal{V}_b(X) = \mathcal{C}^\infty(X;{}^bTX); \tag{2.2.7}
$$

e_b is of rank $n-l$ over $\partial_l X$.

Abusing notation somewhat, by suppressing the map e_b, we can write $\mathcal{V}_b(X) = \mathcal{C}^\infty(X;{}^bTX)$.

PROOF. Since $\mathcal{V}_b(X)\subset\mathcal{V}_E(X)$ the map (2.2.6) is obtained by sending $v\in{}^bT_pX$ to $V(p)$ if $V\in\mathcal{V}_b(X)$ is a section of ${}^bTX$ with value (in ${}^bTX$) v at p. In local coordinates at $p\in\partial_lX$ it is clear that

$$
x_iD_i \longmapsto 0 \quad D_j\longmapsto D_j \quad j>l,\ i\le l. \tag{2.2.8}
$$

Thus the map has rank $n-l$. □

For each $x\in X$ we denote the null space of $e_b\colon {}^bT_xX\longrightarrow T_xX$ by ${}^bN_x\,\mathrm{Fa}(x)$ where $\mathrm{Fa}(x)$ is the boundary face containing x in its interior. These b-normal spaces form important natural subbundles of ${}^bT_FX$ for each boundary face F. If $p\in\partial_kX$ is a boundary point at which X has a corner of codimension k then the k elements

$$
x_i\partial_{x_i}\in{}^bN_pX\subset{}^bT_pX \ \ i=1,\dots,k \tag{2.2.9}
$$

are well defined, except that they may be interchanged by relabelling. From the local coordinate form, (2.2.5), of the bundle transformations we find:

LEMMA 2.2.3. *Over the interior of each $F \in M_k(X)$ the b-normal spaces, ${}^bN_xF$, form a bundle which is the sum of k naturally trivial 1-dimensional subbundles spanned in local coordinates by the vector fields (2.2.9); this subbundle extends smoothly to a trivial bundle ${}^bNF \subset {}^bT_FX$ over the whole of the boundary face F.*

When a general boundary face, F, is given as a component of an intersection of boundary hypersurfaces as in (1.8.7) we get the decomposition described in Lemma 2.2.3 in the form

$$ {}^bNF = \bigoplus_{H \in I} {}^bN_FH. \tag{2.2.10}$$

Since the elements of $\mathcal{V}_b(X)$ are tangent to all the boundary faces there are well-defined restriction maps

$$\mathcal{V}_b(X) \longrightarrow \mathcal{V}_b(F) \quad \forall\ F \in M(X) \tag{2.2.11}$$

which define projections ${}^bT_FX \longrightarrow {}^bTF$.

LEMMA 2.2.4. *For any manifold with corners, X, and any $F \in M(X)$ there is a short exact sequence of vector bundles*

$$0 \hookrightarrow {}^bNF \longrightarrow {}^bT_FX \longrightarrow {}^bTF \longrightarrow 0. \tag{2.2.12}$$

PROOF. Suppose $F \in M_l(X)$ and $p \in F \cap \partial_k X$, so $k \geq l$. Local coordinates near p can be chosen so that $F = \{x_i = 0, i = 1, \dots, l\}$. In the local basis of $\mathcal{V}_b(X)$ given by (2.2.2) the elements x_iD_i for $i = 1, \dots, l$ span ${}^bN_pF$ and the remaining elements x_iD_i for $i = l+1, \dots, k$ and D_j for $j > k$ span ${}^bT_pF$. This shows the exactness of the sequence (2.2.12). □

The dual map to (2.2.6) has the same rank properties and we use the same notation:

$$e_b \colon T^*X \longrightarrow {}^bT^*X. \tag{2.2.13}$$

The space of smooth 1-forms, in the usual sense, $\mathcal{C}^\infty(X; T^*X)$ is thus a subspace of $\mathcal{C}^\infty(X; {}^bT^*X)$. By duality from (2.2.3) we find:

LEMMA 2.2.5. *For any manifold with corners $\mathcal{C}^\infty(X; {}^bT^*X)$ is spanned over $\mathcal{C}^\infty(X)$ by $\mathcal{C}^\infty(X; T^*X)$ and the logarithmic differentials, $d\rho_i/\rho_i$, of defining functions for the boundary hypersurfaces.*

The dual short exact sequence to (2.2.12)

$$0 \hookrightarrow {}^bT^*F \longrightarrow {}^bT^*_FX \longrightarrow {}^bN^*F \longrightarrow 0 \tag{2.2.14}$$

embeds the b-cotangent bundle of the boundary face as a subbundle of the restriction to the face of the b-cotangent bundle of the manifold. This is in marked contrast to the behaviour of the ordinary tangent and cotangent bundles where there are short exact sequences

$$0 \longrightarrow TF \hookrightarrow T_XF \longrightarrow NF \longrightarrow 0 \text{ and} \tag{2.2.15}$$

$$0 \longrightarrow N^*F \hookrightarrow T^*_FX \longrightarrow T^*F \longrightarrow 0 \tag{2.2.16}$$

in which the tangent bundle of the boundary face is embedded as a subbundle of the restriction to the tangent bundle to the manifold, whereas the cotangent bundle to the boundary face is naturally a quotient (by the conormal bundle) of the restriction of the cotangent bundle of the manifold.

2.3. The b-differential

Consider again the definition of a b-map given in §1.12. If $F\colon X \longrightarrow Y$ is an interior b-map then the definition (1.12.2) shows that the logarithmic derivatives of boundary defining functions can be pulled back under it. This translates to the existence of a b-differential.

LEMMA 2.3.1. *For any interior b-map $F\colon X \longrightarrow Y$ the differential and its transpose*

$$F_*\colon T_xX \longrightarrow T_{F(x)}Y,\ F^*\colon T^*_{F(x)} \longrightarrow T^*_xX \tag{2.3.1}$$

extend by continuity from $x \in X^\circ$ to define the b-differentials

$$\begin{aligned} {}^bF^*\colon {}^bT^*_{F(x)}Y &\longrightarrow {}^bT^*_xX \\ {}^bF_*\colon {}^bT_xX &\longrightarrow {}^bT_{F(x)}Y \end{aligned} \quad \forall\ x \in X. \tag{2.3.2}$$

For any interior b-map

$${}^bF_*\colon {}^bN_xX \longrightarrow {}^bN_{F(x)}Y \quad \forall\ x \in X \tag{2.3.3}$$

is given in terms of the basis of logaritmic differentials of defining functions by the matrix of boundary exponent in (1.12.3).

PROOF. The maps (2.3.2) are transposes of each other. To define ${}^bF^*$ we use the fact that, near $p \in \partial_kX$, ${}^bT^*Y$ is spanned by

$$\frac{d\rho'_i}{\rho'_i}, dy_j,\ i = 1, \dots, k,\ j = k+1, \dots n \tag{2.3.4}$$

for $\mathcal{C}^\infty$ functions y_j with ρ'_i the (local) boundary defining functions. By the definition of a b-map, (1.12.2),

$$F^*\left(\frac{d\rho'_i}{\rho'_i}\right) = \frac{dF^*\rho'_i}{F^*\rho'_i} = \frac{da_i}{a_i} + \sum_{l=1}^{N} e(i,l)\frac{d\rho_l}{\rho_l}. \tag{2.3.5}$$

Clearly this shows that ${}^bF^*$ is well-defined in (2.3.2) by continuity from the interior.

Since the b-differential is defined by continuity from the ordinary differential in the interior the diagram

$$\begin{tikzcd} {}^bT_xX \arrow[r, "e_b"] \arrow[d, "{}^bF_*"'] & T_xX \arrow[d, "F_*"] \\ {}^bT_{F(x)}Y \arrow[r, "e_b"'] & T_{F(x)}Y \end{tikzcd} \tag{2.3.6}$$

surely commutes. This shows that ${}^bF_*$ maps the null space of e_b, in ${}^bT_xX$, into the null space of e_b, in ${}^bT_{F(x)}$, i.e. proves (2.3.3). □

For any b-map there is always a minimal boundary face of the image space, $F \in M(Y)$ such that $f(X) \subset F$. The map $f\colon X \longrightarrow F$ is then an interior b-map. Notice that $F = \mathrm{Fa}(f(x))$ for any point x in the interior of X. By the b-differential of a boundary, i.e. non-interior, b-map we shall mean the b-differential of this related interior b-map.

We shall classify b-maps according to the behaviour of their b-differentials. An important class of maps below are the *blow-down maps.* These will not be defined precisely for the moment but they are b-maps $f\colon X \longrightarrow Y$ which are isomorphisms from the interior of X to an open dense subset of Y and which are such that the set of elements of $\mathcal{V}_b(X)$ which are f-related to elements of $\mathcal{V}_b(Y)$ span $\mathcal{V}_b(X)$ over $\mathcal{C}^\infty(X)$. For the moment we shall concentrate more on the maps with surjective b-differentials.

2.4. b-submersions and b-fibrations

Let X be a compact manifold with corners. A smooth surjective map $\phi\colon X \longrightarrow Y$ onto a manifold with corners Y is a *fibration* if to each $y \in Y$ there corresponds a compact manifold with corners F_y and a neighbourhood Ω, of y in Y, such that there is a diffeomorphism

$$\phi^{-1}(\Omega) \stackrel{\tau}{\longleftrightarrow} \Omega \times F_y \text{with } \phi(\tau^{-1}(y,z)) = y \tag{2.4.1}$$

meaning that, locally in Y, X can be reduced to a product so that ϕ is projection onto the first factor. Provided Y is connected, the fibre $F_y = \phi^{-1}(y)$ is diffeomorphic to a fixed manifold F. We shall generally write such a fibration in the form

$$\begin{array}{ccc} F & \text{——} & X \\ & & \Big\downarrow \\ & & Y \end{array} \tag{2.4.2}$$

to indicate that F is the *model fibre.* The fibres of a fibration are always p-submanifolds, in the sense of Definition 1.7.4.

If $\phi\colon X \longrightarrow Y$ is a fibration consider the space, $\mathcal{V}_\phi(X)$, of all $\mathcal{C}^\infty$ vector fields on X which are tangent to each of the fibres of ϕ. These are, in any local trivialization (2.4.1), just vector fields on F_y depending smoothly on y as a parameter. That is, if we let ${}^\phi TX \subset TX$ denote the subbundle of fibre-tangents,

$$\mathcal{V}_\phi = \mathcal{C}^\infty(X; {}^\phi TX),\ \ {}^\phi TX \cong \phi^* TY. \tag{2.4.3}$$

The notion of a fibration, whilst familiar, is too restrictive for our purposes. Rather we shall work with the two successively weaker notions of a b-fibration and a b-submersion. If X and Y are manifolds without boundary then any submersion, i.e. $\mathcal{C}^\infty$ map $\phi\colon X \longrightarrow Y$ with differential $\phi_*\colon T_xX \longrightarrow T_{\phi(x)}Y$ surjective at each point, is a fibration if both X and Y are compact (see Proposition 2.4.1 below.) We consider the weaker notion:

Definition 2.4.13. A *b-submersion* between manifolds with corners is an interior b-map $\phi\colon X \longrightarrow Y$ with ${}^bF_*$, in (2.3.2), surjective for each $x \in X$.

It is immediately clear that a fibration of manifolds with boundary is indeed a b-submersion. However, even if both X and Y are compact, the converse statement is not correct. Consider for example

$$\psi\colon [0,\infty)\times[-1,1] \longrightarrow [0,\infty),\ \psi(r,\tau)=\frac{1}{2}r(1+\tau). \tag{2.4.4}$$

This is a b-submersion, but is not a fibration. Of course the spaces are not compact, but the 'non-fibration' difficulty occurs in a compact set. In the constructions below we shall encounter many examples of b-fibrations which are not fibrations.

Exercise 2.4.13. Check that the map in (2.4.4) has the properties of a blow-down map as listed at the end of §2.3.

Exercise 2.4.14. By modifying (2.4.4), or starting from scratch, find an example of a b-submersion between compact manifolds with corners which is not a fibration.

Proposition 2.4.1. *If $\phi\colon X \longrightarrow Y$ is a b-submersion with Y a manifold without boundary and the leaves of ϕ are compact then ϕ is a fibration.*

Proof. Consider a point $\bar y \in Y$. Since Y has no boundary, ${}^bT^*_yY = T^*_yY$. Directly from (2.3.3) this shows that at each point $x\in\phi^{-1}(\bar y)\cap B$, where $B\in M_kX$, the b-differential annihilates ${}^bN_xB$. Thus the ordinary differential must be surjective even when restricted to T_xB. From this, and standard arguments using the implicit function theorem, it follows that $\phi^{-1}(\bar y)$ is a p-submanifold of X (i.e. any subset of the boundary defining functions are independent on $\phi^{-1}(\bar y)$ at any point where they all vanish), that X is locally a product with $\phi^{-1}(\bar y)$ and that ϕ is locally the projection off $\phi^{-1}(\bar y)$. □

Notice that the condition that ϕ be a b-map is important here, since without it the conclusion need not hold. See Figure 4.

An immediate consequence of Proposition 2.4.1 is that any b-submersion of compact manifolds with corners is a fibration over the interior of its range. The structure is not quite so restricted at the boundary and in various circumstances we need an extra condition; namely the notion of b-normality.

Definition 2.4.14. An interior b-map between manifolds with corners is said to be b-normal if at each point the b-differential is surjective as a map between the b-normal spaces in (2.3.3). A b-fibration $\phi\colon X\longrightarrow Y$ between manifolds with corners is a surjective b-map which is both b-normal and a b-submersion.

To see that the condition of b-normality is not automatic consider the map

$$\begin{aligned} X = &[0,1]\times[0,2\pi]\ni(r,\theta)\longmapsto \\ &(r\cos\theta, r\sin\theta)\in Y=\{(x,y)\in\mathbb{R}^2; x^2+y^2\le 1, x\ge 0, y\ge 0\}. \end{aligned} \tag{2.4.5}$$

Clearly this is a b-map and is easily seen to be a b-submersion but not a b-fibration.

Proposition 2.4.2. *A b-submersion, $\phi\colon X\longrightarrow Y$, between manifolds with corners is a b-fibration if and only if the boundary exponents are limited by the condition*

$$\text{for each } H\in M_1(X),\ e(H,G)\neq 0 \text{ for at most one } G\in M_1(Y). \tag{2.4.6}$$

Under a b-fibration the inverse image of any boundary face of Y is a finite union of boundary faces of X, restricted to each one of which ϕ gives a b-fibration. The fibre $\phi^{-1}(p)$ above each point $p\in Y$ is a finite union of p-submanifolds of X.

PROOF. If (2.4.6) holds for a b-submersion then at each $x \in X$ such that $\phi(x) \in \partial_k X$ the ${}^bF^*(d\rho_i'/\rho_i')$ are independent when restricted to ${}^bN_x X$. Thus the differential in (2.3.3) must be injective and the map must be a b-fibration. Conversely if (2.4.6) does not hold then (2.3.3) cannot always be surjective, even though ${}^bF_*$ might be.

For an interior b-map the inverse image of a boundary face is a union of boundary faces, as follows from (1.12.2)

$$\phi^{-1}(B') = \bigcup_{i=1}^{q} B_i. \tag{2.4.7}$$

Still for a general b-map the restriction

$$\phi \colon B_i \longrightarrow B' \tag{2.4.8}$$

is always a b-map. However even if ϕ is a b-submersion it does not follow that (2.4.8) is a b-submersion. Notice that for any boundary face

$${}^bT_x B = {}^bT_x X / {}^bN_x B. \tag{2.4.9}$$

If ϕ is a b-fibration then this and the surjectivity of (2.3.3) together show that the b-differential of the restricted map, (2.4.8), must be surjective. The condition (2.4.6) clearly survives restriction. The last part of the proposition now follows, since any $p \in Y$ lies in the interior of some boundary face. $\square$

For any point p of the domain of a b-fibration local coordinates x_i can be introduced at p and local coordinates x_i' at $f(p)$ in terms of which

$$f^*x_i' = \prod_j x_j^{e(i,j)},\ i \le k',\ f^*x_j' = x_{j+k'-k} \tag{2.4.10}$$

where the $e(i,j)$ are the boundary exponents of f and k, k' are the local boundary codimensions at p and $f(p)$.

The four distinct types of maps between compact $\mathcal{C}^\infty$ manifolds with corner discussed above mirror the four types of submanifolds discussed in Chapter 1, see Definition 1.12.1 and (1.12.6). Thus

$$\text{fibration} \Longrightarrow b\text{-fibration} \Longrightarrow b\text{-submersion} \Longrightarrow b\text{-map}. \tag{2.4.11}$$

The b-cotangent bundle also gives another interpretation of the transversality condition for an interior p-submanifold. Namely under the natural bundle map $T^*X \longrightarrow {}^bT^*X$ the normal bundle to a p-submanifold, Y, is mapped to a subbundle of ${}^bT^*_Y X$; this is the b-conormal bundle to Y :

$${}^bN^*Y \subset {}^bT^*_Y X. \tag{2.4.12}$$

The normal bundle to Y, $N_y Y = T_y X / T_y Y$ can also be identified with ${}^bT_y X / T_y Y$, i.e. is spanned by the classes of vectors tangent to ∂X. This is the basis of our proof of the existence of a normal fibration in §2.10.

2.5. Boundary-fibration structures

Now we shall define what is the underlying object of study in the sequel. For the moment we do little more than give the bald definition and then discuss some examples. The main point of this definition is that it includes a large variety of cases of 'degenerating vector fields' of geometric origin and interest. We generally consider boundary-fibration structures only on compact manifolds with corners.

From an analytic point of view the main virtue of the definition is that it allows the development of an associated ring of pseudodifferential operators (the $\mathcal{V}$-calculus.) To actually apply this calculus to a concrete problem more conditions (of integrability type) are needed. As we shall see eventually, this is related to the solvability of model problems.

If X is a compact manifold with corners consider a subspace $\mathcal{V} \subset \mathcal{V}_b(X)$. The most basic condition we impose on $\mathcal{V}$ is

$$\mathcal{V} \text{ is a Lie sub-algebra of } \mathcal{V}_b(X). \tag{2.5.1}$$

We shall also require that

$$\mathcal{V} \text{ is a } \mathcal{C}^\infty(X)\text{-module.} \tag{2.5.2}$$

This is not in itself restrictive, since any Lie sub-algebra of $\mathcal{V}_b(X)$ can be extended to a $\mathcal{C}^\infty(X)$-module simply by replacing it by its $\mathcal{C}^\infty(X)$-span, this again being a Lie algebra.

Consider the local structure of $\mathcal{V}$. We shall require of $\mathcal{V}$ that near each $p \in \partial_k X$ there is a basis V_i of $\mathcal{V}_b(X)$ such that

$$\begin{gathered} U_i = x^{\alpha(i)} V_i,\ i \in J(p) \subset \{1, \dots, n\} \\ \text{span } \mathcal{V} \text{ locally for some } \alpha(i) \in \mathbb{N}_0^k. \end{gathered} \tag{2.5.3}$$

Notice that this is a non-trivial condition even in the interior where it requires that $\mathcal{V}$ give a foliation. Suppose that $x^{\beta(i)} W_i$ is another basis of $\mathcal{V}$ near p with the W_i part of a local basis of $\mathcal{V}_b(X)$. Consider the decomposition of the new basis in terms of (2.5.3):

$$x^{\beta(l)} W_l = \sum_{i \in J(p)} a_{li}(x) x^{\alpha(i)} V_i. \tag{2.5.4}$$

In the interior (2.5.4) gives the decomposition of W_l, as an element of $\mathcal{V}_b(X)$, in terms of the basis V_i. Since this is unique and extends smoothly up to the boundary sufficiently near p the coefficients in (2.5.4) must satisfy:

$$\begin{gathered} a_{li}(x) = x^{\gamma(l,i)} b_{li}(x) \text{ where} \\ \gamma(l,i) + \alpha(i) \geq \beta(l) \text{ and the } b_{li} \text{ are } \mathcal{C}^\infty. \end{gathered} \tag{2.5.5}$$

Here the partial ordering of multiindices is the same inequality on all the components.

Proposition 2.5.1. *If $\mathcal{V} \subset \mathcal{V}_b(X)$ is a Lie algebra of $\mathcal{C}^\infty$ vector fields on a manifold with corners for which* (2.5.1), (2.5.2) *and* (2.5.3) *hold then there is a well-defined $\mathcal{C}^\infty$ vector bundle ${}^{\mathcal{V}}TX$ (the structure bundle) and a vector bundle map*

$$e_{\mathcal{V}} \colon {}^{\mathcal{V}}TX \longrightarrow {}^bTX \tag{2.5.6}$$

such that

$$\mathcal{V} = e_{\mathcal{V}}^* \mathcal{C}^\infty(X; {}^{\mathcal{V}}TX). \tag{2.5.7}$$

Proof. Since (2.5.3) gives a local basis for $\mathcal{V}$ the fibre at p can be defined by

$${}^{\mathcal{V}}T_pX = \mathcal{V} / (\mathcal{I}_p(X) \cdot \mathcal{V}) \tag{2.5.8}$$

where $\mathcal{I}_p(X) \subset \mathcal{C}^\infty(X)$ is the ideal of functions vanishing at p. This space always has dimension equal to the dimension of the leaves of the foliation in the interior with local coordinates given by the coefficients in terms of the basis. Moreover, by definition, the elements of any one basis of $\mathcal{V}$ can be expressed as a linear combinations with smooth coefficients of the elements of another basis as in (2.5.4) and then (2.5.5) holds. It follows that the transition functions are smooth, so ${}^{\mathcal{V}}TX$ is a vector bundle satisfying (2.5.7). $\square$

We shall strengthen (2.5.3) both as regards the non-vanishing elements at each boundary face and also consider the manner in which elements can vanish at the boundary.

We first demand that

$$\text{Each } B \in M(X) \text{ has a b-fibration such that } \mathcal{V} \restriction B^\circ \text{ spans the leaves.} \tag{2.5.9}$$

Even over the interior this is a global refinement of (2.5.3). Notice that the b-fibration can change with the boundary face. We call the leaves which meet the interior of B the leaves of the boundary-fibration structure. In general their dimension can vary between boundary face.

From the restriction of the structure bundle to a boundary face, ${}^{\mathcal{V}}T_BX$, there is a natural vector bundle map to ${}^bTB$, just given by restriction. Then (2.5.9) is a condition on the structure of the image. If B is a component of H_I given by (1.8.7) we can label the trivial subspaces of ${}^bNB$ by the subsets of I. Thus put

$$ {}^bN^JB = \bigoplus_{H\in J} {}^bN_BH, \quad J \subset I. \tag{2.5.10}$$

For $\alpha \in \mathbb{N}_0^I$ let $\rho^\alpha = \rho^\alpha$ be the product of the powers of defining functions for the H.

Lemma 2.5.1. *For any Lie algebra satisfying* (2.5.1), (2.5.2) *and* (2.5.3) *and any boundary face, B,*

$$\begin{aligned} {}^{\mathcal{V}}T_p^{(\alpha)}B = \{v \in {}^bT_pX; \exists\ V \in \mathcal{V}_b(X),\\ \textit{with } V(p) = v \textit{ and } \rho^\alpha V \in \mathcal{V}\},\ p \in B^\circ \end{aligned} \tag{2.5.11}$$

extends smoothly to define a subbundle of ${}^bT_B$.

Proof. This is essentially the same as the proof of Proposition 2.5.1. The elements of $\mathcal{V}$ which are of the form $x^\alpha V$ with $V \in \mathcal{V}_b(X)$ locally are those which are smooth linear combinations of the basis elements in (2.5.3) of the form

$$\sum_{\{j\in J(p);\alpha(j)\le\alpha\}} x^\alpha a_i(x)V_i \tag{2.5.12}$$

with the a_i smooth. Thus at any point in the interior of B the space ${}^{\mathcal{V}}T_p^\alpha B$ is the span of the $V_i(p)$ in (2.5.3) for which the corresponding $\alpha(i)$ has all components at least as large as those of α. $\square$

To make sure that the Lie algebra $\mathcal{V}$ does not change behaviour at points in the interior of a boundary face we shall impose the additional conditions:

$$ {}^{\mathcal{V}}T^{(\alpha)}B \cap {}^bN^JB \subset {}^bNB \text{ is a subbundle } \ \forall\ \alpha \in \mathbb{N}^J,\ J \subset I. \tag{2.5.13}$$

Notice that the ${}^{\mathcal{V}}T^{(\alpha)}B$ are increasing with α. One way to ensure (2.5.13) is simply to require

$$\begin{gathered} {}^{\mathcal{V}}T^{(0)}B = {}^{\mathcal{V}}T^{(\alpha)}B \quad \forall\ \alpha \in \mathbb{N}^k \\ {}^{\mathcal{V}}T^{(0)}B \cap {}^bN^JB \text{ is a subbundle of } {}^bNB \quad \forall\ J \subset I = \mathrm{Hu}(B). \end{gathered} \tag{2.5.14}$$

DEFINITION 2.5.15. If X is a compact manifold with corners then a boundary-fibration structure on X is a subspace $\mathcal{V} \subset \mathcal{V}_b(X)$ for which the conditions (2.5.1), (2.5.2), (2.5.3), (2.5.9) and (2.5.13) hold. If (2.5.14) holds the boundary-fibration structure is said to be regular.

The rather overworked 'regular' is used here because (2.5.14) gives a natural extension of the notion of a regular-singular point for ordinary differential equations.

Let us examine the possibilities in dimension one. The manifold with corners, taken to be compact and connected, has to be $\mathbb{S}^1$ or $[-1,1]$. In the former case the only possibilities for $\mathcal{V}$ are 0 and $\mathcal{V}(\mathbb{S}^1) = \mathcal{C}^\infty(\mathbb{S}^1; T\mathbb{S}^1)$. For the interval the possibilities, apart from 0, are determined by two positive integers, k and l, as

$$\mathcal{V} \text{ is spanned by } \begin{cases} (x+1)^k \partial_x \text{ near } -1 \\ (1-x)^l \partial_x \text{ near } 1 \end{cases} \tag{2.5.15}$$

and only in case $k = l = 1$ is the structure regular.

2.6. Basic examples

We first consider three examples where the fibration structures are trivial in that they consist either of the point fibration $X \xrightarrow{\mathrm{Id}} X$ or the single leaf fibration $X \longrightarrow \{pt\}$. The fundamental example is:

$$\mathcal{V}_b \text{ (tangency structure).} \tag{2.6.1}$$

The first and second part of the definition have already been checked. The third is trivially true. The fibrations in (2.5.9) are single leaf. Although, from the point of view of the definition, this example is trivial it should not be taken too lightly. The corresponding pseudodifferential operators are called 'totally characteristic' in [**5**] and [**4**]; they will be called b-pseudodifferential here. This boundary structure might also reasonably be called cylindrical. It is discussed extensively in Chapter 7.

The next basic example is

$$\mathcal{V}_0(X) = \{V \in \mathcal{V}_b(X);\ V = 0 \text{ at } \partial X \text{ (in } \mathcal{V}_E(X))\}. \tag{2.6.2}$$

The global structure of these vector fields is clearly

$$\mathcal{V}_0 \ni V \Longleftrightarrow V = \rho W,\ W \in \mathcal{V}_E(X). \tag{2.6.3}$$

Thus we can write $\mathcal{V}_0(X) = \rho\mathcal{V}_E(X)$ where ρ is a total boundary defining function. Clearly $\mathcal{V}_0(X)$ is a Lie algebra, since

$$\begin{aligned} &= \rho V \cdot \rho W - \rho W \cdot \rho V \\ &= \rho(V\rho)W - \rho(W\rho)V + \rho^2[V,W]. \end{aligned} \tag{2.6.4}$$

The local structure result (2.5.3) follows from (2.6.3). The structure bundle, given by Proposition 2.5.1, with local basis ρD_i, will be denoted ${}^0TX$. The space of vector fields, $\mathcal{V}_0(X)$, can be characterized as the space of infinitesimal diffeomorphisms of X which fix each point of the boundary.

These first two examples are completely canonical; $\mathcal{V}_b(X)$ is always regular and $\mathcal{V}_0(X)$ is regular on a manifold with boundary but not on a manifold with corners of codimension two or more. The next example is neither canonical nor regular.

Let ρ be a total boundary defining function, for simplicity on a manifold with boundary. Then consider the *cusp algebra*

$$\mathcal{V}_\rho(X) = \{V \in \mathcal{V}_b(X);\ V\rho \in \rho^2\mathcal{C}^\infty(X)\}.$$

Near any point $p \in \partial X$ coordinates can always be chosen with $\rho = x_1$. A general element of $\mathcal{V}_b(X)$ is then locally of the form

$$V = a_1 x_1 \partial_1 + \sum_{j \geq 2} a_j \partial_j .$$

Since $V\rho = Vx_1 = ax_1$ we see that $V \in \mathcal{V}_\rho(X)$ if and only if $a_1 = x_1 b$ with b being $\mathcal{C}^\infty$. Thus locally

$$\mathcal{V}_\rho(X) = \operatorname{sp}\{x_1^2 \partial_1,\ \partial_j\ j \geq 2\} \tag{2.6.5}$$

in these (special) local coordinates. This proves the first part, (2.5.3), of the definition of a boundary-fibration structure. The fibrations in (2.5.9) are all the single leaf fibration. It is again easy to check that $\mathcal{V}_\rho(X)$ is a Lie algebra:

$$[V, W]\rho = V(W\rho) - W(V\rho) \in \rho^3 \mathcal{C}^\infty(X)$$

and it is also clearly a $\mathcal{C}^\infty(X)$-module satisfying

$$\mathcal{V}_\rho(X) \supset \rho \mathcal{V}_b(X). \tag{2.6.6}$$

The choice of defining function, ρ, *does* affect the algebra:

Exercise 2.6.15. Show that on a manifold with boundary $\mathcal{V}_\rho(X) = \mathcal{V}_{\rho'}(X)$ for two boundary defining functions ρ and ρ' if and only if there is a constant $c > 0$ such that

$$\rho' = c\rho + a\rho^2,\ a \in \mathcal{C}^\infty(X). \tag{2.6.7}$$

Exercise 2.6.16. If ρ and ρ' are two global defining functions for the boundary of a manifold with boundary, X, show that there is a global diffeomorphism $\chi \colon X \longrightarrow X$ such that

$$\chi_* \colon \mathcal{V}_\rho(X) \longleftrightarrow \mathcal{V}_{\rho'}(X). \tag{2.6.8}$$

This shows that the cusp structure is *natural* i.e. well-defined up to diffeomorphism.

One can also mix these various structures. For example, on a compact manifold with corners one can consider a boundary-fibration structure defined by the vector fields which are tangent to certain of the boundary hypersurfaces and vanish at the remainder.

Exercise 2.6.17. Show that if $\mathcal{V}$ is a boundary-fibration structure on a compact manifold with corners and ρ^α is a product of defining functions of boundary hypersurfaces then $\rho^\alpha \mathcal{V}$ is also a boundary-fibration structure.

2.7. More Examples

Suppose $\phi \colon X \longrightarrow Y$ is a b-fibration of compact manifolds with corners. Then set

$$\begin{aligned} \mathcal{V}_\phi(X) = \{V \in \mathcal{V}_b(X); V \text{ is tangent to each leaf of } \phi \\ \text{and the leaves of its restriction to each of the boundary faces of } X\}. \end{aligned} \tag{2.7.1}$$

Since a b-fibration has surjective b-differential its null space at each point forms a vector subbundle ${}^{b\phi}TX \subset {}^bTX$. The condition (2.7.1) is just that

$$V \text{ is a section of } {}^{b\phi}TX. \tag{2.7.2}$$

Of the conditions (2.5.1), (2.5.2), (2.5.3), (2.5.9) and (2.5.13) for a boundary-fibration structure, $\mathcal{V}_\phi(X)$ clearly satisfies the first two. From the local structure of a b-fibration given in (2.4.10) it follows that $\mathcal{V}_\phi(X)$ is locally spanned by the vector fields

$$x_r\frac{\partial}{\partial x_r},\ x_p\frac{\partial}{\partial x_p}-x_q\frac{\partial}{\partial x_q},\ \frac{\partial}{\partial y_l} \tag{2.7.3}$$

where the indices r are those for which $e(i,r)=0$ for all i, the indices p and q run over those indices for which $p<q$ and both $e(i,p)\neq 0$ and $e(i,q)\neq 0$ for some i and the y_l are those of the non-boundary coordinates which are not the pull back of coordinates on the base. From this (2.5.3) and (2.5.13) follow immediately. Clearly (2.5.9) also holds with the b-fibration given by ϕ itself. The structure bundle is just ${}^{b\phi}TX$. Certainly $\mathcal{V}_\phi(X)$ is always regular.

A closely related example occurs in the 'adiabatic limit.' Let M be (for simplicity) a manifold without boundary, but with a fibration

$$\begin{array}{ccc} F & \text{———} & M \\ & & \downarrow \\ & & Y. \end{array} \tag{2.7.4}$$

Let $X=[0,1]\times M$ be the product manifold. On X we can consider the space $\mathcal{V}_\psi(X)$ where $\psi\colon X\longrightarrow[0,1]$ is the product fibration, with fibres M. Within this consider

$$\mathcal{V}_a(X)=\{V\in\mathcal{V}_\psi(X);\ V \text{ is tangent to the fibres of } \phi \text{ over } x=0\}. \tag{2.7.5}$$

In local coordinates z in F and y in Y it is easily seen that

$$\mathcal{V}_a(X)=\operatorname{sp}\{xD_{y_i},\ D_{z_j}\}. \tag{2.7.6}$$

This proves the local structure conditions, the fibration condition is then obviously valid. Thus this is again a regular boundary-fibration structure.

One particularly simple, and frequently encountered, case of the adiabatic limit structure occurs when the fibration (2.7.4) is taken to have $F=M$. Then we shall denote $\mathcal{V}_a(X)$ given by (2.7.6) as $\mathcal{V}_\hbar(X)$ and call it the 'semiclassical' struture. If the (small) parameter is written $\hbar$ instead of x then $\mathcal{V}_\hbar(X)$, with $X=[0,1]\times M$ just consists of the vector fields $\hbar W$ where W is a smooth vector field on M depending smoothly on $\hbar$.

Another class of examples is provided by 'Θ-structures.' Let X be a compact manifold with boundary and suppose that Θ is a $\mathcal{C}^\infty$ 1-form on X which is non-vanishing even when pulled back to the boundary

$$\iota^*_{\partial X}\Theta\neq 0. \tag{2.7.7}$$

Then consider the space of smooth vector fields

$$\mathcal{V}_\Theta(X)\subset\mathcal{V}_0(X),\ V\in\mathcal{V}_\Theta(X)\Longleftrightarrow\Theta(V)=O(\rho^2) \text{ at } \partial X. \tag{2.7.8}$$

It is straightforward to check that this is indeed a boundary-fibration structure (see [**2**]). The Θ-structure only depends on the conformal class of Θ at the boundary (but not just on the boundary, i.e. as a section of $T^*_{\partial X}X$). One important special case of Θ-structures is given by strictly pseudoconvex domains in complex manifolds.

As we shall discuss below there are many other interesting cases of boundary-fibration structures.

2.8. The ring $\mathrm{Diff}^*_{\mathcal{V}}(X)$

Since $\mathcal{V}$ is a Lie algebra and $\mathcal{C}^\infty$-module we can easily pass to its enveloping algebra. That is, we consider differential operators

$$P\colon \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \tag{2.8.1}$$

which can be written as sums of up to m-fold products of elements of $\mathcal{V}$:

$$P = \sum_{\text{finite}} V_{i,1}\cdots V_{i,j} + a, \quad V_{i,j} \in \mathcal{V},\ j \le m,\ a \in \mathcal{C}^\infty(X). \tag{2.8.2}$$

We denote by $\mathrm{Diff}^m_{\mathcal{V}}(X)$ the space of such differential operators. In particular $\mathrm{Diff}^0_{\mathcal{V}}(X) = \mathcal{C}^\infty(X)$ acting by multiplication.

Let ${}^{\mathcal{V}}T^*X$ be the dual bundle to the structure bundle ${}^{\mathcal{V}}TX$. Thus any element $V \in \mathcal{V}$ defines a smooth linear map on ${}^{\mathcal{V}}T^*X$, the symbol of V is then defined by

$${}^{\mathcal{V}}\sigma_1(V) = iV \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}T^*X). \tag{2.8.3}$$

Let $P^m({}^{\mathcal{V}}T^*X)$ be the space of $\mathcal{C}^\infty$ functions on ${}^{\mathcal{V}}T^*X$ which are homogeneous polynomials of degree m on the fibres. The symbol in (2.8.3) extends to define ${}^{\mathcal{V}}\sigma_1(P) \in P^1({}^{\mathcal{V}}T^*X)$ for any $P \in \mathrm{Diff}^1_{\mathcal{V}}(X) = \mathcal{V} + \mathcal{C}^\infty(X)$. The symbol of $P \in \mathrm{Diff}^m_{\mathcal{V}}(X)$, of the form (2.8.2), is the sum of the m-fold products

$${}^{\mathcal{V}}\sigma_m(P) = \sum_{\substack{\text{finite} \\ m \text{ factors}}} {}^{\mathcal{V}}\sigma_1(V_{i,1})\dots{}^{\mathcal{V}}\sigma_1(V_{i,m}) \tag{2.8.4}$$

LEMMA 2.8.1. *The symbol of an element $P \in \mathrm{Diff}^m_{\mathcal{V}}(X)$ given by* (2.8.4) *is independent of the presentation* (2.8.2). *The resulting symbol map gives a short exact sequence*

$$0 \longrightarrow \mathrm{Diff}^{m-1}_{\mathcal{V}}(X) \hookrightarrow \mathrm{Diff}^m_{\mathcal{V}}(X) \xrightarrow{{}^{\mathcal{V}}\sigma_m} P^m({}^{\mathcal{V}}T^*X) \longrightarrow 0. \tag{2.8.5}$$

PROOF. Let U_i be a basis for $\mathcal{V}$ as in (2.5.3). Since $\mathcal{V}$ is a Lie algebra the ordered products of up to m of the basis elements form a local basis for $\mathrm{Diff}^m_{\mathcal{V}}(X)$ over $\mathcal{C}^\infty(X)$. In terms of this basis the symbol is therefore well defined by (2.8.4) and vanishes precisely on the subspace of operators of order $m-1$, so (2.8.5) holds. It therefore suffices to show that the symbol is independent of the basis used to defined it but this is an immediate consequence of the decomposition of one basis in terms of another. $\square$

The main initial problem, from an analytic point of view, is the investigation of the *elliptic* elements of $\mathrm{Diff}^m_{\mathcal{V}}(X)$. Ellipticity in the space $\mathrm{Diff}^m_{\mathcal{V}}(X)$ is the condition that the symbol should be non-vanishing on ${}^{\mathcal{V}}T^*X \setminus 0$. Typical questions we examine include the mapping, invertibility, Fredholm and index properties of the elliptic elements. We shall carry this investigation forward by 'microlocalizing' $\mathcal{V}$. This construction introduces spaces, $\Psi^*_{\mathcal{V}}(X)$, of $\mathcal{V}$-pseudodifferential operators which at least start the process of inverting the elliptic operators. To carry out this microlocalization we need to consider some basic analysis on X, as well as the geometry of the boundary-fibration structure $\mathcal{V}$.

The space $\mathrm{Diff}^m_{\mathcal{V}}(X)$ can be generalized to define the notion of a $\mathcal{V}$-differential operator acting from sections of one vector bundle, K, to sections of another, L. Namely $\mathrm{Diff}^m_{\mathcal{V}}(X; K, L)$ consists of those linear operators

$$P\colon \mathcal{C}^\infty(X; K) \longrightarrow \mathcal{C}^\infty(X; L) \tag{2.8.6}$$

such that whenever $u \in \mathcal{C}^\infty(X;K)$ and $g \in \mathcal{C}^\infty(X;L^*)$

$$\mathcal{C}^\infty(X) \ni \phi \longmapsto Q\phi = \langle g, P(\phi u)\rangle \in \mathcal{C}^\infty(X) \tag{2.8.7}$$

is an element of $\mathrm{Diff}^m_{\mathcal{V}}(X)$. If $U \subset X$ is an open set on which both K and L have trivializations with $e_i \in \mathcal{C}^\infty(X;K)$ and $f_j \in \mathcal{C}^\infty(X;L)$ giving bases over U then it follows from (2.8.7) that

$$P(\sum_i \phi_i e_i) = \sum_j (P_{ij}\phi_i) f_j \ \forall\ \phi_i \in \mathcal{C}^\infty_c(U) \text{ with } P_{ij} \in \mathrm{Diff}^m_{\mathcal{V}}(X). \tag{2.8.8}$$

Moreover if the P_{ij} are given for a covering of X by trivializations and corresponding (global) sections then P is determined by (2.8.8) and summation over a partition of unity.

For any vector bundles K and L over X let $P^m(K;L)$ be the space of smooth maps from the total space of K to that of L which map the fibre K_x into L_x for each $x \in X$ and restrict to homogeneous polynomials of degree m on each fibre. This is consistent with the notation above if $P^m(K)$ is identified with $P^m(K;L)$ with L the trivial line bundle $X \times \mathbb{C}$.

Proposition 2.8.1. *For any boundary-fibration structure, $\mathcal{V}$, on a manifold with corners X and vector bundles K and L over X the symbols of the local operators in* (2.8.8) *combine to give a well-defined symbol map*

$$^{\mathcal{V}}\sigma_m \colon \mathrm{Diff}^m_{\mathcal{V}}(X;K,L) \longrightarrow P^m(^{\mathcal{V}}T^*X;\hom(K;L)) \tag{2.8.9}$$

and corresponding short exact (symbol) sequence

(2.8.10)

$$0 \longrightarrow \mathrm{Diff}^{m-1}_{\mathcal{V}}(X;K,L) \hookrightarrow \mathrm{Diff}^m_{\mathcal{V}}(X;K,L) \xrightarrow{^{\mathcal{V}}\sigma_m} P^m(^{\mathcal{V}}T^*X;\hom(K;L)) \longrightarrow 0.$$

Proof. Let $U^{(r)}$ be a covering of X by open sets on which K and L both have trivializations. Take $e^{(r)}_i \in \mathcal{C}^\infty(X;K)$, $f^{(r)}_j \in \mathcal{C}^\infty(X;L)$ and $g^{(r)}_j \in \mathcal{C}^\infty(X;L^*)$ giving bases and with $\langle f^{(r)}_j, g^{(r)}_k\rangle = \delta_{jk}$ on $U^{(r)}$. For $P \in \mathrm{Diff}^m_{\mathcal{V}}(X;K,L)$ let $P^{(r)}_{ij} \in \mathrm{Diff}^m_{\mathcal{V}}(X)$ be defined by (2.8.8) for each r. If $\phi_r \in \mathcal{C}^\infty(X)$ is a partition of unity subordinate to the covering set

$$^{\mathcal{V}}\sigma(P) = \sum_r \sum_{i,j} \phi_r {}^{\mathcal{V}}\sigma(P^r_{ij} g^{(r)}_j \otimes e^{(r)}_j \in P^m(^{\mathcal{V}}T^*X;\hom(K,L)). \tag{2.8.11}$$

Here we use the identification

$$\hom(K,L) \cong L^* \otimes K$$

as bundles over X. The $g^{(r)}_j$ are fixed over $U^{(r)}$ as the dual basis to $f^{(r)}_j$ and since ϕ_r has support in $U^{(r)}$ it follows that (2.8.11) is independent of the choice of the $g^{(r)}_k$. In fact it is easy to see, using Lemma 2.8.1, that this defines $^{\mathcal{V}}\sigma(P)$ independently of all choices and that (2.8.10) holds. □

One simple consequence of our assumption that $\mathcal{V} \subset \mathcal{V}_b(X)$ is that conjugation by powers of boundary defining functions always preserves these spaces:

Lemma 2.8.2. *For any boundary-fibration structure on a compact manifold with corners and any product ρ^β of boundary defining functions*

$$\mathrm{Diff}^m_{\mathcal{V}}(X;K,L) \ni P \longmapsto \rho^{-\beta} P \rho^\beta \in \mathrm{Diff}^m_{\mathcal{V}}(X;K,L) \tag{2.8.12}$$

is always an isomorphism.

PROOF. Certainly this is true for $m = 0$. The definition of $\mathrm{Diff}^m_{\mathcal{V}}(X; K, L)$ shows that it is enough to check the case $m = 1$ and then

$$\rho^{-\beta} V \rho^{\beta} u = Vu + q_V u \text{ with } q_V \in \mathcal{C}^\infty(X) \ \forall\ V \in \mathcal{V}$$

since $\mathcal{V} \subset \mathcal{V}_b(X)$. □

2.9. Forms and metrics

There are many geometric examples of $\mathcal{V}$-differential operators. In particular to any boundary-fibration structure there corresponds an operator of exterior differentiation. This acts on the $\mathcal{V}$-form bundles, ${}^{\mathcal{V}}\!\Lambda^k$, which are the k-fold antisymmetric powers of ${}^{\mathcal{V}}T^*X$. Over the interior of X, ${}^{\mathcal{V}}T^*X$ can be identified with the fibre forms on the integral submanifolds of $\mathcal{V}$. In particular this means that exterior differentiation,

$$d \colon \mathcal{C}^\infty(X^\circ; {}^{\mathcal{V}}\!\Lambda^j X) \longrightarrow \mathcal{C}^\infty(X^\circ; {}^{\mathcal{V}}\!\Lambda^{j+1} X), \tag{2.9.1}$$

is well defined by restriction to each fibre.

PROPOSITION 2.9.1. *For any boundary-fibration structure on a compact manifold with corners, X, exterior differentiation restricts from* (2.9.1) *to give*

$$d_{\mathcal{V}} : \mathcal{C}^\infty(X; {}^{\mathcal{V}}\!\Lambda^j X) \longrightarrow \mathcal{C}^\infty(X; {}^{\mathcal{V}}\!\Lambda^{j+1} X)\ \forall\ j,\ d_{\mathcal{V}} \in \mathrm{Diff}^1_{\mathcal{V}}(X; {}^{\mathcal{V}}\!\Lambda^j X, {}^{\mathcal{V}}\!\Lambda^{j+1} X). \tag{2.9.2}$$

PROOF. Let $V_1, \dots, V_N$ be a local frame for ${}^{\mathcal{V}}TX$, i.e. a local basis for $\mathcal{V}$. Let $\alpha_1, \dots \alpha_N$ be the dual frame for ${}^{\mathcal{V}}T^*X$. The action of d on functions in (2.9.1) can be written

$$d_{\mathcal{V}}\phi = \sum_{i=1}^{N} (V_i\phi)\alpha_i \ \forall\ \phi \in \mathcal{C}^\infty(X). \tag{2.9.3}$$

This proves (2.9.2) for $j = 1$. Cartan's formula for the exterior derivative of a 1-form

$$d\alpha(V, W) = V\alpha(W) - W\alpha(V) - \alpha([V, W])$$

shows that $d_{\mathcal{V}}\alpha_i \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}\!\Lambda^2 X)$. The general distribution formula

$$d_{\mathcal{V}}(\alpha \wedge \beta) = d_{\mathcal{V}}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_{\mathcal{V}}\beta$$

then can be used to show that (2.9.2) holds for all j. First it follows that

$$\alpha_I = \alpha_{i_1} \wedge \dots \wedge \alpha_{i_j} \longmapsto d_{\mathcal{V}}(\alpha_I) \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}\!\Lambda^{j+1} X) \tag{2.9.4}$$

for any choice of indices $I = i_1, \dots, i_j$, and hence that

$$d_{\mathcal{V}}(\phi\alpha_I) = d_{\mathcal{V}}\phi \wedge \alpha_I + \phi d_{\mathcal{V}}(\alpha_I) \tag{2.9.5}$$

which is the general case. □

Now consider a smooth and positive-definite section of the symmetric part, $\mathrm{Sym}^2({}^{\mathcal{V}}T^*X)$ of the tensor product ${}^{\mathcal{V}}T^*X \otimes {}^{\mathcal{V}}T^*X$. Over the interior this is a metric on the leaves of the fibration of X°. In general we call it a $\mathcal{V}$-metric; it is just a fibre metric on ${}^{\mathcal{V}}TX$. Such a metric induces a fibre inner product on each of the bundle ${}^{\mathcal{V}}\!\Lambda^j X$ in the usual way: If the α_i are chosen to be orthonormal then

$$\langle \alpha_I, \alpha_{I'} \rangle = \delta_{II'},\ i_1 \le i_2 \dots \le i_j,\ i'_1 \le i'_2 \dots \le i'_j. \tag{2.9.6}$$

Moreover, at least in the interior, there is a metric density on each leaf.

PROPOSITION 2.9.2. *If, in* X°, δ_g *is the transpose of* $d_{\mathcal{V}}$ *with respect to the inner product* (2.9.6) *and metric volume form then* $\delta_g \in \mathrm{Diff}^1_{\mathcal{V}}(X; {}^{\mathcal{V}}\Lambda^{j+1}X, {}^{\mathcal{V}}\Lambda^j X)$ *for each* j *and the* $\mathcal{V}$*-Laplacian*

$$\Delta_g = \delta_g d_{\mathcal{V}} + d_{\mathcal{V}}\delta_g \in \mathrm{Diff}^2_{\mathcal{V}}(X; {}^{\mathcal{V}}\Lambda^j X) \tag{2.9.7}$$

is elliptic with symbol given by the metric function $|\zeta|^2$ *on* ${}^{\mathcal{V}}T^*X$.

PROOF. Certainly δ_g is well-defined in the interior, to show that it is a $\mathcal{V}$-differential operator it suffices to compute the form of $\delta_g(\psi\alpha_I)$ for each $(j+1)$-index I and with $\psi \in \mathcal{C}^\infty(X)$. Now writing

$$\delta_g(\psi\alpha_I) = \sum_{|J|=j} [\delta_g(\psi\alpha_I)]_J \alpha_J$$

and assuming that all functions are smooth with supports in the interior

$$\begin{aligned} &\int [\delta_g(\psi\alpha_I)]_J \psi dg \\ &\quad = \int \langle \delta_g(\psi\alpha_I), \phi\alpha_J\rangle dg \\ &\quad = \int \langle \psi\alpha_I, d_{\mathcal{V}}(\phi\alpha_J)\rangle dg \\ &\quad = \int \psi[d_{\mathcal{V}}(\phi\alpha_J)]_I dg. \end{aligned} \tag{2.9.8}$$

From (2.9.5) we can easily compute the form of the last integral. The second term in (2.9.5) contributes a $\mathcal{C}^\infty$ tensorial term, which contributes a $\mathcal{C}^\infty$ tensorial term, μ, to δ_g. From (2.9.3) we see that

$$\int [(\delta_g - \mu)(\psi\alpha_I)]_J \psi dg = \pm \int \psi V_i \phi dg \cdot \delta_{\{I,i\},J}. \tag{2.9.9}$$

Thus it is enough to show that the transpose, with respect to dg, of any element $V \in \mathcal{V}$ is in $\mathrm{Diff}^1_{\mathcal{V}}(X)$. Of course the differential part of the transpose is just $-V$. The multiplicative term is μ' which is the divergence of V with respect to dg, i.e.

$$d[\iota_V(\alpha_{\{1,\dots,N\}})] = \mu'(\alpha_{\{1,\dots,N\}}). \tag{2.9.10}$$

This shows that $\delta_g \in \mathrm{Diff}^1_{\mathcal{V}}(X; {}^{\mathcal{V}}\Lambda^{j+1}X, {}^{\mathcal{V}}\Lambda^j X)$.

If $G\colon {}^{\mathcal{V}}T^*X \longleftrightarrow {}^{\mathcal{V}}TX$ is the metric isomorphism then the symbol of δ_g at $\zeta \in {}^{\mathcal{V}}T^*X$ is clearly given by the contraction $\iota_{G(\zeta)}$. Thus, from the definition (2.9.7),

$$\Delta_g \in \mathrm{Diff}^2_{\mathcal{V}}(X; {}^{\mathcal{V}}\Lambda^j X) \text{ and } {}^{\mathcal{V}}\sigma(\Delta_g) = \iota_{G(\zeta)}(\zeta\wedge) + (\zeta\wedge)\iota_{G(\zeta)} = |\zeta|^2 \tag{2.9.11}$$

as stated. □

This computation explains in large measure why the treatment of elliptic $\mathcal{V}$-differential operators can lead to results on, for example, L^2-cohomology.

There is a further useful extension of Proposition 2.9.2 . Namely suppose that g is a $\mathcal{V}$-metric on a manifold with corners. As usual let ρ_i be defining functions for the boundary hypersurfaces and consider a (singularly) conformal metric

$$g' = \rho^{-2\beta} g. \tag{2.9.12}$$

Here β is a multi-index, which we take to be integral but with possibly negative elements. Before analyzing such metrics let us consider an example.

On $\mathbb{R}^n$ consider the Euclidean metric

$$g_e = dx_1^2 + \cdots + dx_n^2. \tag{2.9.13}$$

To talk about a boundary-fibration structure we need to have a compact space (admittedly only by fiat.) Let us take the stereographic compactification of $\mathbb{R}^n$. Thus introduce polar coordinates near infinity:

$$r^2 = x_1^2 + \cdots + x_n^2,\ \omega = x/r$$

in terms of which

$$g_e = dr^2 + r^2 d\omega^2 \tag{2.9.14}$$

where $d\omega^2$ is the standard metric on the sphere. Now to compactify the space set $t = 1/r$. Then a neighbourhood of infinity looks like $\{0 < t < \epsilon\} \times \mathbb{S}^{n-1}$ and the metric is

$$g_e = \left(\frac{dt}{t^2}\right)^2 + t^{-2} d\omega^2 = t^{-2}\left[\left(\frac{dt}{t}\right)^2 + d\omega^2\right]. \tag{2.9.15}$$

This reduces the Euclidean metric to a conformal multiple of a b-metric.

Proposition 2.9.3. *For a conformal multiple,* (2.9.12)*, of a $\mathcal{V}$-metric the co-differential*

$$\delta_g = \rho^{2\beta}\delta'_g,\ \delta'_g \in \operatorname{Diff}^1_{\mathcal{V}}(X; {}^{\mathcal{V}}\Lambda^{j+1}X, {}^{\mathcal{V}}\Lambda^j X) \tag{2.9.16}$$

and the Laplacian

$$\Delta_g = \rho^{2\beta}\Delta'_g,\ \Delta'_g \in \operatorname{Diff}^2_{\mathcal{V}}(X; {}^{\mathcal{V}}\Lambda^j X). \tag{2.9.17}$$

Proof. Since γ' is a conformal multiple of g we see that the metric densities are related by

$$dg' = \rho^{-N\beta} dg. \tag{2.9.18}$$

Similar g' induces an inner product on the $\mathcal{V}$-form bundles; on j-forms it is just

$$\langle , \rangle' = \rho^{2j\beta}\langle , \rangle. \tag{2.9.19}$$

If we think of $\delta_{g'}$ as acting on ${}^{\mathcal{V}}\Lambda^j X$ it is therefore

$$\delta_{g'} = \rho^{(-2j+N)\beta}\delta_g \rho^{(2j+2-N)\beta} \tag{2.9.20}$$

and (2.9.16) follows from Lemma 2.8.2 .

Similarly for the Laplacian we see that

$$\Delta_{g'} = \rho^{(-2j-2+N)\beta}\delta_g \rho^{(2j+4-N)\beta} d + d\rho^{(-2j+N)\beta}\delta_g\rho^{(2j+2-N)\beta} \tag{2.9.21}$$

acting on $\mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^j X)$. This gives (2.9.17). □

This result allows us to apply the theory of elliptic $\mathcal{V}$-differential operators to analyze the mapping and solvability properties of the Laplacian of a conformal $\mathcal{V}$-metric (2.9.12). However if $\beta_i > 0$ for some i it is not possible to analyze the resolvent in this way since

$$[\rho^{2\beta}\Delta'_g - \lambda^2]^{-1} = \rho^{-2\beta}[\Delta'_g - \rho^{-2\beta}\lambda^2]^{-1} \tag{2.9.22}$$

cannot then be expressed in terms of the inverse of a $\mathcal{V}$-differential operator. In that case one can of course consider it as a $\rho^\beta\mathcal{V}$-metric i.e. associated to the boundary-fibration structure $\rho^\beta\mathcal{V}$. Cases with some positive and some negative powers can then be handled too.

Thus to analyze the resolvent of the Laplacian (2.9.15) on $\mathbb{R}^n$ by the methods below one needs to consider it as a $\mathcal{V}$-differential operator for the 'scattering' type metric on the ball (half-sphere) associated to an appropriate boundary-fibration structure, $\mathcal{V}_s(\mathbb{S}^n)$. If x is a normal variable near the boundary and y_i are tangential variables the vector fields in $\mathcal{V}_s$ are those of the form

$$x^2\partial_x,\, x\partial_{y_i}. \tag{2.9.23}$$

Thus the 'scattering structure' is just $x\mathcal{V}_b(X)$ so this is actually well-defined on any compact manifold with boundary, $\mathcal{V}_s(X) = x\mathcal{V}_b(X)$.

2.10. Normal fibrations

A diffeomorphism f from a neighbourhood $\Gamma \supset Y$ of a submanifold $Y \subset X$ is said to preserve Y if $f(Y) \subset Y$. In that case f_* maps T_yY onto $T_{f(y)}Y$ for each $y \in Y$ and so induces a map, which we also denote f_* from $N_yY = T_yX/T_yY$ to $N_{f(y)}Y$. Then f is said to preserve (or fix) Y to first order if

$$\begin{gathered} f(y) = y \quad \forall\ y \in Y \\ f_*\colon N_yY \longrightarrow N_yY \text{ is the identity } \forall\ y \in Y. \end{gathered} \tag{2.10.1}$$

As NY is a vector bundle over Y the zero section is canonically identified with Y and the normal bundle to the zero section, in NY, is naturally identified with the vector bundle itself. This means that there is a similar condition to (2.10.1) for a diffeomorphism from a neighbourhood of the zero section of NY to X :

$$\begin{gathered} f\colon \Gamma \longrightarrow X,\ Y \subset \Gamma \subset NY,\ \Gamma \text{ open} \\ f(y) = y \quad \forall\ y \in Y \text{ the zero section} \\ f_*\colon NY \longrightarrow NY \text{ is the identity.} \end{gathered} \tag{2.10.2}$$

We call such a diffeomorphism a normal fibration of X near Y; often the term 'collar neighbourhood' is used. The basic result we wish to establish is the existence of such a normal fibration for a p-submanifold. In that case N^+Y, the inward pointing part of the normal bundle, is itself a manifold with corners so we require the map to be a diffeomorphism in this sense.

Proposition 2.10.1. *If $Y \subset X$ is a closed p-submanifold of a compact manifold with corners then there is a normal fibration of X near Y; if f_1 and f_2 are normal fibrations then f_2f^{-1} satisfies* (2.10.1) *on a small enough neighbourhood of Y.*

Proof. We use the standard method of proof in the case $\partial X = \emptyset$, namely geodesic flow. However, to ensure that f maps the boundary hypersurfaces of N^+Y into those of X we shall use a b-metric and the singular symplectic structure on ${}^bT^*X$. As noted above a b-metric is just a positive definite inner product on the fibres of ${}^bTX$. On $\mathbb{R}^n_k$

$$\sum_{j=1}^{k}(\frac{dx_j}{x_j})^2 + \sum_{j>k} dx_j^2 \tag{2.10.3}$$

is an example and we can always find a global b-metric by using this form in local coordinates and summing over a partition of unity.

As a non-degenerate inner product the metric, g, fixes a vector bundle isomorphism

$$G\colon {}^bT^*X \longrightarrow {}^bTX,\quad v = G(\gamma) \text{ if } \gamma(w) = g(v,w) \quad \forall\ w \in {}^bT_pX. \tag{2.10.4}$$

Just as in the familiar case g lifts to a quadratic function on the fibres of ${}^bT^*X$, the square of the length. We set

$$e = \frac{1}{2}|\xi|^2,\quad \forall\ \xi \in {}^bT^*X. \tag{2.10.5}$$

The total space, Z, of the bundle ${}^bT^*X$ carries a singular canonical 1-form, a section of ${}^bT^*Z$:

$$\alpha(v) = \gamma(\pi_* v) \quad \forall\ v \in {}^bT^*_\gamma Z. \tag{2.10.6}$$

Local coordinates $x_1, \dots, x_n$ in X induce canonically dual coordinates $x_1, \dots, x_n$, $\lambda_1, \dots, \lambda_k$ and $\xi_{k+1}, \dots, \xi_n$ in Z. In these coordinates

$$\alpha = \sum_{j=1}^k \lambda_j \frac{dx_j}{x_j} + \sum_{j>k} \xi_j\, dx_j \tag{2.10.7}$$

as follows from (2.10.6). The canonical 2-form on Z is

$$\omega = -d\alpha = \sum_{j=1}^k \frac{dx_j}{x_j} \wedge d\lambda_j + \sum_{j>k} dx_j \wedge d\xi_j. \tag{2.10.8}$$

This is a non-degenerate antisymmetric bilinear form on ${}^bTX$. Thus the b-Hamilton vector field, on Z, is well-defined by

$$\omega(v, H_e) = de(v) \quad \forall\ v \in {}^bTZ. \tag{2.10.9}$$

The integral curves of H_e on the unit sphere bundle, $e = 1$, are the geodesics of g parameterized by arclength.

If $Y \subset X$ is a closed interior p-submanifold the normal bundle can be identified with the inverse image of ${}^bN^*Y \subset {}^bT^*X$ under G. The map

$$NY \supset \Gamma \xrightarrow{G} {}^bN^*Y \subset {}^bT^*X \xrightarrow{\exp(H_e)} {}^bT^*X \xrightarrow{\pi} X \tag{2.10.10}$$

is a normal fibration if Γ is a small enough neighbourhood of the zero section.

This proof does not work directly for a boundary p-submanifold, because the b-conormal bundle is not naturally identified with the conormal bundle in the usual sense. However, Y *is* an interior p-submanifold of some boundary face B so B has a normal fibration around Y. To get a normal fibration of X from one of B it is only necessary to use defining functions for the boundary hypersurfaces meeting B. Thus B has a neighbourhood in X of the form $[0,1)^k \times B$, where k is the codimension of B. Such a product decomposition gives a normal fibration. □

The importance of such normal fibrations is that they allow NY to be used as a good local model for X near Y. All normal fibrations are homotopic. Indeed it is enough to note that a map (2.10.1) (say on N^+Y) can be connected, possibly in a smaller neighbourhood of the zero section, to the identity by a smooth homotopy of such maps. Indeed this can be done locally in coordinate patches. Thus if f satisfies (2.10.1) there is a $\mathcal{C}^\infty$ 1-parameter family f_t of maps with $f_0 = \mathrm{Id}$ and

$f_1 = f$ all satisfying (2.10.1). Such a family is obtained by the integration of a uniquely determined (parameter dependent) vector field W_t through

$$\frac{d}{dt} f_t^* \phi = f_t^*(W_t \phi) \quad \forall\ \phi \in \mathcal{C}^\infty(\Gamma). \tag{2.10.11}$$

It is particularly important to note that the fact that each f_t fixes Y to first order is reflected in the fact that

$$W_t = \sum_{\text{finite}} \psi_j W_j,\ \psi_j = 0 \text{ on } Y,\ W_j \text{ tangent to } Y. \tag{2.10.12}$$

2.11. Localization and normal operators

To analyze the elliptic differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(X)$ for a boundary-fibration structure we need to be able to solve (generally rather explicitly) certain model problems. Indeed the main point of the pseudodifferential calculus, associated to $\mathcal{V}$, which is developed below is that it allows the model problems to be isolated and then the solutions of the models to be directly incorporated in the solution of the full problem. For the moment we construct the model, or normal, operators abstractly. The construction of appropriate representations will be tackled later.

Thus suppose $\mathcal{V}$ is a boundary-fibration structure on a compact manifold with corners, X. The condition (2.5.9) means, in particular, that every point of X lies in an integral submanifold of $\mathcal{V}$. The Lie algebra can be localized at any such submanifold. If $B \in M(X)$ is a boundary face and $F \subset B$ is a leaf of the fibration (2.5.9) which meets the interior of B we set

$$\mathcal{I}_F = \{f \in \mathcal{C}^\infty(X); f = 0 \text{ on } F\} \text{ and } \mathcal{W}^F = \mathcal{V}/\left[\mathcal{I}_F \cdot \mathcal{V}\right]. \tag{2.11.1}$$

Since the elements of $\mathcal{V}$ are tangent to F the subspace $\mathcal{I}_F \cdot \mathcal{V} \subset \mathcal{V}$ is an ideal so the quotient is itself a Lie algebra. We call $\mathcal{W}^F$ the normal algebra at F. It is a $\mathcal{C}^\infty(F)$-module since if $[V] \in \mathcal{W}^F$ and $\phi \in \mathcal{C}^\infty(F)$, $[\tilde{\phi}V] \in \mathcal{W}^F$ is independent of the extension of ϕ to $\tilde{\phi} \in \mathcal{C}^\infty(X)$ and the choice of V.

Consider the enveloping algebra $\mathcal{D}^*[\mathcal{W}^F]$ of $\mathcal{W}^F$. This is, by definition, the quotient of the full tensor algebra (over $\mathcal{C}^\infty(F)$) by the relations

$$V \cdot \phi - \phi \cdot V = V_{|F}\phi,\ \phi \in \mathcal{C}^\infty(F), V \in \mathcal{W}^F \tag{2.11.2}$$

$$V \cdot W - W \cdot V = [V, W],\ V, W \in \mathcal{W}^F. \tag{2.11.3}$$

It is filtered by the tensor order. There is an induced map of the enveloping algebras

$$\mathrm{Diff}^m_{\mathcal{V}}(X) \longrightarrow \mathcal{D}^m[\mathcal{W}^F]. \tag{2.11.4}$$

LEMMA 2.11.1. *The map of enveloping algebras* (2.11.4) *gives a short exact sequence*

$$0 \hookrightarrow \mathcal{I}_F \cdot \mathrm{Diff}^m_{\mathcal{V}}(X) \hookrightarrow \mathrm{Diff}^m_{\mathcal{V}}(X) \longrightarrow \mathcal{D}^m[\mathcal{W}^F] \longrightarrow 0, \quad \forall\ m. \tag{2.11.5}$$

PROOF. The map (2.11.4) is well defined since the relations on $\mathrm{Diff}^*_{\mathcal{V}}(X)$, defined as the quotient of the full tensor algebra of $\mathcal{V}$ over $\mathcal{C}^\infty(X)$, map into the relations on $\mathcal{D}^*[\mathcal{W}^F]$. Certainly the map is surjective and its null space consists of the image in $\mathrm{Diff}^*_{\mathcal{V}}(X)$ of the elements of the tensor algebra in which at least one element is mapped to zero in $\mathcal{W}^F$. This gives (2.11.5). □

The image of $P \in \operatorname{Diff}^m_{\mathcal{V}}(X)$ is called the normal operator, of P, at F. In this form it is not immediately given as a differential operator. The construction of representations in Chapter 5 shows that the normal operator can indeed be considered as a differential operator.

We can also get a Lie algebra simply by restricting to F :

$$\mathcal{V}^F = \mathcal{V} \restriction_F \subset \mathcal{V}_b(F). \tag{2.11.6}$$

Thus induces a boundary-fibration structure on F.

Lemma 2.11.2. *The restriction,* (2.11.6)*, of a boundary-fibration structure to an integral leaf F, meeting the interior of the boundary face B, is a boundary-fibration structure on F the structure bundle of which can be identified with the image ${}^{\mathcal{V}}TF \subset {}^bT_FB$ of ${}^{\mathcal{V}}T_FX$ under $e_{\mathcal{V}}$.*

Proof. All the conditions in Definition 2.5.15 clearly persist on restriction. ☐

We consider in ${}^{\mathcal{V}}T_xX$, for $x \in B^\circ$, the null space of the restriction to B. Again because of (2.5.9) this extends to a subbundle over B, which we denote ${}^{\mathcal{V}}NB$. Naturally the image of ${}^{\mathcal{V}}N_FB$ in ${}^{\mathcal{V}}TF$ is zero, in fact:

Lemma 2.11.3. *For any integral leaf, F, of a boundary-fibration structure there is a natural short exact sequence of vector bundles*

$$0 \longrightarrow {}^{\mathcal{V}}N_FB \longrightarrow {}^{\mathcal{V}}T_FX \longrightarrow {}^{\mathcal{V}}TF \longrightarrow 0 \tag{2.11.7}$$

which arises from a short exact sequence of Lie algebras

$$0 \longrightarrow \mathcal{N}^F = \frac{\{V \in \mathcal{V}; V = 0 \text{ on } B\}}{\mathcal{I}_F \cdot \{V \in \mathcal{V}; V = 0 \text{ on } B\}} \longrightarrow \mathcal{W}^F \longrightarrow \mathcal{V}^F \longrightarrow 0. \tag{2.11.8}$$

Proof. As defined in (2.11.8) $\mathcal{N}^F$ is a Lie algebra since it is the quotient of the ideal in $\mathcal{V}$ consisting of the elements which vanish when restricted to B by the smaller ideal obtained by adding factors of $\mathcal{I}_F$. The null space of the map from $\mathcal{W}^F$ to $\mathcal{V}^F$ consists of the vector fields which vanish when restricted to F. Thus to show that the first map in (2.11.8) is surjective we need to check that each element $V \in \mathcal{V}$ which vanishes on F can be written

$$V = V' + \sum_{\text{finite}} a_i W_i,\ a_i \in \mathcal{I}_F,\ W_i \in \mathcal{V}\ V'_F = 0. \tag{2.11.9}$$

Away from F this is trivial and it suffices to prove it locally near each point of F and then to use a partition of unity. Locally the decomposition follows from the existence of a basis as in (2.5.3) since in the decomposition of V the coefficients of the basis elements not vanishing on F must be in $\mathcal{I}_F$. The same decomposition shows that the first map in (2.11.8) is also injective. The exactness of ☐

We call $\mathcal{N}^F$ the null algebra of $\mathcal{V}$ at F. We shall examine its structure in more detail below; (2.11.8) indicates that the normal operator at F should be regarded as a differential operator on F, corresponding to the induced boundary-fibration structure, with additional 'hidden' variables in the fibres of ${}^{\mathcal{V}}N_FB$.

Proposition 2.11.1. *For any integral leaf F of a boundary-fibration structure $\mathcal{V}$ the fibres of the bundle ${}^{\mathcal{V}}NF$ over F have the structure of finite dimensional solvable Lie algebras, with natural filtration, ${}^{\mathcal{V}}N^\alpha B$ for $\alpha \in \mathbb{N}^k$ given by*

$$[{}^{\mathcal{V}}N^\alpha_x, {}^{\mathcal{V}}N^\beta_x] \subset {}^{\mathcal{V}}N^{\alpha+\beta}_x \quad \forall\ x \in B,\ [{}^{\mathcal{V}}N_x, {}^{\mathcal{V}}N_x] \subset \sum_{|\alpha|=1} {}^{\mathcal{V}}N^\alpha_x \tag{2.11.10}$$

and with structure constants varying smoothly in any local trivialization.

PROOF. The filtration of $\mathcal{N}^F$ is given by the degree of vanishing of the generators of V at B°. Thus for $\alpha \in \mathbb{N}^k$ set

$$\mathcal{N}^{\alpha,F} = \frac{\{V \in \mathcal{V}; V = 0 \text{ on } B\},\ V \in \mathcal{V}_b(X) \text{ near } F}{\mathcal{I}_F \cdot \{V \in \mathcal{V}; V = 0 \text{ on } B\}}. \tag{2.11.11}$$

Thus $\mathcal{N}^F = \mathcal{N}^{0,F}$. In fact each $\mathcal{N}^{\alpha,F}$ is an ideal in $\mathcal{N}^F$ and by (2.5.3) $\mathcal{N}^{\alpha,F} = 0$ if $\alpha \geq \alpha'$ for some α'. In general ${}^{\mathcal{V}}N^\alpha F \subset {}^{\mathcal{V}}N^\beta F$ if $\alpha \geq \beta$. Again from (2.5.3) each of the ${}^{\mathcal{V}}N^\alpha F$ generates a subbundle of ${}^{\mathcal{V}}NF$ by continuous extension from the interior. □

2.12. Quasi-homogeneous normal fibrations

PROPOSITION 2.12.1. *Any bundle filtration,* (1.15.10)*, of the conormal bundle to a p-submanifold which satisfies the first part of* (1.15.14) *can be extended to a quasi-homogeneous structure with this filtration occuring as in* (1.15.7).

PROOF. Consider first the case of a point in $X = \mathbb{R}^n$ so $N^*Y = \mathbb{R}^n$ is naturally the dual space. With each $S_{k(j)}$ is associated the integer $k(j)$ and we assign weight (precisely) $k(j)$ to the linear functions in the space $S_{k(j)} \ominus S_{k(j+1)}$. This notion of weight extends uniquely to the polynomials on $\mathbb{R}^n$ and then $\mathcal{F}^{(q)}(Y)$ can be taken to be the space of $\mathcal{C}^\infty$ functions on X with Taylor series having all terms of weight at least q. It is straightforward to check that this is a quasi-homogeneity structure at $Y = \{0\}$ which corresponds to the given subspaces.

In the general case take a metric on the fibres of N^*Y and consider the decomposition

$$N^*Y = U_1 \oplus U_2 \oplus \cdots \oplus U_r \tag{2.12.1}$$

where $U_r = S_{k(r)}$ and the successive U_j's are for $j = r-1, r-2, \dots, 1$, the orthocomplements of $S_{k(j+1)}$ in $S_{k(j)}$. Using a normal fibration of X around Y the subbundles U_j each defines a local manifold, which we again denote U_j, near Y. Then define the successive ideals $\mathcal{F}^{(q)}(Y)$ as consisting of those $\mathcal{C}^\infty$ functions of weight q along the fibres in the linear sense discussed above. Again this gives a quasi-homogeneity structure which corresponds to the given bundles. □

The quasi-homogeneity structure associated to a filtration of the conormal bundle of a p-submanifold by the choice of a normal fibration, used in this proof, is actually a general model:

PROPOSITION 2.12.2. *For any quasi-homogeneity structure at a p-submanifold of a manifold with corners there is a normal fibration which linearizes it in the sense that if S_j° are the annihliators bundles of the S_j in NY then*

$$\mathcal{F}^{(q)}(Y) = \left\{u \in \mathcal{C}^\infty(X); u_{|S_j^\circ} \text{ vanishes to order } q-j \text{ at } Y\right\}\ \forall\ j. \tag{2.12.2}$$

PROOF. □

We shall make use of this linearization in the treatment of quasi-radial blow up in Chapter 5.

COROLLARY 2.12.1. *Let Y be a p-submanifold of X, then for any two quasi-homogeneity strutctures, $\mathcal{F}_i^{(*)}(Y)$, $i = 1, 2$, which give the same bundle filtration of*

N^*Y there is a 1-parameter family of diffemorphisms F_t of X fixing Y and NY such that $F_0 = \mathrm{Id}$ and

$$F_1^* \mathcal{F}_2^{(q)}(Y) = \mathcal{F}_1^{(q)}(Y) \ \forall\ q. \tag{2.12.3}$$

2.13. Quasi-homogeneity at leaves

We shall show that a boundary-fibration structure induces a quasi-homogeneity structure on each leaf, F, for which there are no local true parameters. By this we mean that the interior b-fibration, ϕ, is such that

$$\phi_* \colon T_p \operatorname{Fa}(p) \longrightarrow T_\gamma(p) Y \text{ vanishes } \ \forall\ p \in F. \tag{2.13.1}$$

This condition can be expressed in terms of the local basis in (2.5.3):

Lemma 2.13.1. *There are no local true parameters at a leaf F of a boundary-fibration structure $\mathcal{V}$ if and only if at each point $p \in F$ the $V_i(p) \in T_pX$ for $i \in J(p)$ in* (2.5.3) *span $T_p \operatorname{Fa}(p)$.*

Proof. As in (2.13.1) let $\phi\colon X \longrightarrow Y$ be the b-fibration for the interior, the existence of which is required in (2.5.9). Since the coefficients in (2.5.3) expressing the U_i in terms of the V_i vanish only at the boundary, the null space of the differential of ϕ must be exactly the span of the V_i for $i \in J(p)$ in the interior near p. Since the V_i are independent in $\mathcal{V}_b(X)$ and the null space of ${}^b\phi_*$ is a subbundle of ${}^bTX$, by continuity the V_i must actually span this null space everywhere near p. The relationship between ϕ_* and ${}^b\phi_*$ is such that if $v \in T_p \operatorname{Fa}(p)$ and $w \in {}^bT_pX$ restricts to v then $\phi_*(v) \in T_{\phi(p)}Y$ is the restriction to $\operatorname{Fa}(\phi(p))$ in Y of ${}^b\phi_*(w)$. Thus the null space of ϕ_* on $T_p \operatorname{Fa}(p)$ is exactly the span of the projections into $T_p \operatorname{Fa}(p)$ of the $V_i(p)$. This shows the equivalence to (2.13.1) and the condition in the statement of the lemma. □

Consider the space

$$\widetilde{\mathcal{F}}^{(q)}(F) = \{u \in \mathcal{I}(F); \mathcal{V}u \subset \mathcal{I}(F)^q\},\ q \geq 1. \tag{2.13.2}$$

This is an ideal but to conform with the property (1.15.7) we enlarge it to include appropriate products of the larger ideals:

$$\mathcal{F}^{(q)}(F) = \widetilde{\mathcal{F}}^{(q)} + \sum_{\substack{r_1 \dots r_t \geq q,\\ r_j \geq 1,\ j \leq q}} \widetilde{\mathcal{F}}^{(r_1)} \cdot \widetilde{\mathcal{F}}^{(r_2)} \cdots \widetilde{\mathcal{F}}^{(r_q)}, q \geq 1. \tag{2.13.3}$$

The sum can be taken to be finite since $\mathcal{I}(F)^q \subset \widetilde{\mathcal{F}}^{(q)}$.

Proposition 2.13.1. *If F is a leaf of a boundary-fibration structure $\mathcal{V}$ with no local true parameters, in the sense of* (2.13.1), *then* (2.13.2) *and* (2.13.3) *define a quasi-homogeneity structure at F.*

Proof. Certainly the spaces $\mathcal{F}^{(q)}(F)$ form a decreasing sequence of ideals, with $\mathcal{F}^{(1)}(F) = \mathcal{I}(F)$, by definition. Consider the map

$$\mathcal{F}^{(q)}(F) \ni u \longmapsto (du)_{|F} \in \mathcal{C}^\infty(F; N^*F),\ q \geq 1. \tag{2.13.4}$$

The additional terms in (2.13.3) have vanishing differentials at F so the image in (2.13.4) is the same if the domain is restricted to $\widetilde{\mathcal{F}}^{(q)}(F)$ given by (2.13.2). Consider a point $p \in F$ The assumed local form, (2.5.3), of $\mathcal{V}$ shows that if $u \in \mathcal{F}^{(q)}(F)$ then u must satisfy

$$\langle du, V_i\rangle = 0 \text{ on } F \text{ near } p \text{ if } i \in J(p) \text{ and } |\alpha_i| < q. \tag{2.13.5}$$

since otherwise $U_i u \notin \mathcal{I}^q(F)$. Set

$$k = \max |\alpha_i|. \tag{2.13.6}$$

Notice that if $k = 0$ then $F = \mathrm{Fa}(F)$ and (2.13.5) implies that $u = 0$ on F. Even if $k \ge 1$ Lemma 2.13.1 shows that for $q > k$ (2.13.5) implies that the restriction of u to $\mathrm{Fa}(p)$ is stationary at p, i.e. that $(du)_{\restriction F} \in \mathcal{C}^\infty(F; N^* \mathrm{Fa}(F))$. Recall the restriction (2.5.13) on a boundary-fibration structure. □