CHAPTER 1

# Manifolds with corners

This chapter contains material preliminary to, and surrounding, the notion of a manifold with corners. The precise definition is delayed until §1.8 because embedded submanifolds have to be discussed first, as we insist (mainly for the sake of simplicity) on the boundary faces of a manifold with corners being embedded. This ensures that 'passage to the boundary' stays within the class of manifolds with corner but requires us to deal with the preliminary notion of a t-manifold ('$t$' for tied). The material is quite standard with the exception of the concept of a $b$-map, introduced in §1.12. In the sequel the 'category' in which we usually work is that of manifolds with corners and $b$-maps.

Manifolds with corners arise, as will be shown later, in various ways. Constructions leading to manifolds with corners include the desingularization of singular varieties (blow-up) and the compactification of non-compact spaces.

An example of the first type of construction is the introduction of polar coordinates around the tip of a cone

$$C = \{(t, x, y) \in \mathbb{R}^3; t^2 = x^2 + y^2,\ t \geq 0\},\ (t, x, y) = t(1, \omega),\ |\omega|^2 = 1. \tag{1.0.1}$$

Clearly this reduces the cone to a half-cylinder

$$\beta\colon [0, \infty) \times \mathbb{S}^1 \ni (t, \omega) \longmapsto (t, t\omega) \in C. \tag{1.0.2}$$

Such blow-up constructions will be examined in more detail in Chapter 5. They play a very fundamental rôle below.

The second type of construction is exemplified by the stereographic projection of Euclidean space onto a half-sphere. If the half-sphere, $\mathbb{S}^n_+$, of dimension $n$ is embedded as the upper half (i.e. $x_{n+1} \geq 0$) of the unit sphere in $\mathbb{R}^{n+1}$ and the Euclidean space of dimension $n$ is embedded as the plane $x_{n+1} = 1$ then stereographic projection identifies the unit vector $\omega = (\omega', \omega_{n+1}) \in \mathbb{S}^n_+$ with the vector $(\omega'/\omega_{n+1}, 1)$. The boundary of the half-sphere, which is just $\mathbb{S}^{n-1}$, can be identified with the sphere at infinity of $\mathbb{R}^n$. This sort of compactification underlies many of the model constructions below, it also occurs in scattering theory and other questions which involve 'boundary conditions at infinity.'

## 1.1. Model spaces

Both of these examples lead to manifolds with boundary. As we shall see below the product of two or more manifolds with boundary is a manifold with corners. Indeed the definition of a manifold with corners below is based on the model spaces

$$\mathbb{R}^n_k = [0, \infty)^k \times \mathbb{R}^{n-k} = \{x \in \mathbb{R}^n;\ x_i \geq 0,\ 1 \leq i \leq k\} \tag{1.1.1}$$

which are products of half-lines and lines. The topology on $\mathbb{R}^n_k$ is inherited from $\mathbb{R}^n$. In particular a subset $\Omega \subset \mathbb{R}^n_k$ is (relatively) open if there exists $\Omega' \subset \mathbb{R}^n$, open, such that

$$\Omega = \Omega' \cap \mathbb{R}^n_k. \tag{1.1.2}$$

In place of the spaces $\mathbb{R}^n_k$ in (1.1.1) one could use compact models such as $[0,1]^k \times \mathbb{T}^{n-k}$ where $\mathbb{T}^{n-k}$ is the product of $n-k$ circles. Being compact these models are more typical of the spaces with which we are primarily concerned. However it is computationally more convenient to work with the non-compact models.

Stability under products is of primary importance, enough alone to justify the detailed study of manifolds with corners, because of the Schwartz kernel theorem (see Chapter 3). This result is of great (if largely philosophical) significance in that it shows that linear operators, say from smooth functions to distributions, on a manifold with boundary can be identified with distributions, namely their Schwartz kernels, on the product of the manifold with itself.

Before giving the definition of a manifold with corners in detail we shall discuss the spaces of smooth functions on $\mathbb{R}^n_k$.

## 1.2. Smooth functions

If $\Omega \subset \mathbb{R}^n_k$ is an open subset we define

$$\begin{aligned}\mathcal{C}^\infty(\Omega) = \{u\colon \Omega^\circ \longrightarrow \mathbb{C};\ u \text{ is } \mathcal{C}^\infty \text{ in } \Omega^\circ \text{ with all derivatives}\\ \text{bounded on } K \cap \Omega^\circ \text{ for all } K \Subset \Omega\}.\end{aligned} \tag{1.2.1}$$

Here $\Omega^\circ = \Omega \cap (\mathbb{R}^n_k)^\circ$,

$$(\mathbb{R}^n_k)^\circ = (0,\infty)^k \times \mathbb{R}^{n-k} = \{x; x_i > 0,\ 1 \le i \le k\}. \tag{1.2.2}$$

Even though an element $u \in \mathcal{C}^\infty(\Omega)$ is, according to (1.2.1), a function on $\Omega^\circ$ it follows that each derivative $D^\alpha u$ extends to be continuous on $\Omega$. Here we use the standard multi-index notation for derivatives

$$D^\alpha u = (-i)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x),\ \ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Indeed by definition $u \in \mathcal{C}^\infty(\Omega)$ implies $D^\alpha u \in \mathcal{C}^\infty(\Omega)$, so it is enough to check that $u$ is continuous on $\Omega$. Since all derivatives are bounded near any point $\bar{x} \in \partial\Omega$, that is where $\bar{x}_i = 0$ for some $i \le k$, the continuity follows by writing

$$u(x) = \int_{\tilde{x}_1}^{x_1} \cdots \int_{\tilde{x}_n}^{x_n} u(s) ds + u(\tilde{x})$$

where $\tilde{x} \in \Omega^\circ$ is near $\bar{x}$.

The boundedness in (1.2.1) can then be written

$$\|u\|_{p,K} = \sup_{\substack{|\alpha| \le p\\ x \in K}} |D^\alpha u(x)| < \infty\ \ \forall\ p,\ K \Subset \Omega. \tag{1.2.3}$$

Since there is a sequence of compact subsets of $\Omega$, $\{K_j\}$, exhausting $\Omega$ (i.e. $K \Subset \Omega \Longrightarrow K \subset K_j$ for some $j$) these estimates can be expressed as the finiteness of countably many of the semi-norms in (1.2.3). These semi-norms give $\mathcal{C}^\infty(\Omega)$ a complete locally convex topology, such matters are discussed further in Chapter 3.

## 1.3. Partitions of unity

If $\Omega' \subset \mathbb{R}^n$ is open we denote by $\mathcal{C}^\infty(\Omega')$ the usual space of $\mathcal{C}^\infty$ functions. Thus if $\Omega \subset \mathbb{R}^n_k$ is (relatively) open then

$$\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^\infty(\Omega^\circ) \tag{1.3.1}$$

in general without equality. Equality in fact occurs only when $\Omega = \Omega^\circ$, so the notation is consistent.

If $\Omega_a$, for $a \in A$, is an open covering of $\Omega \subset \mathbb{R}^n_k$, itself an open set, then there is a partition of unity, $\{\phi_j\}$, subordinate to the covering. The functions $\{\phi_j\}$, $\phi_j \in \mathcal{C}^\infty(\Omega)$ form a partition of unity if the following three conditions hold.

(i) Each $\phi_j \in \mathcal{C}^\infty(\Omega)$ has compact support, so if $O \subset \Omega$ is the largest open set on which $\phi_j$ vanishes then $\operatorname{supp}(\phi_j) = \Omega \setminus O$ is compact.

(ii) These supports are locally finite, so only a finite number of them meet any given compact set in $\Omega$.

(ii)

$$\sum_j \phi_j(x) = 1 \quad \forall\ x \in \Omega. \tag{1.3.2}$$

That the partition of unity be subordinate to the given covering of $\Omega$ is just the extra condition that each of the supports $\operatorname{supp}(\phi_j)$ should be contained in (at least) one of the $\Omega_a$.

The existence of a partition of unity subordinate to a given covering of $\Omega \subset \mathbb{R}^n_k$ follows from the standard case of open sets in $\mathbb{R}^n$ as follows. Open sets $\Omega'_a \subset \mathbb{R}^n$ can be chosen with $\Omega_a = \Omega'_a \cap \mathbb{R}^n_k$. If $\Omega'$ denotes the union of these sets it is an open extension of $\Omega$ into $\mathbb{R}^n$. Let $\phi'_j$ be a partition of unity in $\Omega'$ subordinate to the $\Omega'_a$, then the restrictions $\phi_j = \phi_j \restriction \Omega$ give a partition in $\Omega$ subordinate to the covering $\Omega_a$.

It is sometimes convenient to have a partition of unity satisfy additional conditions. For example it is straightforward in the standard case to arrange that each element is the square of a $\mathcal{C}^\infty$ function. Clearly this construction allows such properties to be arranged for relatively open subsets of $\mathbb{R}^n_k$ as well.

## 1.4. Seeley extension

If $\Omega \subset \mathbb{R}^n_k$ is relatively open and $\Omega' \subset \mathbb{R}^n$ is open and such that $\Omega = \Omega' \cap \mathbb{R}^n_k$ then the restriction of any element of $\mathcal{C}^\infty(\Omega')$ to $\Omega$ is an element of $\mathcal{C}^\infty(\Omega)$. The converse is also true.

Theorem 1.4.1. *[Seeley extension] If $\Omega' \subset \mathbb{R}^n$ is open and $\Omega = \Omega' \cap \mathbb{R}^n_k$ then there is a continuous linear extension map*

$$\begin{cases} E\colon \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega') \\ E(f)|_\Omega = f \quad \forall\ f \in \mathcal{C}^\infty(\Omega). \end{cases} \tag{1.4.1}$$

This is a strengthening, in the special case of $\mathbb{R}^n_k$ as a closed subset of $\mathbb{R}^n$, of a general (non-linear) extension result of Whitney. The linearity, and continuity, of $E$ will be quite useful later. The proof will use induction over $k$ and reduction to the case $\Omega = \mathbb{R}^n_k$. To begin consider the simplest case.

Lemma 1.4.1. *There is a continuous linear extension map*

$$E\colon \mathcal{C}^\infty([0,\infty)) \longrightarrow \mathcal{C}^\infty(\mathbb{R}). \tag{1.4.2}$$

PROOF. We start by looking at a map

$$\tilde{E}(f)(x) = \sum_{l=0}^{\infty} C_l \cdot (\phi f)(\frac{x}{a_l}). \tag{1.4.3}$$

Here $\{C_l\}$ is a sequence of real numbers and $\{a_l\}$ is a positive sequence with $a_l \longrightarrow 0$ as $l \longrightarrow \infty$. The function $\phi$ is inserted to reduce the support, we shall take it to satisfy

$$\phi(x) = \begin{cases} 0 \text{ in } |x| > 2 \\ 1 \text{ in } |x| < 1 \end{cases} \qquad \phi \in \mathcal{C}^\infty(\mathbb{R}). \tag{1.4.4}$$

Then, for each $x > 0$ only finitely many of the terms on the right are non-zero, since $x/a_l \longrightarrow \infty$. Thus, without further assumptions

$$\tilde{E} \colon \mathcal{C}^\infty([0,\infty)) \longrightarrow \mathcal{C}^\infty((0,\infty)). \tag{1.4.5}$$

We shall choose the sequences to satisfy

$$\sum_{l=0}^{\infty} |C_l| a_l^{-p} < \infty \quad \forall\ p \in \mathbb{N} \tag{1.4.6}$$

and

$$\sum_{l=0}^{\infty} C_l a_l^{-p} = (-1)^p \quad \forall\ p \in \mathbb{N}. \tag{1.4.7}$$

Assume for the moment that (1.4.6) and (1.4.7) hold. By differentiation of (1.4.3)

$$(\frac{d}{dx})^p \tilde{E}(f)(x) = \sum_{l=0}^{\infty} C_l a_l^{-p} \frac{d^p(\phi f)}{dx^p}(\frac{x}{a_l}). \tag{1.4.8}$$

Since $\phi f$ is a $\mathcal{C}^\infty$ function of compact support, all its derivatives are bounded. Thus

$$\sup_{x \in [0,\infty)} \left| \frac{d^p}{dx^p} \tilde{E}(f)(x) \right| \le C_p \sum_{0 \le r \le p} \sup_{0 \le x < 2} \left| \frac{d^r f}{dx^r}(x) \right|. \tag{1.4.9}$$

In particular this boundedness, which uses (1.4.6), shows that

$$\tilde{E} \colon \mathcal{C}^\infty([0,\infty)) \longrightarrow \mathcal{C}^\infty([0,\infty)) \text{ is continuous.} \tag{1.4.10}$$

Now consider the limit

$$\lim_{x \downarrow 0} (\frac{d}{dx})^p \tilde{E}(f)(x) = (\sum_{l=0}^{\infty} C_l a_l^{-p})(\frac{d^p}{dx^p} f)(0)$$

where we use the fact that $d^r\phi/dx^r(0) = 0$ if $r \ge 1$, in view of (1.4.4). Thus (1.4.7) implies

$$[(\frac{d}{dx})^p \tilde{E}(f)](0) = (-1)^p \frac{d^p f}{dx^p}(0). \tag{1.4.11}$$

If we set

$$E(f)(x) = \begin{cases} f(x) & x > 0 \\ \tilde{E}f(-x) & x \le 0 \end{cases} \tag{1.4.12}$$

then $E(f)(x)$ and all its derivatives are continuous. The continuity of $E$ follows from (1.4.9).

The lemma is thereby reduced to showing the existence of sequences satisfying (1.4.6) and (1.4.7). To do so consider the entire function

$$h(z) = \cos(\pi(\frac{3^z - 1}{2})). \tag{1.4.13}$$

Since, for $p \in \mathbb{N}$, $(3^p - 1)/2$ is odd or even with $p$

$$h(p) = (-1)^p \quad p \in \mathbb{N}. \tag{1.4.14}$$

Now as $\cos w$ is entire the Taylor series of $h$ as a function of $3^z$ converges (absolutely) everywhere, so constants $C_l$ can be defined by

$$h(z) = \sum_{l=0}^{\infty} C_l (3^l)^z. \tag{1.4.15}$$

Thus the choice $a_l = 3^{-l}$ gives (1.4.7). The estimates (1.4.6) follow from the absolute convergence. □

Proof of Theorem 1.4.1. First we consider the case $\Omega = \mathbb{R}^n_k$. Clearly it suffices to give a continuous linear extension map

$$E_k : \mathcal{C}^\infty(\mathbb{R}^n_k) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n_{k-1}) \quad \text{for each } 1 \le k \le n. \tag{1.4.16}$$

An element of $g \in \mathcal{C}^\infty(\mathbb{R}^n_k)$ can be identified with the smooth map

$$\mathbb{R}^{n-1}_{k-1} \ni (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \longmapsto g(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, x_n) \subset \mathcal{C}^\infty([0, \infty)) \tag{1.4.17}$$

and conversely. The map $E_k$ can then be defined by using the one-dimensional extension map of Lemma 1.4.1 in the variable $x_k$. That the extension is an element of $\mathcal{C}^\infty(\mathbb{R}^n_k)$ follows from the fact that linear maps are $\mathcal{C}^\infty$ and that composition of $\mathcal{C}^\infty$ maps gives a $\mathcal{C}^\infty$ map. Continuity also follows directly.

The general case of $\Omega \subset \mathbb{R}^n_k$, an open subset, can be handled by the use of a partition of unity. Thus, following the remarks in §1.3, we can choose $\phi_i \in \mathcal{C}^\infty_c(\mathbb{R}^n_k)$ having locally finite supports contained in $\Omega$ and such that

$$\sum_i \phi_i^2(x) = 1 \quad \forall\ x \in \Omega. \tag{1.4.18}$$

Then put

$$E(f) = \sum_{i=1}^{\infty} \phi_i E_k(\phi_i f).$$

It is straightforward to check that this has all the desired properties. □

## 1.5. Diffeomorphisms

If $\Omega_i \subset \mathbb{R}^n_{k_i}$ $i = 1, 2$ are (relatively) open then a map $f : \Omega_1 \longrightarrow \Omega_2$ is a *diffeomorphism* if it is a homeomorphism with inverse $g : \Omega_2 \longrightarrow \Omega_1$ and the components of $f$ and $g$ are in $\mathcal{C}^\infty(\Omega_1)$ and $\mathcal{C}^\infty(\Omega_2)$ respectively.

Lemma 1.5.1. *If $f : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism from an open subset of $\mathbb{R}^n_{k_1}$ to an open subset of $\mathbb{R}^n_{k_2}$ then there are open subsets $\Omega'_i \subset \mathbb{R}^n$ with $\Omega_i = \Omega'_i \cap \mathbb{R}^n_{k_i}$, $i = 1, 2$, and a diffeomorphism $F : \Omega'_1 \longrightarrow \Omega'_2$ such that $f = F|_{\Omega_1}$.*

PROOF. Choose open sets $\Omega_i'' \subset \mathbb{R}^n$ such that $\Omega_i = \Omega_i'' \cap \mathbb{R}^n_{k_i}$. The extension theorem, Theorem 1.4.1, allows the components of $f$ and $g$ to be extended from $\Omega_i$ to $\Omega_i''$. Thus there are $\mathcal{C}^\infty$ maps $f' : \Omega_1 \longrightarrow \mathbb{R}^n$, $g' : \Omega_2 \longrightarrow \mathbb{R}^n$ such that

$$g' \circ f' = \mathrm{Id} \text{ on } \Omega_1,\ f' \circ g' = \mathrm{Id} \text{ on } \Omega_2. \tag{1.5.1}$$

From this it follows that the Jacobian matrix of $f'$ is invertible at each point of $\Omega_1$. The inverse function theorem shows $f'$ to be invertible in some $\Omega_1' \supset \Omega_1$, open in $\mathbb{R}^n$. This proves the lemma. □

For any $l$, $0 \le l \le k$, set

$$\partial_l \mathbb{R}^n_k = \{x \in \mathbb{R}^n_k;\ x_i = 0 \text{ for exactly } l \text{ of the first } k \text{ indices}\}. \tag{1.5.2}$$

For any open subset $\Omega \subset \mathbb{R}^n_k$ set

$$\partial_l \Omega = \partial_l \mathbb{R}^n_k \cap \Omega \quad 0 \le l \le k. \tag{1.5.3}$$

COROLLARY 1.5.1. *If $f : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism of an open subset of $\mathbb{R}^n_{k_1}$ onto an open subset of $\mathbb{R}^n_{k_2}$ then*

$$f : \partial_l \Omega_1 \longleftrightarrow \partial_l \Omega_2 \quad \forall\ 0 \le l \le k. \tag{1.5.4}$$

PROOF. If $F : \Omega_1' \longrightarrow \Omega_2'$ is an extension of $f$ to a diffeomorphism of open subsets of $\mathbb{R}^n$, with $\Omega_i = \Omega_i' \cap \mathbb{R}^n_k$ for $i = 1, 2$ then $\Omega_i \subset \Omega_i'$ is closed. Thus $F$ maps the boundary of $\Omega_1$ onto the boundary of $\Omega_2$. Since diffeomorphisms preserve the smoothness of hypersurfaces it follows that $F$, and hence $f$, must map each $\Omega_1 \cap \{x_r = 0\}$ into some $\Omega_2 \cap \{x_{\sigma(r)} = 0\}$ where $\sigma$ is a permutation of $\{1, \dots, k\}$. As the $\partial_l \Omega_i$ are just the unions of the $l$-fold intersections of these hypersurfaces they must be mapped into each other by $f$. □

As already noted for $\mathbb{R}^n$, the support of a continuous function on a topological space is the complement of the largest open set on which the function vanishes identically. It is denoted $\operatorname{supp}(f)$. If $U \subset X$ is an open set in a topological space $X$ and $u : U \longrightarrow \mathbb{C}$ is continuous on $U$ with compact support $\operatorname{supp}(u) \Subset U$ then there is a unique continuous function $\tilde{u} : X \longrightarrow \mathbb{C}$ with $\tilde{u} = u$ on $U$ and $\operatorname{supp}(\tilde{u}) \subset U$. This 'extension as zero' is so natural that we generally identify $u$ and $\tilde{u}$ without comment. We denote by $\mathcal{C}^\infty_c(U)$, for $U \subset \mathbb{R}^n_k$, the subspace of $\mathcal{C}^\infty(U)$ consisting of the elements of compact support. For any $K \Subset U$ we can always find $u \in \mathcal{C}^\infty_c(U)$ which is equal to 1 on $K$; in particular $\mathcal{C}^\infty_c(U)$ separates points.

One useful characterization of diffeomorphisms between open subsets of $\mathbb{R}^n_k$ and $\mathbb{R}^n_{k'}$ is in terms of the pull-back map.

LEMMA 1.5.2. *If $U \subset \mathbb{R}^n_k$ and $U' \subset \mathbb{R}^n_{k'}$ are open subsets and $f : U \longrightarrow U'$ is a map then $f$ is a diffeomorphism if and only if the pull-back map $f^*(\phi) = \psi$ defined by $\psi = \phi \cdot f : U \longrightarrow \mathbb{C}$ if $\phi : U' \longrightarrow \mathbb{C}$ gives an isomorphism*

$$f^* : \mathcal{C}^\infty_c(U') \longleftrightarrow \mathcal{C}^\infty_c(U). \tag{1.5.5}$$

PROOF. Suppose that $f^*$ is an isomorphism as in (1.5.5). Since $\mathcal{C}^\infty_c(U)$ separates points it follows that $f$ must be 1-1 and onto. The components $f_j$ of $f$ are the pull-backs of the coordinate functions $f_j = f^* x_j$. For any point $p \in U$ one can choose $\phi \in \mathcal{C}^\infty(U')$ with compact support and $\phi = 1$ near $f(p)$. Then $f^*(x_j \phi)$ is $\mathcal{C}^\infty$ and equal to $f_j$ near $p$. Thus all the components of $f$ are $\mathcal{C}^\infty$. The same argument applies to the inverse so $f$ is a diffeomorphism. The converse is obvious, proving the lemma. □

## 1.6. $\mathcal{C}^\infty$ structures

Let $X$ be a Hausdorff topological space. A chart (with corners) on $X$ is a map

$$\phi : \Omega \longrightarrow \mathbb{R}^n_k \tag{1.6.1}$$

which is a homeomorphism from an open set $\Omega \subset X$ onto an open subset of $\mathbb{R}^n_k$, for some $k$. Two charts $(\phi_i, \Omega_i)$ are said to be *compatible* if either $\Omega_1 \cap \Omega_2 = \emptyset$ or

$$\phi_2 \circ \phi_1^{-1} : \phi_1(\Omega_1 \cap \Omega_2) \longrightarrow \phi_2(\Omega_1 \cap \Omega_2)$$

is a diffeomorphism of open subsets of $\mathbb{R}^n_{k_1}$ and $\mathbb{R}^n_{k_2}$. An *atlas* on $X$ is a system of charts $(\phi_a, \Omega_a)$, $a \in A$, which are compatible in pairs and which cover $X$ :

$$X = \bigcup_{a \in A} \Omega_a. \tag{1.6.2}$$

A $\mathcal{C}^\infty$ *structure (with corners)* on $X$ is a maximal atlas, i.e. an atlas which contains any chart compatible with each element of the atlas.

By a *coordinate system* in a space $X$, with $\mathcal{C}^\infty$ structure with corners, we shall mean a *pointed chart* i.e. a chart $(\phi, \Omega)$ and a point $p \in X$ such that

$$\phi(p) = 0. \tag{1.6.3}$$

This normalization is made so that the corresponding $k$ in (1.6.1) is minimal near $p$. The *dimension* of the $\mathcal{C}^\infty$ structure (near $p$) is the integer $n$.

If $X$ has a $\mathcal{C}^\infty$ structure with corners we denote by $\mathcal{C}^\infty(X)$ the space of all functions

$$u : X \longrightarrow \mathbb{C}$$

which are such that $u \circ \phi^{-1}$ is $\mathcal{C}^\infty$ on $\phi(\Omega) \subset \mathbb{R}^n_k$ for each chart $(\phi, U)$.

Definition 1.6.1. A t-manifold is a pair $(X, \mathcal{F})$ where $X$ is a paracompact Hausdorff space and $\mathcal{F} = \mathcal{C}^\infty(X)$ for some $\mathcal{C}^\infty$ structure with corners on $X$.

Usually $X$ will be compact. In §1.8 below we introduce the more refined notion of a manifold with corners.

Lemma 1.6.1. *Any t-manifold admits partitions of unity, i.e. given any open cover $\Omega_a$, $a \in A$, of the t-manifold, $X$, there are countably many elements $\phi_j \in \mathcal{C}^\infty_c(X)$ such that the support of each of the $\phi_j$ is a compact subset of one of the $\Omega_a$, only finitely many of these supports have non-empty intersection and*

$$\sum_j \phi_j = 1 \text{ on } X. \tag{1.6.4}$$

Proof. Since $X$ is paracompact we can find a countable and locally finite cover by coordinate neighbourhoods which is subordinate to the given cover. It suffices therefore to assume that the $\Omega_a$ are coordinate neighbourhoods, that $A$ is countable and that only finitely many of the $\Omega_a$ have non-empty intersection. Proceeding by induction it can be seen that there are compact subsets $K_a \Subset \Omega_a$ with the property that the interiors of the $K_a$ still cover $X$. Since $\Omega_a$ is a coordinate neighbourhood there exists $\phi'_a \in \mathcal{C}^\infty_c(\Omega_a)$ which is equal to 1 on $K_a$. The sum in

$$\phi = \sum_{a \in A} \phi'_a$$

is locally finite, so $\phi \in \mathcal{C}^\infty(X)$ is everywhere positive. Then $\phi_a = \phi'_a/\phi$ is a partition of unity subordinate to the given cover. $\square$

As on $\mathbb{R}^n_k$ we can arrange that $\phi_j = \psi_j^2$ with the $\psi_j \in \mathcal{C}^\infty(X)$.

Using a partition of unity we can construct a *total boundary defining function* on any t-manifold.

LEMMA 1.6.2. *On any t-manifold there exist $\rho \in \mathcal{C}^\infty(X)$ such that*

$$\begin{gathered}\rho > 0 \text{ on } X^\circ, \rho = 0 \text{ on } \partial X \text{ and} \\ \text{in local coordinates at } p \in \partial X,\ \rho = a(x)x_1 \dots x_k,\ a(p) > 0\end{gathered} \tag{1.6.5}$$

*where a is smooth.*

PROOF. Take any covering of $X$ by coordinate systems and a partition of unity $\phi_i^2$ subordinate to this cover. Let $\rho_a$ be the $\mathcal{C}^\infty$ function in each coordinate system given as $x_1 \dots x_k$, just the product of the first $k$ coordinates. Then

$$\rho = \sum_j \phi_j \rho_{a(j)} \tag{1.6.6}$$

satisfies all the requirements if $j \longmapsto a(j)$ associates to $j$ a coordinate patch in which $\phi_j$ has support. □

EXERCISE 1.6.1. Let $X$ be a t-manifold. Suppose that $H_1, \dots, H_p$ are any number of boundary hypersurfaces (see the discussion following (1.8.4).) Show that there is a joint defining function $\rho_H$ satisfying, in place of (1.6.5),

$$\begin{gathered}\rho > 0 \text{ on } X \setminus \bigcup_{j=1}^p H_p, \rho = 0 \text{ on each } H_j \text{ and} \\ \text{in local coordinates at } p \in \partial X,\ \rho = a(x) \prod_{j \in J(p)} x_j,\ a(p) > 0,\ a\ \mathcal{C}^\infty,\end{gathered} \tag{1.6.7}$$

where $J(p)$ is the subset of $\{1, \dots, k\}$ such that $x_j = 0$ locally near $p$ on some $H_i$ if and only if $j \in J(p)$. Note that the $H_i$ need not be embedded.

It is easy to see that the $\mathcal{C}^\infty$ structure can be recovered from $\mathcal{F} = \mathcal{C}^\infty(X)$. In fact the following three properties are enough to characterize a linear space, $\mathcal{F}$, of continuous functions on a topological manifold as $\mathcal{C}^\infty(X)$ for some $\mathcal{C}^\infty$ structure. First $\mathcal{F}$ must separate points:

$$\forall\ p, p' \in X\ \exists\ f \in \mathcal{F} \text{ s.t. } f(p) \neq f(p') \tag{1.6.8}$$

so $X$ is Hausdorff. Secondly $\mathcal{F}$ should be local in the sense that

$$\begin{gathered}\text{if } X = \bigcup_{a \in A} U_a, \text{ with each } U_a \text{ open and} \\ f \colon X \longrightarrow \mathbb{C} \text{ is such that } \exists\ g_a \in \mathcal{F} \text{ with } f = g_a \text{ on } U_a \text{ then } f \in \mathcal{F}.\end{gathered} \tag{1.6.9}$$

Thirdly $\mathcal{F}$ must admit coordinate systems in the intrinsic sense that

$$\begin{gathered}\text{for each } p \in X \text{ there exist functions } f_1, \dots, f_n \in \mathcal{F} \text{ and } k \in \mathbb{N} \\ \text{such that } f = (f_1, \dots, f_n) \colon X \longrightarrow \mathbb{R}^n_k \text{ restricts to a homeomorphism of} \\ \text{an open neighbourhood } U \text{ of } p \text{ onto an open set } f(U) \subset \mathbb{R}^n_k \text{ with} \\ \{\psi \circ f; \psi \in \mathcal{C}^\infty(\mathbb{R}^n_k),\ \operatorname{supp}(\psi) \Subset f(U)\} = \{\phi \in \mathcal{F}; \operatorname{supp}(\phi) \Subset U\}.\end{gathered} \tag{1.6.10}$$

Of course (1.6.10) is the essential condition. The only advantage of this type of characterization of $\mathcal{C}^\infty$ structures, as opposed to the usual covering definition given above, is that one does not need to directly discuss the transition between coordinate systems.

PROPOSITION 1.6.1. *Suppose* $(X, \mathcal{F})$ *is a pair consisting of a paracompact topological space and a space of continuous functions,* $\mathcal{F}$*, on* $X$ *and that* $\mathcal{F}$ *has the three properties* (1.6.8), (1.6.9) *and* (1.6.10) *then there is a unique* $\mathcal{C}^\infty$ *structure with corners on* $X$ *for which* $\mathcal{F} = \mathcal{C}^\infty(X)$.

PROOF. By assumption the neighbourhoods in (1.6.10) cover $X$. We shall show that these open sets, and the restrictions to them of the maps:

$$\phi = f \restriction_U : U \longrightarrow \mathbb{R}^n_k,$$

give an atlas for $X$. If $(\phi', U')$ is another such pair, with $\tilde{U} = U \cap U' \neq \emptyset$, then applying (1.6.10) twice it follows that

$$\psi = \phi' \cdot \phi^{-1} \colon V = f(\tilde{U}) \longrightarrow V' = f'(\tilde{U})$$

is such that $\psi^*(\mathcal{C}^\infty_c(V')) = \mathcal{C}^\infty_c(V)$. By Lemma 1.5.2 this implies that $\psi \colon V \longrightarrow V'$ is a diffeomorphism of open sets of $\mathbb{R}^n_k$. Thus the covering gives an atlas.

It then follows from (1.6.8) that $X$ is Hausdorff. The $\mathcal{C}^\infty$ structure with corners defined by this atlas has $\mathcal{F} = \mathcal{C}^\infty(X)$ because of (1.6.9) and the existence of partitions of unity subordinate to the covering by the open sets of the atlas. □

DEFINITION 1.6.2. If $X$ is a t-manifold and $\mathcal{F} \subset \mathcal{C}^\infty(X)$ has the separation property (1.6.8) and satisfies the following variant of (1.6.10):

$$\begin{gathered}\text{For each } p \in X \text{ there exist functions } f_1, \dots, f_n \in \mathcal{F} \text{ and } k \in \mathbb{N} \\ \text{such that } f = (f_1, \dots, f_n) \colon X \longrightarrow \mathbb{R}^n_k \text{ restricts to a homeomorphism of} \\ \text{some neighbourhood } U \text{ of } p \text{ onto an open set } f(U) \subset \mathbb{R}^n_k \text{ with} \\ \{\psi \circ f; \psi \in \mathcal{C}^\infty(\mathbb{R}^n_k),\ \operatorname{supp}(\psi) \Subset f(U)\} = \{\phi \in \mathcal{C}^\infty(X); \operatorname{supp}(\phi) \Subset U\}.\end{gathered} \tag{1.6.11}$$

then we shall say that $\mathcal{F}$ *generates* $\mathcal{C}^\infty(X)$.

Notice that the $\mathcal{C}^\infty$ structure is indeed uniquely fixed by $\mathcal{F}$ in this case.

EXERCISE 1.6.2. Consider the reflections

$$\begin{gathered} R_i \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\ R_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n). \end{gathered} \tag{1.6.12}$$

Let $G_k = \{\mathrm{Id}, R_1, \dots, R_k\}$ be the group generated by the first $k$ reflections (which commute). Set $X = \mathbb{R}^n / G_k$. The space

$$\mathcal{F} = \{u \colon \mathbb{R}^n \longrightarrow \mathbb{C}; u \circ R_i = u,\ i = 1, \dots, k\} \tag{1.6.13}$$

can be identified as a space of functions on $X$. Show that $(X, \mathcal{F})$ is a $\mathcal{C}^\infty$ t-manifold.

EXERCISE 1.6.3. Consider the region

$$B = \{(x, y) \in \mathbb{R}^2;\ 0 \le y \le x^2,\ x \ge 0\}.$$

Set $\mathcal{B} = \mathcal{C}^\infty(\mathbb{R}^2)|_B$. Show that $(B, \mathcal{B})$ is *not* a $\mathcal{C}^\infty$ t-manifold.

EXERCISE 1.6.4. Take $X$ to be the set

$$X = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1,\ y, z \ge 0\} \tag{1.6.14}$$

and let

$$\mathcal{F} = \mathcal{C}^\infty(\mathbb{R}^3) \restriction X. \tag{1.6.15}$$

Show that $(X, \mathcal{F})$ is a t-manifold.

## 1.7. Submanifolds

One of the standard definitions of a submanifold (meaning an embedded submanifold) of a manifold without boundary is a subset which can, near each of its points, be reduced to a linear subspace in some local coordinates. We shall define a submanifold of a t-manifold similarly by reduction to the linear case. A linear submanifold with corners of $\mathbb{R}^n_k$ is a subset of the form

$$G \cdot \left(\mathbb{R}^{n'}_{k'} \times \{0\}\right) \subset \mathbb{R}^n_k \tag{1.7.1}$$

for some $G \in GL(n, \mathbb{R})$. Note that we are requiring that the left side of (1.7.1) be contained in the right side, not taking the intersection. Here $GL(n, \mathbb{R})$ is the group of all invertible real $n \times n$ matrices. In the case of a manifold without boundary, $k = k' = 0$ and the linear transformation $G$ can always be taken as the identity, by appropriate choice of coordinates. However this is not the case in general since the space $\mathbb{R}^n_k$ is not invariant under all linear transformations. This leads to a variety of notions of a submanifold (always embedded) depending on the intersection properties with the boundary. We start with the weakest condition:

Definition 1.7.3. If $X$ is a t-manifold then a *submanifold* $S \subset X$ is a (connected) subset with the property that for each $\bar{s} \in S$ there is a coordinate system $(\phi, U)$ based at $\bar{s}$, a linear transformation $G \in GL(n, \mathbb{R})$ and an open neighbourhood $\Omega' \subset \mathbb{R}^n$ of 0 in terms of which

$$\phi \restriction U : S \cap U \longleftrightarrow G \cdot \left(\mathbb{R}^{n'}_{k'} \times \{0\}\right) \cap \Omega'. \tag{1.7.2}$$

for some integers $n'$, $k' = k'(\bar{s})$.

Notice that the assumption that $(\phi, U)$ is a coordinate system forces the right side of (1.7.2) to be a subset of $\mathbb{R}^n_k$ for the appropriate local boudary codimension $k$.

If $S$ is not (or is not known to be) connected then this will be explicitly noted. The minimum value of the integer $n - n'$ is the *codimension* of $S$, $\operatorname{codim}_X S$. In fact it follows directly that $n'$ can be taken independent of $\bar{s}$.

Examples include the interior of any t-manifold and the subset

$$S = \{x \in \mathbb{R}^n_k; x_{k+1} = 0, x_{k+2} \geq 0\} \subset \mathbb{R}^n_k \quad (n \geq k + 2). \tag{1.7.3}$$

We show below that any t-manifold can be embedded as a submanifold, in this sense, of a manifold without boundary of the same dimension.

Another example, and one which is fundamental in the theory of pseudodifferential operators developed below, is the diagonal in the product of a manifold with boundary with itself. For the case of $[0, 1)$

$$\begin{gathered} \Delta = \{(x, x); x \in [0, 1)\} = A \cdot \{(x, 0); x \in [0, 1)\} \\ A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \end{gathered} \tag{1.7.4}$$

Thus $\Delta$ is a submanifold in this sense. The general case is discussed below.

Lemma 1.7.1. *If $S \subset X$ is a submanifold of a t-manifold then the covering of $S$ by the coordinate systems $G \cdot \phi$ in* (1.7.2) *is an atlas on $S$, giving it the structure of a t-manifold of dimension* $\dim X - \operatorname{codim}_X S$.

Proof. As a sub-basis of open sets on $S$ we take the sets $U \cap S$ for which (1.7.2) holds. Observe that this topology must be Hausdorff, since the restriction to $S$ of $\mathcal{C}^\infty(X)$ is a set of continuous functions separating points. The inclusion $S \hookrightarrow X$ is continuous when $S$ is given this topology (which is the weakest topology with this property.) Hence the paracompactness of $S$ follows from that of $X$. Thus it is only necessary to check that these coordinate systems form an atlas.

Suppose $(\phi, U, G, \Omega)$ and $(\phi', U', G', \Omega')$ are two systems as in the definition. By the assumption that these arise from coordinate systems on $X$ the composite map $\phi' \cdot \phi^{-1}$ is a diffeomorphism between open sets in $\mathbb{R}^n_k$ and $\mathbb{R}^n_{k'}$. Hence $G' \cdot \phi' \cdot \phi^{-1} \cdot G^{-1}$ must be $\mathcal{C}^\infty$ on its domain $G \cdot \phi(U)$. Since $\Omega$ is necessarily the intersection of this set with $\{x_1, \dots, x_q \geq 0,\ x_{n-q+1} = \dots x_n = 0\}$ the restriction to $\Omega'$ must be $\mathcal{C}^\infty$. Since the same argument applies to the inverse the coordinate systems are indeed compatible. □

Exercise 1.7.5. Show that $S$ in (1.7.3) is diffeomorphic to $\mathbb{R}^{n-1}_{k+1}$.

The rôle of the linear transformation $G$ in this definition is simply to allow the boundary points of $S$ to be rather unrelated to the boundary points of $X$. For example it is perfectly possible for the boundary of $S$ to lie in the interior of $X$. On the other hand it is clearly the case that

$$\text{either } S \cap \partial X \subset \partial S \text{ or } S \subset \partial X. \tag{1.7.5}$$

The subset $S$ need not be closed as a subset of $X$. This is reflected in the fact that in general the restriction to $S$ of the smooth functions on $X$ does not give all the $\mathcal{C}^\infty$ functions on $S$.

Lemma 1.7.2. *For any submanifold of $X$, a t-manifold, restriction gives a linear map*

$$\restriction_S : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(S); \tag{1.7.6}$$

*this is surjective if and only if $S \subset X$ is closed, but in any case the range of* (1.7.6) *always generates the $\mathcal{C}^\infty$ structure on $S$ in the sense of Definition 1.6.2.*

Proof. If $S$ is closed then the topology given by the sets in (1.7.2) as a subbasis is the topology induced from $X$. The existence of an extension to $\mathcal{C}^\infty(X)$ of each element of $\mathcal{C}^\infty(S)$ then follows from the extendibility of functions with compact support in a coordinate patch as in (1.7.2). Using the map $\phi$ to pull back the extension this is equivalent to the statement that if $G \colon \mathbb{R}^n \longleftrightarrow \mathbb{R}^n$ is an invertible linear map such that

$$G^{-1}\{x_l \geq 0,\ l = 1, \dots, p,\ x_i = 0,\ i = n-q+1, \dots, n\} \subset \mathbb{R}^n_k \tag{1.7.7}$$

then for any $f \in \mathcal{C}^\infty_c(\mathbb{R}^{n-q}_p)$ there exists $g \in \mathcal{C}^\infty_c(\mathbb{R}^n_k)$ such that

$$f = g \circ G^{-1} \quad \text{on } x_{n-q+1} = \dots = x_n = 0. \tag{1.7.8}$$

Using Seeley's extension operator $g$ can be constructed by first extending $f$ to $\mathbb{R}^{n-q}$ then to $\tilde{f}$ on $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$ and then setting $g = \tilde{f} \circ G$.

Conversely if $S$ is not closed then there is a sequence in $S$ with no convergent subsequence (in $S$) yet which converges in $X$. It is easy, using a partition of unity, to construct a $\mathcal{C}^\infty$ function on $S$ which is unbounded along this sequence. It cannot then be the restriction to $S$ of a $\mathcal{C}^\infty$ function on $X$. □

We shall also introduce more restrictive conditions on submanifolds. First we consider the notion of a d-submanifold, the '*d*' being for decomposable. The linear d-submanifolds of $\mathbb{R}^n_k$ are those which are everywhere locally the image, under a linear transformation preserving $\mathbb{R}^n_k$, of a subset of the form

$$L = \{x \in \mathbb{R}^n_k; x_{l+1} = \cdots = x_k = 0, x_{k+1} \geq 0, \dots, x_{k+j} \geq 0, \\ x_{k+j+1} = \cdots = x_{k+j+r} = 0\}, \tag{1.7.9}$$

where $l, r, j \geq 0$, $l \leq k$ and $r + j + k \leq n$. We think of these as 'decomposable' submanifolds because $\mathbb{R}^n_k$ can be written as a product of half-lines and lines with $L$ simultaneously a product of points, half-lines and lines. However, notice that the half-line factors of $L$ may be subsets of the linear factors of $\mathbb{R}^n_k$.

Definition 1.7.4. A submanifold $S \subset X$ of a t-manifold is a d-submanifold if for each $p \in S$ there are local coordinates $\phi$ at $p$, with coordinate neighbourhood $\Omega \subset X$, such that

$$\phi(\Omega \cap S) = L \cap \phi(\Omega), \tag{1.7.10}$$

with $L$ of the form (1.7.9). The submanifold $S$ is said to be a p-submanifold if $L$ can always be taken of the form (1.7.9) with $j = 0$.

Notice that for a p-submanifold half-line factors can appear in $S$ only if they appear in $X$, i.e.

$$S \text{ a } p\text{-submanifold} \implies \partial S \subset \partial X. \tag{1.7.11}$$

The '$p$' is for product since a p-submanifold can always, locally, be brought to the normal form

$$\{x_{k-j+1} = \cdots = x_k = 0,\ x_{k+1} = \cdots = x_{k+r} = 0\} \tag{1.7.12}$$

where $j + r$ is the codimension of the submanifold, so $X$ and $S$ have a common local product decomposition. In (1.7.12) either $j$ or $r$ can be zero, if $j = 0$ then the submanifold meets the interior of $X$ and we call it an interior p-submanifold; if $j > 0$ we call it a boundary p-submanifold. If $r = 0$ then the p-submanifold is a boundary face.

## 1.8. Manifolds with corners

Recall that for the model spaces we set

$$\partial_l \mathbb{R}^n_k = \{x \in \mathbb{R}^n_k; x_i = 0 \text{ for exactly } l \text{ of the first } k \text{ indices}\}. \tag{1.8.1}$$

For a general t-manifold set

$$\partial_l X = \{p \in X;\ \text{coordinates at } p \text{ map to } \partial_l \mathbb{R}^n_k\}. \tag{1.8.2}$$

Then $X^\circ = \partial_0 X$. More generally we shall set

$$\partial^l X = \overline{\partial_l X} = \bigcup_{r \geq l} \partial_r X. \tag{1.8.3}$$

Thus $\partial_l X$ consists precisely of the points in the boundary of $X$ at which the boundary has codimension $l$ while $\partial^l X$ consists of the points at which the boundary has codimension at least $l$. We also use the notation

$$\partial X = \partial^1 X \ \text{ so } X^\circ = X \backslash \partial X. \tag{1.8.4}$$

A boundary hypersurface of a t-manifold, $X$, is the closure of a component of $\partial_1 X$; the collection of boundary hypersurfaces will be denoted $M_1(X)$. It is somewhat important to note that even hypersurface boundary faces need not be submanifolds in the sense of Definition 1.7.3. This is precisely the reason we have carried along the otherwise unattractive notation of 't-manifold.' The spaces we shall mainly be considering have embedded boundary faces.

DEFINITION 1.8.5. A manifold with corners is a Hausdorff space with a $\mathcal{C}^\infty$ structure with corners (a t-manifold) such that each boundary hypersurface is a submanifold in the sense of Definition 1.7.3.

If a boundary hypersurface is a submanifold in the weak sense of Definition 1.7.3 it is automatically a p-submanifold.

Obviously any point in a t-manifold has an open neighborhood which is a manifold with corners. The only thing that can stop a t-manifold from being a manifold with corners is that two of the local boundary hypersurfaces near some point are in the closure of the same component of $\partial_1 X$.

LEMMA 1.8.1. *In a manifold with corners each of the boundary hypersurfaces has a global defining function in the sense that there exists $\rho \in \mathcal{C}^\infty(X)$ such that*

$$\rho \geq 0, \;\; H = \{\rho = 0\}$$

*is the boundary hypersurface in question and near each point of $H$ there are local coordinates with $\rho$ as first element.*

PROOF. Let $H$ be a boundary hypersurface. Given that it is the closure of a component of $\partial_1 X$, the assumption that $H$ is a submanifold means that near each of its points there is a local defining function for $H$ and this can be taken to be $x_1$ in some local coordinate system. Take a covering of $H$ by coordinate neighbourhoods of $X$ based at points of $H$, let $M$ be the union of these neighbourhoods. Take a covering of $X \setminus M$ by coordinate neighbourhoods not meeting $H$; together these give a covering of $X$. Let $\phi_i$, $\phi'_j$ be a partition of unity subordinate to this covering with the $\phi_i$ supported in the first set of neighbourhoods and the $\phi'_j$ in the second. Then if $x_1^{(i)}$ is a local defining function for $H$ in the coordinate chart containing the support of $\phi_i$ set

$$\rho = \sum_i x_1^{(i)} \phi_i + \sum_j \phi_j. \tag{1.8.5}$$

Certainly $\rho \in \mathcal{C}^\infty(X)$ is non-negative. Moreover $\rho$ vanishes precisely at $H$. It only remains to show that near each point of $H$ there are local coordinates with $\rho$ as first element. In fact if $x_1$ is a local defining function for $H$ then $\rho = a x_1$ locally with $a > 0$ and smooth. Thus $\rho$ can be used in place of $x_1$ as first coordinate. □

In general $\rho_H$ will denote a defining function for the boundary hypersurface $H \in M_1(X)$. Sometimes the boundary hypersurfaces will be labelled $H_1, \dots, H_N$ and then the corresponding defining functions will be denoted $\rho_i$, $i = 1, \dots N$.

By a *boundary face* of $X$ we mean the closure, in $X$, of a component of $\partial_l X$ for some $l$, which is the codimension. Let $M_l(X)$ be the set of boundary faces of codimension $l$ and set

$$M(X) = \bigsqcup_{l \geq 0} M_l(X). \tag{1.8.6}$$

Thus, for instance, $X \in M(X)$ is the sole 'boundary face' of codimension 0. For a manifold with corners, these are all $\mathcal{C}^\infty$ embedded submanifolds, in fact they are p-submanifolds in the sense of Definition 1.7.4. Each boundary face of codimension $k$ is a component of (precisely) one of the intersections

$$H_I = \bigcap_{i \in I} H_i, \ I \subset M_1(X), \ \#I = k. \tag{1.8.7}$$

For a given $I$, $H_I$ is a union, possibly empty, of boundary faces.

If $S \subset X$ is a p-submanifold of a manifold with corners there is a unique boundary face $B \in M(X)$ such that

$$S \subset B \text{ and } \partial S = \partial B \cap S. \tag{1.8.8}$$

This will be written $B = \mathrm{Fa}(S)$. In fact $\mathrm{Fa}(S)$ is necessarily the smallest boundary face containing $S$. If $S$ is an interior p-submanifold then $\mathrm{Fa}(S) = X$. In this sense (1.7.11) is really the distinguishing condition between d-submanifolds and p-submanifolds. It has the following important consequence:

Lemma 1.8.2. *A p-submanifold of a manifold with corners is a manifold with corners.*

Proof. Consider first the case of an interior p-submanifold, $S \subset X$. Near any boundary point $p \in \partial S \subset \partial X$, the submanifold is locally defined by the vanishing of interior coordiantes. Thus the boundary defining functions $\rho_H$ for the boundary hypersurfaces of $X$ through $p$ must be independent (i.e. can be taken as part of a coordinate system) on $S$. It follows that the boundary hypersurfaces of $S$, which are just the components of the intersections $S \cap H$ for $H \in M_1(X)$ all have defining functions and are therefore embedded. Thus $S$ is a manifold with corners.

For a boundary p-submanifold simply observe that $S \subset \mathrm{Fa}(S)$ is an interior p-submanifold and $\mathrm{Fa}(S)$, being a boundary face of a manifold with corners, is a manifold with corners. □

Together with this result for p-submanifolds one of the most important properties of the class of manifolds with corners is that it is preserved under taking products.

Lemma 1.8.3. *The direct product $X \times Y$ of two manifolds with corners is a manifold with corners with the $\mathcal{C}^\infty$ structure generated by $\pi_X^*\mathcal{C}^\infty(X) \cup \pi_Y^*\mathcal{C}^\infty(Y)$ where $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are the projections.*

Proof. The products of coordinate patches from an atlas on $X$ and an atlas on $Y$ clearly give coordinate patches forming an atlas on $X \times Y$ with a $\mathcal{C}^\infty$ structure generated as stated. The boundary hypersurfaces of $X \times Y$ are all of the form $H \times Y$ or $X \times G$ where $H \in M_1(X)$ and $G \subset M_1(Y)$. Lifting a defining function from $X$ or $Y$ for the appropriate boundary hypersurface gives a defining function on $X \times Y$, so all boundary hypersurfaces are embedded and hence $X \times Y$ is a manifold with corners. □

## 1.9. Nesting and intersection

In the sequel we often consider nested submanifolds

$$Y_1 \subset Y_2 \subset \cdots \subset Y_r \subset X \tag{1.9.1}$$

of strictly increasing dimension. If $X$ is a manifold without boundary then at any point of $X$ all the submanifolds in (1.9.1) can be linearized simultaneously in local coordinates. In fact for any $p \in X$ we can choose local coordinates $x_i$ based at $p$ in terms of which

$$Y_k \cap \Omega = \{x_i = 0,\ i \le \operatorname{codim}(Y_k)\} \tag{1.9.2}$$

where $\Omega$ is a small coordinate neighbourhood of $p$. For p-submanifolds of a manifold with corners something very similar is true provided we require each $Y_i$ to be a p-submanifold of $Y_{i+1}$, with $X = Y_{r+1}$, rather than just a p-submanifold of $X$.

LEMMA 1.9.1. *If, for $i = r, \dots, 1$, $Y_i \subset Y_{i+1}$ are p-submanifolds, with $X = Y_{r+1}$ a manifold with corners, then near any point of $X$ there are (product) coordinates in terms of which all the $Y_i$ through that point are given by intersections of coordinate planes.*

PROOF. We cannot quite arrange (1.9.2), in general, since the coordinates are to be product coordinates so the first $k$ must always define the boundary hypersurfaces through the point in question. In fact the lemma follows by applying the definition of a p-submanifold inductively. Certainly if $r = 1$ then $Y_r$ and $X$ have a common product decomposition near any point of $Y_r$, which means that in appropriate local coordinates $Y_r$ is the intersection of coordinate planes in $X$. So assume the result for $r'$ and consider $r = r'+1$. Dropping the smallest submanifold (and renumbering) the inductive hypothesis can be applied to give coordinates in which all the $Y_j$ for $j > 1$ are the intersections of coordinate planes. Moreover any coordinate change in $Y_2$ can be extended as a product coordinate change with the other variables unaltered to give the same result. Then applying the definition to $Y_1 \subset Y_2$ as a p-submanifold proves the inductive step and hence the full result. $\square$

DEFINITION 1.9.6. A chain of p-submanifolds of a manifold with corners is a sequence of submanifolds as in (1.9.1), satisfying the condition of Lemma 1.9.1.

EXERCISE 1.9.6. Write down an example which shows that, for any $r \ge 2$, the condition that the $Y_i$ form a chain of p-submanifolds is strictly stronger than the requirement that they form a chain under inclusion and that each $Y_i$ be a p-submanifold of $X$.

The opposite extreme, in terms of the intersection properties of submanifolds, of inclusion is the condition that $Y_1$ and $Y_2$ should meet transversally. Although this is usually expressed differentially in the boundaryless case there is again a formulation in terms of product decompositions. Thus $Y_1$ and $Y_2$ meet transversally, written $Y_1 \pitchfork Y_2$, if local coordinates $x_i$ can be introduced at each point of the intersection in terms of which

$$\begin{gathered} Y_1 = \bigcap_{j \in S_1} \{x_j = 0\} \text{ and } Y_2 = \bigcap_{j \in S_2} \{x_j = 0\} \\ \text{with } S_1 \cap S_2 = \emptyset. \end{gathered} \tag{1.9.3}$$

It follows that $Y_1 \cap Y_2$, also conventionally denoted $Y_1 \pitchfork Y_2$, is a submanifold.

In the case of manifolds with corners we adopt the same condition, (1.9.3), for two p-submanifolds to intersect transversally in $X$, where we also insist that the coordinates be product coordinates. Then the intersection is a p-submanifold. More generally we need to consider the transversality of chains of p-submanifolds.

Definition 1.9.7. A finite collection $Y^{(j)}$ for $j = 1, \dots, k$ of chains of p-submanifolds of a manifold with corners $X$, where the $j$th chain is

$$Y_1^j \subset Y_2^{(j)} \subset \cdots \subset Y_{r(j)}^{(j)} \subset X,$$

are said to intersect transversally if each point $p \in X$ has a neighbourhood $\Omega$ with a product decomposition $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and there is a chain of p-submanifolds $Z^{(j)}$ of each $\Omega_j$ such that

$$Y_i^{(j)} = \Omega_1 \times \cdots \times \Omega_{j-1} \times Z_i^{(j)} \times \cdots \times \Omega_k,\ j = 1, \dots, k. \tag{1.9.4}$$

Of course we must allow zero dimensional factors and empty chains in this definition.

The local condition in Definition 1.9.7 can be expressed as the existence of a local coordinate system near each point in terms of which each element of each chain is given as the intersection of coordinate hyperplanes and no one hyperplane appears in the definition of elements of more than one chain:

$$\begin{gathered} Y_l^j = \{x_r = 0, r \in E(l,j)\},\ E(l,j) \subset \{1, \dots, n\}, \\ E(l,j) \cap E(l',j') = \emptyset \text{ if } j \neq j',\ E(l,j) \subset E(l',j) \text{ if } l \geq l'. \end{gathered} \tag{1.9.5}$$

## 1.10. Cotangent and tangent bundles

We shall describe next the construction of the (extension) cotangent bundle of a manifold with corners. It is again naturally a manifold with corners. For each $p \in X$ set

$$I(p) = \{f \in \mathcal{C}^\infty(X); f(p) = 0\}. \tag{1.10.1}$$

This is an ideal in $\mathcal{C}^\infty(X)$. Set

$$I^2(p) = I(p) \cdot I(p) = \{u \in \mathcal{C}^\infty(X); u = \sum_{\text{finite}} f_i g_i,\ f_i, g_i \in I(p)\}. \tag{1.10.2}$$

The (extension) cotangent fibre at $p$ is

$$T_p^* X = I(p)/I^2(p). \tag{1.10.3}$$

It is easy to see, by reference to local coordinates and use of Taylor's formula, that this is a vector space of dimension equal to that of $X$. As a set the cotangent bundle of $X$ is

$$T^*X = \bigcup_{p \in X} T_p^* X. \tag{1.10.4}$$

If $f \in \mathcal{C}^\infty(X)$ and $p \in X$ then $f - f(p) \in I(p)$. Thus

$$df(p) = [f - f(p)] \in T_p^* X \tag{1.10.5}$$

defines a *section* of $T^*X$, that is a map

$$df \colon X \longrightarrow T^*X,\ \pi \circ df = Id. \tag{1.10.6}$$

Here $\pi \colon T^*X \longrightarrow X$ is the natural projection, mapping $T_p^*X$ to $p$. If $Y$ is another manifold and $f \in \mathcal{C}^\infty(X \times Y)$ let $d_X f \colon X \times Y \longrightarrow T^*X$ be the family of sections in which $Y$ is just a space of parameters.

Using these *exact* sections of $T^*X$ we define the following space of functions on $T^*X$ :

$$\mathcal{F} = \{u \colon T^*X \longrightarrow \mathbb{C};\ u \circ d_X f \colon X \longrightarrow \mathbb{C} \text{ is } \mathcal{C}^\infty \text{ for all } f \in \mathcal{C}^\infty(X \times Y)\}. \tag{1.10.7}$$

Proposition 1.10.1. *$(T^*X, \mathcal{F})$, with $\mathcal{F}$ defined by (1.10.7), is a $\mathcal{C}^\infty$ manifold with corners.*

Proof. We start with the case $X = \mathbb{R}^n_k$. Then the natural coordinate system $x_1, x_2, \dots, x_n$ generates a global coordinate system on $T^*\mathbb{R}^n_k$ :

$$\begin{gathered} I(p)/I^2(p) \cong \mathbb{R}^n,\ f \longmapsto a \text{ where} \\ f(p) = 0 \Longrightarrow f(x) = \sum_{i=1}^n a_i(x-p)_i + g,\ g \in I^2(p). \end{gathered} \tag{1.10.8}$$

Thus $T^*\mathbb{R}^n_k \cong \mathbb{R}^n_k \times \mathbb{R}^n = \mathbb{R}^{2n}_k$. Moreover the condition (1.10.7) does characterize $\mathcal{C}^\infty(\mathbb{R}^{2n}_k)$ by taking $f = \sum\limits_i a_i x_i \in \mathcal{C}^\infty(X \times \mathbb{R}^n)$ with the $a_i$ parameters. Indeed then

$$u \in \mathcal{F} \Longleftrightarrow u(x_1, \dots, x_n, a_1, \dots, a_n) \in \mathcal{C}^\infty(\mathbb{R}^n_k \times \mathbb{R}^n). \tag{1.10.9}$$

In the general case suppose that $p \in X$ is any point. Let $f_i$, $i = 1, \dots, n$ be functions as in (1.6.10). Then for $q$ near $p$

$$I(q) = \mathbb{R} \cdot \{f_1 - f_1(q), \dots, f_n - f_n(q)\} + I^2(q) \tag{1.10.10}$$

by Taylor's theorem. Thus if $\Omega$ is a sufficiently small neighbourhood of $p$ and

$$\mathcal{F}_c(\Omega) = \{u \in \mathcal{F}; u = 0 \text{ outside a compact subset of } T^*\Omega\} \tag{1.10.11}$$

then $\mathcal{F}_c(\Omega) = \Phi^*\mathcal{C}^\infty_c(\Phi(\Omega) \times \mathbb{R}^n)$ as follows from the discussion above. Thus the proposition follows from Proposition 1.6.1. □

We remark that if $x_1, \cdots, x_n$ are local coordinates at $p \in X$ then

$$df = \sum_{i=1}^n \xi_i dx_i \tag{1.10.12}$$

give *canonically dual* local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ in $\pi^{-1}(\Omega) \subset T^*X$, where $\Omega$ is the coordinate patch.

Similarly it is straightforward to show that the dual spaces

$$T_pX = (T_p^*X)^* \tag{1.10.13}$$

fit together to form the (extension) tangent bundle

$$TX = \bigcup_p T_pX, \tag{1.10.14}$$

which is also naturally a $\mathcal{C}^\infty$ manifold with corners. The $\mathcal{C}^\infty$ structure on $TX$ is, for instance, characterized as follows. First set

$$\begin{aligned} \mathcal{G} = \big\{\phi\colon X \times Y \longrightarrow TX; \phi \text{ is a section and} \\ X \times Y \ni (x,y) \longmapsto \phi(x,y) \circ d_X f(x,y) \text{ is } \mathcal{C}^\infty\ \forall\ f \in \mathcal{C}^\infty(X \times Y)\big\}. \end{aligned} \tag{1.10.15}$$

Then set

$$\mathcal{F} = \{v\colon TX \longrightarrow \mathbb{C}; v \cdot \phi\colon X \times Y \longrightarrow \mathbb{C} \text{ is } \mathcal{C}^\infty\ \forall\ \phi \in \mathcal{G}\}. \tag{1.10.16}$$

Exercise 1.10.7. Check that, with $\mathcal{F}$ defined as in (1.10.15) and (1.10.16), $(TX, \mathcal{F})$ is always a manifold with corners.

If $X$ and $Y$ are $\mathcal{C}^\infty$ manifolds with corners then a map $F\colon X \longrightarrow Y$ is $\mathcal{C}^\infty$ if the pull-back operator,

$$F^*(f)\colon X \longrightarrow \mathbb{C},\ \text{defined by } F^*(f) = f \circ F, \tag{1.10.17}$$

for $f\colon Y \longrightarrow \mathbb{C}$, maps $\mathcal{C}^\infty$ functions to $\mathcal{C}^\infty$ functions:

$$F^*\colon \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X). \tag{1.10.18}$$

Of course if $F(p) = q$ for a $\mathcal{C}^\infty$ map $F\colon X \longrightarrow Y$ then $F^*I(q) \subset I(p)$. Thus $F^*\colon T^*_qY \longrightarrow T^*_pX$, where for any $f \in \mathcal{C}^\infty(Y)$ we always have $F^*df(q) = d(F^*f)(p)$. Dually the differential of $F$ is $F_*\colon T_pX \longrightarrow T_qY$, defined by $\alpha(F_*v) = F^*\alpha(v)$ if $v \in T_pX$ and $\alpha \in T^*_qY$.

For a t-manifold we shall often write the space $\mathcal{C}^\infty(X;TX)$ of all smooth sections of $TX$ as $\mathcal{V}_E(X)$; the space of smooth extension vector fields. Each element of $\mathcal{V}_E(X)$ acts on $\mathcal{C}^\infty(X)$, essentially by definition:

(1.10.19)
$$V\colon \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X),\ Vf(p) = \langle V(p), df(p)\rangle,\ V \in \mathcal{V}_E(X),\ f \in \mathcal{C}^\infty(X).$$

Here $\langle,\rangle$ is the pairing between $T_pX$ and $T^*_pX$. Moreover this action shows that $\mathcal{V}_E(X)$ is a Lie algebra with

$$[V,W]f(p) = (VW - WV)f(p)\ \forall\ f \in \mathcal{C}^\infty(X). \tag{1.10.20}$$

In the sequel we study a variety of Lie subalgebras of $\mathcal{V}_E(X)$ which we often denote by $\mathcal{V}_\#(X)$ with $\#$ distinguishing the particular algebra. For manifolds without boundary there are no 'universal' proper subalgebras of the type discussed below, so $\mathcal{V}_E(X)$ will generally be written simply $\mathcal{V}(X)$ in that case.

## 1.11. Vector bundles

Having defined the tangent and cotangent bundles we shall briefly recall the general notion of a vector bundle over a manifold with corners. A *vector bundle* over $X$ is a triple $(Y,\pi,\mathcal{G})$ where $Y$ is a $\mathcal{C}^\infty$ manifold with corners, $\pi\colon Y \longrightarrow X$ is a surjective $\mathcal{C}^\infty$ map with surjective differential at each point and $\mathcal{G} \subset \mathcal{C}^\infty(Y)$ is a linear subspace and $\pi^*\mathcal{C}^\infty(X)$-module with the following properties:

$$\mathcal{G} \restriction \pi^{-1}(p) \text{ has dimension } \dim Y - \dim X \quad \forall\ p \in X, \tag{1.11.1}$$

$$\begin{gathered}\mathcal{G} \restriction \pi^{-1}(p) \text{ induces a linear structure on } \pi^{-1}(p) \text{ i.e.}\\ \pi^{-1}(p) \cong \left(\mathcal{G} \restriction \pi^{-1}(p)\right)^* \text{ as manifolds}\end{gathered} \tag{1.11.2}$$

and finally

$$\mathcal{G} \cup \pi^*\mathcal{C}^\infty(X) \text{ generates } \mathcal{C}^\infty(Y) \text{ in the sense of Definition 1.6.2.} \tag{1.11.3}$$

Lemma 1.11.1. *If $(Y,\pi,\mathcal{G})$ is a vector bundle then*

$$\mathcal{G} = \{u \in \mathcal{C}^\infty(Y); u \textit{ is linear on each fibre of } \pi\} \tag{1.11.4}$$

*and each $p \in X$ has a neighbourhood $\Omega$ with a diffeomorphism*

$$F\colon \Omega \times \mathbb{R}^q \longleftrightarrow \pi^{-1}(\Omega) \tag{1.11.5}$$

*such that $F(p,\cdot) \in \mathrm{Iso}(\mathbb{R}^q;\pi^{-1}(p))$.*

PROOF. By (1.11.3), $\mathcal{G}$ is contained in the space of fibre-linear smooth functions. Conversely if $p \in X$ and $O_p \in Y_p$ is the origin of the fibre, $Y_p$, above $p$ then by (1.11.2) there are elements $\xi_l \in \mathcal{G}$, for $l = 1, \dots, k = \dim Y - n = q$, which restrict to $Y_p$ to give a basis of linear functions on the fibres. The surjectivity of the differential of $\pi$ and (1.11.3) show that there are elements $f_j \in \mathcal{C}^\infty(X)$, for $j = 1, \dots, n = \dim X$, giving coordinates in some neighbourhood $\Omega$ of $p$, such that any $u \in \mathcal{C}^\infty(Y)$ can be written locally uniquely in the form $\tilde{u}(f_1, \dots, f_n, \xi_1, \dots, \xi_k)$ where $\tilde{u}$ is $\mathcal{C}^\infty$. If $u$ is linear on the fibres then $\tilde{u}$ must be linear in the last $k$ variables. Since $\mathcal{G}$ is a $\mathcal{C}^\infty(X)$-module this proves (1.11.4). Moreover the map

$$G\colon \pi^{-1}(\Omega) \ni (x, y) \longmapsto (x, \xi(x, y)) \in \Omega \times \mathbb{R}^q \tag{1.11.6}$$

is an isomorphism the inverse of which gives a map (1.11.5). □

A smooth map (1.11.5), linear on the fibres, is just a local trivialization of the bundle.

EXERCISE 1.11.8. Discuss the relation between different local trivializations of a vector bundle and hence check that this definition is consistent with your favourite definition of a vector bundle over a manifold without boundary.

Generally $\pi$ and $\mathcal{G}$ are taken as obvious and the vector bundle is identified with its 'total space' $Y$; the fibre $\pi^{-1}(p)$ is, as above, often denoted $Y_p$. A vector bundle map $F\colon Y' \longrightarrow Y$ between two vector bundles $Y$ over $X$ and $Y'$ over $X'$ is a $\mathcal{C}^\infty$ map such that, for some $f\colon X' \longrightarrow X$, $F\colon Y'_p \longrightarrow Y_{f(p)}$ is a linear map for each $p \in X'$. Often $X = X'$ and the map $f$ is the identity.

Without going into irrelevant detail concerning categories and functors we recall that any smooth functor on the category of vector spaces allows us to construct new vector bundles from old. A functor, with $l$ covariant indices and $q$ contravariant indices, on the category of finite-dimensional vector spaces assigns to any $l + q$ vector spaces $V_i$, $W_j$ where $i = 1, \dots, l$ and $j = 1, \dots, q$ a vector space $U = \Phi(V_1, \dots, V_l; w_1, \dots, W_q)$ and similarly assigns morphisms, so if $S_i \in \mathrm{Iso}(V_i; V'_i)$ and $T_j \in \mathrm{Iso}(W'_j; W_j)$ are linear isomorphism then $\Phi(S_1, \dots, S_l; T_1, \dots, T_q) \in \mathrm{Iso}(U, U')$ is determined where $U' = \Phi(V'_1, \dots, V'_l; W'_1, \dots, W'_q)$. Furthermore a naturality property under composition is also demanded, namely if $S'_1 \in \mathrm{Iso}(V'_1; V''_1)$ and $T'_1 \in \mathrm{Iso}(W''_1, W'_1)$ then

$$\begin{aligned} &\Phi(S'_1 \circ S_1, S_2, \dots, S_l; T_1, \dots, T_q) \\ &\qquad = \Phi(S'_1, S_2, \dots, S_l; T_1, \dots, T_q) \circ \Phi(S_1, S_2, \dots, S_l; T_1, \dots, T_q) \\ &\Phi(S_1, S_2, \dots, S_l; T'_1 \circ T_1, \dots, T_q) \\ &\qquad = \Phi(S_1, S_2, \dots, S_l; T'_1, \dots, T_q) \circ \Phi(S_1, S_2, \dots, S_l; T_1, \dots, T_q) \end{aligned} \tag{1.11.7}$$

and similarly in the other arguments. Such a functor is said to be smooth if $\Phi$ defines smooth maps and the composition maps in (1.11.7) are smooth (jointly in all variables).

The cases of particular interest in differential geometry and differential analysis are the functors:

(1) Dual of $W : W^*$
(2) Exterior sum (or product) of $V_1$ and $V_2 : V_1 \oplus V_2$
(3) Tensor product of $V_1$ and $V_2 : V_1 \otimes V_2$
(4) Exterior powers of $V : \bigwedge^l V$

(5) Symmetric powers of $V : \mathrm{Sym}^l(V)$
(6) Homomorphism space of $V$ to $W : \hom(V;W)$
(7) Density spaces of $V : |V|^\alpha$, $\alpha \in \mathbb{R}$,

where in each case $V$ denotes a covariant and $W$ a contravariant space. Since it is perhaps the least familiar we shall discuss the density bundles as an example.

If $V$ is a finite-dimensional vector space (say over $\mathbb{R}$) with dual space $V^*$ consider the maps which are totally antisymmetric in $d = \dim V$ factors of $V^*$ and absolutely homogeneous of degree $\alpha \in \mathbb{R}$ :

$$v\colon \bigwedge\nolimits^d V^* \setminus 0 \longrightarrow \mathbb{R},\ v(r\mu) = |r|^\alpha v(\mu),\ r \in \mathbb{R}\setminus 0,\ \mu \in \bigwedge\nolimits^d V^* \setminus 0. \tag{1.11.8}$$

Since $\bigwedge^d V^*$ has dimension one, these form a one-dimensional vector space, which we denote $|V|^\alpha$. If $v_i \in V$ for $i = 1, \dots, d$ is a basis and $v_i^*$ is the dual basis of $V^*$ then $v$ is fixed by the value of $v(v_1^* \wedge \cdots \wedge v_d^*) \in \mathbb{R}$. If $S \in \mathrm{Iso}(V, V')$ is an isomorphism onto a second vector space then the $Sv_i$ give a basis and the induced transformation $S \in \mathrm{Iso}(|V|^\alpha, |V'|^\alpha)$ is fixed by

$$(Sv)((Sv_1)^*, \dots, (Sv_d)^*) = v(v_1^*, \dots, v_d^*). \tag{1.11.9}$$

If $V = V' = \mathbb{R}^d$ so that $|V|^\alpha$ and $|V'|^\alpha$ are identified and $S \in \mathrm{Iso}(\mathbb{R}^d;\mathbb{R}^d)$ is an invertible matrix then $Sv = |\det S|^\alpha v$ which shows that the density functor is smooth.

Exercise 1.11.9. Check that the other functors listed above are smooth.

Proposition 1.11.1. *If $\Phi$ is a smooth functor of vector spaces and $V_i$, $W_j$ are smooth vector bundles over a manifold with corners $X$ then*

$$\Phi_p(V_1, \dots, V_l; W_1, \dots, W_q) = \Phi((V_1)_p, \dots (V_l)_p; (W_1)_p, \dots, (W_q)_p)$$
$$\Phi(V;W) = \Phi(V_1, \dots, V_l; W_1, \dots, W_q) = \bigsqcup_{p\in X} \Phi_p(V_1, \dots, V_l; W_1, \dots, W_q)$$

*has a natural structure as a $\mathcal{C}^\infty$ vector bundle over $X$.*

Proof. Let $U(p) = \Phi(V_1(p), \dots, V_l(p); W_1(p), \dots, W_q(p))$ be the fibres over $p$. Each point $p \in X$ has an open neighbourhood $\Omega$ on which all the bundles $V_i$, $W_j$ have trivializations as in (1.11.5), $F_i$ for the $V_i$ and $G_j$ for the $W_j$. Thus $F_i'(p') = F_i(p')F^{-1}(p) \in \mathrm{Iso}(V_i(p), V_i(p'))$ is an isomorphism depending smoothly on $p' \in \Omega$ and similarly for $G_j'(p') = G_j{-}1(p)G_j(p') \in \mathrm{Iso}(W_j(p'); W_j(p))$. The functor provides an isomorphism $\Phi(F_1'(p'), \dots, F_l'(p'); G_1'(p'), \dots, G_q'(p'))\,\mathrm{Iso}(U(p), U(p'))$ for $p' \in \Omega$ which, by assumption, depends smoothly on $p'$. This gives a local trivialization of the bundle $U$ over $\Omega$, let $\mathcal{G}$ be the space of functions on

$$U = \bigsqcup_{p\in X} U(p)$$

which are linear on the fibres and smooth on the product $\Omega \times U(p)$ when pulled back under any such local trivialization. The smoothness of the functor implies that this gives $U$ the structure of a $\mathcal{C}^\infty$ vector bundle. $\square$

## 1.12. $b$-maps

A general $\mathcal{C}^\infty$ map has no relation to the boundaries of the manifolds with corners it maps. A useful condition to impose, corresponding to (1.7.5), is

$$\text{either } F^{-1}(\partial Y) \subset \partial X \text{ or } F^{-1}(\partial Y) = X. \tag{1.12.1}$$

In fact we impose a differential version of (1.12.1) in defining the fundamental notion of a $b$-map.

Definition 1.12.8. Suppose $X$ and $Y$ are $\mathcal{C}^\infty$ manifolds with corners and $\rho_H$, for $H \in M_1(X)$, and $\rho'_G$, for $G \in M_1(Y)$, are complete sets of defining functions for the boundary hypersurfaces of $X$ and $Y$ respectively. Then a $\mathcal{C}^\infty$ map $F\colon X \longrightarrow Y$ is said to be a $b$-map if for each $G \in M_1(Y)$

$$\begin{gathered}\text{either } F^*\rho'_G \equiv 0 \text{ or}\\ F^*\rho'_G = a_G \prod_{H\in M_1(X)} \rho_H^{e(H,G)},\ 0 < a_G \in \mathcal{C}^\infty(X).\end{gathered} \tag{1.12.2}$$

If the first case in (1.12.2) does not occur for any $G$ so $F^*\rho'_G$ is not identically zero, then $F$ is said to be an interior $b$-map, otherwise it is a boundary $b$-map.

To see that an interior $b$-map satisfies the first condition in (1.12.1) consider a general boundary point $p \in \partial Y$. Thus $p \in G$ for some $G \in M_1(Y)$ so $\rho'_G(p) = 0$. Then $F(x) = p$ implies $F^*\rho'_G(x) = 0$; by (1.12.2) this implies $\rho_H(x) = 0$ for some $H \in M_1(X)$ so $x \in \partial X$. On the other hand if $F$ is a boundary b-map then the first case in (1.12.2) occurs for some $G \in M_1(Y)$. In this case

$$F(X) \subset G,\ G \in M_1(Y) \tag{1.12.3}$$

so the second condition in (1.12.1) holds. Repeating this reduction it follows that there is a unique boundary face $B \in M(Y)$ such that $F(X) \subset B$ and $F\colon X \longrightarrow B$ is an interior b-map.

We call the exponents $e(H,G)$ in (1.12.2) the boundary exponents of $F$, they are necessarily non-negative integers, since $F$ is assumed to be $\mathcal{C}^\infty$. We shall see below that $b$-maps have important analytic properties.

A diffeomorphism between manifolds with corners is always a $b$-map, as can be seen from Corollary 1.5.1. A projection $\pi_X : X \times Y \longrightarrow X$, $\pi_X(x,y) = x$ is always a $b$-map as is any $\mathcal{C}^\infty$ map from a manifold with boundary into a manifold without boundary. On the other hand a $\mathcal{C}^\infty$ map from a manifold without boundary onto a manifold with corners with range meeting the boundary cannot be an interior $b$-map, since it cannot even satisfy (1.12.1). For example the map

$$\mathbb{R} \ni x \longmapsto x^2 \in [0,\infty) \tag{1.12.4}$$

is not a $b$-map.

Lemma 1.12.1. *The composition of two $b$-maps is a $b$-map.*

Proof. This follows directly by inserting (1.12.2) for the first map into the analogous identity for the second. $\square$

Exercise 1.12.10. Show that the boundary exponents of $b$-maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ satisfy the following matrix composition law

$$e_{g\circ f}(K,G) = \sum_{H\in M_1(Y)} e_g(K,H)e_f(H,G),\ K \in M_1(Z),\ G \in M_1(X). \tag{1.12.5}$$

Clearly we also have:

PROPOSITION 1.12.1. *For a d-submanifold $S \subset X$ of a manifold with corners the inclusion $\iota_S\colon S \hookrightarrow X$ is a b-map.*

PROOF. The definition of a d-submanifold is given in Definition 1.7.4. It is only necessary to check (1.12.2) locally near each point of $S$, for the inclusion map. Since (1.12.2) is clearly coordinate invariant we only have to observe that the inclusions for the local models in (1.7.9) are always $b$-maps. □

For example the natural embedding of a boundary face is always a boundary $b$-map. We then add to our list of types of submanifolds:

DEFINITION 1.12.9. A submanifold $S$ of a manifold with corners $X$ is a $b$-submanifold if the inclusion map is a $b$-map.

Then Proposition 1.12.1 takes the form:

$$\begin{aligned} &S \text{ a } p\text{-submanifold} \Longrightarrow S \text{ a } d\text{-submanifold} \Longrightarrow \\ &\qquad\qquad S \text{ a } b\text{-submanifold} \Longrightarrow S \text{ a submanifold.} \end{aligned} \tag{1.12.6}$$

These four classes of submanifolds are in general distinct. Thus in $\mathbb{R}^3_3$ the linear subspace $\{x_3 = x_1 + x_2\}$ is a submanifold but not a $b$-submanifold. On the other hand

LEMMA 1.12.2. *The diagonal $\Delta \subset X^2$ is always a b-submanifold.*

PROOF. If $x_1, \dots, x_n$ are local coordinates in $X$ near a boundary point of codimension $k$ then taking the same coordinates in both factors, but denoted $x'_i$ in the second,

$$\Delta = \{x_i = x'_i, i = 1 \dots, n\}. \tag{1.12.7}$$

Thus we can take $x_i, i = 1, \dots, n$ as coordinates and then the defining relation (1.12.2) of a $b$-map becomes

$$\iota^*(x_i) = x_i,\ \iota^*(x'_i) = x_i,\ i = 1, \dots, k \tag{1.12.8}$$

for the inclusion. □

If $\partial X \neq \emptyset$ it is easy to see that $\Delta \subset X^2$ is not at p-submanifold (nor is it a d-submanifold).

LEMMA 1.12.3. *If $Y \subset X$ is a b-submanifold of a manifold with corners then at each point of $Y$ local (product) coordinates $x_i$, $y_j$ $i = 1, \dots k$, $j = 1, \dots, l$, can be introduced in terms of which $Y$ takes the local form:*

$$\begin{aligned} &x_i = 0,\ 1 \le i \le k',\ A_r(x) = 0,\ 1 \le r \le k'', \\ &\qquad\qquad y_j = 0,\ 1 \le j \le l_1,\ y_j \ge 0,\ l_1 < j \le l_1 + l_2, \end{aligned} \tag{1.12.9}$$

*where $k' + k'' \le k$ and the $A_r$ are independent linear forms in $x'' = (x_{k'+1}, \dots, x_{k''})$ which take on values with both signs on $[0, \infty)^{k''-k'}$. Conversely any submanifold everywhere locally of the form (1.12.9) is a b-submanifold. The condition $l_2 = 0$ at all point of $Y$ is equivalent to $\partial Y \subset \partial X$.*

The assumption that the $A_r$ take values with both signs means that the number, $k'$, of single variable $x_i$ is maximal.

As a consequence of this local normal form $b$-submanifolds have reasonable restriction properties:

LEMMA 1.12.4. *If $Y$ is a b-submanifold of a manifold with corners $X$ then for any boundary face, $B \in M(X)$, $Y \cap B$ is a b-submanifold of $B$ and*

$$T_pY \cap T_pB = T_p(Y \cap B) \;\forall\; p \in B \cap Y. \tag{1.12.10}$$

## 1.13. Integration of vector fields

One of the most fundamental of analytic tools is the integration of vector fields, i.e. the solution of ordinary differential equations. We shall not delve into the basic constructions, but recall the following result for manifolds without boundary.

PROPOSITION 1.13.1. *If $X$ is a manifold without boundary and $V \in \mathcal{V}(X)$ is a real vector field then for any compact subset $K \subset X$ there exists $\epsilon > 0$ and a smooth map*

$$(-\epsilon, \epsilon) \times K \ni (t, p) \longmapsto \exp(tV)(p) \in X \tag{1.13.1}$$

*with $\exp(0)(p) = p$ and such that for each fixed $p$ the curve $t \longmapsto \exp(tV)$ is the unique integral curve of $V$ through $p$ in the sense that*

$$(\exp(tV))_* (\frac{d}{dt}) = V. \tag{1.13.2}$$

It is often important to note that solution map $\exp(tV)$ depends smoothly on parameters in $V$ as well as in $X$. This can be deduced from the result as stated by adding the parameter space, $S$, as part of the base and lifting $V$ to $S \times X$ to act trivially on $S$. Integrating this vector field shows that the solution depends smoothly on the parameters.

The defining condition (1.13.2) can be written in terms of the pull-back of smooth functions as

$$\frac{d}{dt}(\exp(tV))^* f = \exp(tV)^*(Vf). \tag{1.13.3}$$

It follows that if $Vf \geq 0$ on $\{f = 0\}$ then for $t \geq 0$ the integral curves of $V$ starting in the region $\{f \geq 0\} \subset X$ stay in this region. We can use this to integrate the appropriate vector fields on a manifold with corners.

DEFINITION 1.13.10. On a t-manifold a vector field $V \in \mathcal{V}_E(X)$ is said to be inward-pointing if $Vf \geq 0$ on $\partial X$ for every function $f \in \mathcal{C}^\infty(X)$ with $f \geq 0$ and $f = 0$ on $\partial X$.

COROLLARY 1.13.1. *If $X$ is a t-manifold and $V \in \mathcal{V}_E(X)$ is a real inward-pointing vector field then for any compact subset $K \subset X$ there exists $\epsilon > 0$ and a smooth map*

$$[0, \epsilon) \times K \ni (t, p) \longmapsto \exp(tV)(p) \in X \tag{1.13.4}$$

*with $\exp(0)(p) = p$ and such that for each fixed $p$ the curve $t \longmapsto \exp(tV)$ is the unique integral curve of $V$ through $p$ in the sense that*

$$(\exp(tV)_* (\frac{d}{dt}) = V. \tag{1.13.5}$$

PROOF. The uniqueness, and hence coordinate invariance, of the integral curves means that it is enough to construct $\exp(tV)$ in coordinate patches, i.e. it suffices to consider the case $X = \Omega \subset \mathbb{R}^n_k$ is open. A section of $\mathcal{V}_E(\Omega)$ can always be extended to an element of $\mathcal{V}(\Omega')$ for some $\Omega' \subset \mathbb{R}^n$ open with $\Omega = \Omega' \cap \mathbb{R}^n_k$. Since $V$ is assumed to be inward pointing it follows that the integral curves stay in $\Omega = \{x_1 \dots x_k \geq 0\}$ as $t$ increases. This proves the local, and hence the semi-global, result. $\square$

One simple application of these results is the existence of semi-global product decompositions near boundary faces:

LEMMA 1.13.1. *A boundary face, $F \in M_l(X)$, of codimension $l$ in a compact manifold with corners has a neighbourhood diffeomorphic to $[0,1)^l \times F$.*

PROOF. Consider first the case of a boundary hypersurface $H \in M_1(X)$. The exitence of a defining function $\rho_H$ allows us to choose a vector field with is tangent to the other boundary hypersurfaces but is inward-pointing vector across $H$ :

$$V\rho_H > 0 \text{ on } H,\ V\rho_G = 0 \text{ on } G,\ G \in M_1(X),\ G \neq H. \tag{1.13.6}$$

Indeed each point of $H$ has a coordinate neighbourhood with in which $H$ is locally defined by $x_1 = 0$. If $\phi_a$ is a partition of unity for $H$ subordinate to such a coordinate covering and $V_a = \partial/\partial x_1$ in the coordinate patch then $V = \sum_a \phi_a V_a$ satifies (1.13.6). Integration of $V$ starting from $H$ gives an isomorphism of the desired type:

$$\exp(t\delta V)\colon [0,1) \times H \longrightarrow X \tag{1.13.7}$$

for $\delta > 0$ sufficiently small.

The general case then follows by induction, since $F \in M_l(X)$ is, for $l > 1$, always a boundary face of some $G \in M_{l-1}(X)$. □

## 1.14. Extension

It is sometimes convenient to embed a manifold with corners, or even a t-manifold, $X$, as a submanifold of a manifold without boundary, $\tilde{X}$, with $\tilde{X}$ compact if $X$ is compact. One of the difficulties with doing this is the possible existence of non-embedded boundary components.

PROPOSITION 1.14.1. *Any t-manifold, $X$, can be embedded as a d-submanifold of a manifold without boundary, $\tilde{X}$. If $X$ is compact then $\tilde{X}$ can be taken compact. Any diffeomorphism of $X$ can be extended to a diffeomorphism of a neighbourhood of $X$ in $\tilde{X}$.*

PROOF. Using Lemma 1.6.2 we can take a total boundary defining function and set

$$Y_\epsilon = \{p \in X;\ \rho(p) \geq \epsilon\} \tag{1.14.1}$$

for $\epsilon > 0$ small enough. This is a submanifold with boundary, since from (1.12.8) it follows that $d\rho \neq 0$ on $U \setminus \partial X$ for some neighbourhood $U$ of $\partial X$ in $X$. Choose a strictly inward pointing $\mathcal{C}^\infty$ vector field on $X$, i.e. $V$ such that

$$Vx_j(p) > 0, \quad j \leq k$$

whenever the $x_i$ are coordinates at $p \in \partial X$. This is certainly possible locally since $\partial/\partial x_1 + \partial/\partial x_2 + \cdots + \partial/\partial x_k$ is strictly inward-pointing in $\mathbb{R}^n_k$. Then $V$ can be taken as a sum over a partition of unity subordinate to a coordinate cover of the boundary

$$V = \sum_j \phi_j V_j \tag{1.14.2}$$

where the $V_j$ are local choices.

Assuming, for simplicity, that $X$ is compact, integration along $V$ gives a $\mathcal{C}^\infty$ map

$$\exp(tV)\colon X \longrightarrow X \quad (t > 0 \text{ small}) \tag{1.14.3}$$

which is a diffeomorphism onto its range,

$$\exp(tV)X \subset X^\circ. \tag{1.14.4}$$

Given $\epsilon$ small enough, $\exp(tV)X \subset Y_\epsilon$. This embeds $X$ as a submanifold in the interior of $Y_\epsilon$.

Since the extension part of the proposition is straightforward, the proof is completed by the next result. □

EXERCISE 1.14.11. Carry through the proof in the non-compact case.

PROPOSITION 1.14.2. *If $X$ is a manifold with boundary then the double of $X$*

$$2X = (X \sqcup X)\,/\partial X \tag{1.14.5}$$

*can be given a $\mathcal{C}^\infty$ structure, as a manifold without boundary, such that both the inclusions*

$$X \hookrightarrow 2X \tag{1.14.6}$$

*are diffeomorphisms onto their range, embedding $X$ as a d-submanifold, and such that the natural reflection $I\colon 2X \longrightarrow 2X$, is a diffeomorphism.*

PROOF. The existence of a total boundary defining function $\rho$, as discussed in Lemma 1.6.2, shows that some neighbourhood of $\partial X$,

$$\rho^{-1}([0,\epsilon)) \cong \partial X \times [0,\epsilon), \quad \epsilon > 0 \text{ small enough.} \tag{1.14.7}$$

This means that $X$ can be covered by coordinate patches of two types, those in the first class not meeting $\partial X$ and those in the second class being consistent with (1.14.7), i.e. the coordinate patches are products under (1.14.7), $\rho = x_1$ is the first coordinate and the other (boundary) coordinates are constant along the fibres of (1.14.7). Then a complete covering of $2X$ is given by using the first class of coordinate patches twice, once for each half, and reversing the sign of $x_1$ in the second class of coordinates, on the second half $X$, so that each pair or coordinate patches combine to give one coordinate patch consistent with the decomposition

$$\rho^{-1}([0,\epsilon)) \sqcup \rho^{-1}([0,\epsilon))/\partial X \cong \partial X \times (-\epsilon,\epsilon). \tag{1.14.8}$$

Clearly this gives a $\mathcal{C}^\infty$ structure with the desired properties. □

Although we have already noted that the usual functorial constructions apply to vector bundles over manifolds with corners (or indeed t-manifolds) an alternative approach is to use this extension result to define all the usual form bundles by restriction from an extension to a manifold without boundary. Thus

$$TX = T_X\tilde{X}, \text{ etc.} \tag{1.14.9}$$

The naturality of this definition follows from the extendibility of a diffeomorphism to a neighbourhood of $X$ in $\tilde{X}$, i.e. invariance under diffeomorphisms and independence of the extension. We also note that the local form of Theorem 1.4.1 gives a similar Seeley extension map, a continuous linear map

$$E\colon \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(\tilde{X}),\ E(f)_{|X} = f \ \forall\ f \in \mathcal{C}^\infty(X). \tag{1.14.10}$$

For manifolds with corners there is a more direct construction leading to an embedding as in Proposition 1.14.1. Namely, the doubling construction in Proposition 1.14.2 can be carried out across any embedded boundary hypersurface of a t-manifold, and for a manifold with corners this gives a sequence of manifolds with corners, each embedded as an open set in the next, with one less boundary hypersurface at each step.

If $X$ is a manifold with corners and $A \subset M_1X$ is a set of boundary hypersurfaces we shall denote by $2_AX$ a manifold, usually with corners, obtained by successive doubling across the faces in $A$. Up to diffeomorphism, trivial on $X \hookrightarrow 2_AX$, it is independent of the order in which the doubling takes place.

LEMMA 1.14.1. *If $S \subset X$ is a p-submanifold of a t-manifold then for any extension $\widetilde{X}$ of $X$ there is an embedded submanifold $\widetilde{S} \subset \widetilde{X}$ without boundary which restricts to $S$ :*

$$X \cap \widetilde{S} = S. \tag{1.14.11}$$

PROOF. In the case of an interior p-submanifold we can choose the strictly inward-pointing vector field in the retraction construction of Proposition 1.14.1 to be tangent to $S$, since this can clearly be arranged locally near each point of the boundary. Then $S$ provides its own extension. If $S$ is a boundary p-submanifold let $B$ be the smallest boundary face containing $S$. If $X$ is doubled across all the boundary hypersurfaces containing $B$, (i.e. containing $S$) but not across the others then $S$ becomes an interior p-submanifold. Now the interior case can be followed again with the inward-pointing vector field chosen to be tangent both to $S$ and to all the hypersurfaces across which the manifold has been doubled. □

## 1.15. Quasi-homogeneity

If $Y \subset X$ is a p-submanifold of a manifold with corners the conormal bundle, $N^*Y$, to $Y$ in $X$ consists of the differentials of elements of $\mathcal{I}(Y)$, the ideal of smooth functions on $X$ vanishing at all points of $Y$. From the local coordinate form it is clear that $N^*Y$ gives the short exact sequence

$$0 \longrightarrow \mathcal{I}(Y)^2 \longrightarrow \mathcal{I}(Y) \stackrel{d_{\restriction Y}}{\longrightarrow} \mathcal{C}^\infty(Y; N^*Y) \longrightarrow 0, \tag{1.15.1}$$

and thus its sections are just elements of the quotient $\mathcal{I}(Y)/\mathcal{I}(Y)^2$, the elements of which are the first-order germs of functions vanishing on $Y$. Recall that $\mathcal{I}(Y)^q$ is, for any positive integer $q$, the ideal in $\mathcal{C}^\infty(X)$ consisting of the finite sums of $q$-fold products of functions vanishing on $Y$. The higher order quotients can also be identified in terms of the conormal bundle:

LEMMA 1.15.1. *For any $q \in \mathbb{N}$*

$$0 \longrightarrow \mathcal{I}(Y)^{q+1} \longrightarrow \mathcal{I}(Y)^q \longrightarrow \mathcal{C}^\infty(Y; S^q(N^*Y)) \longrightarrow 0 \tag{1.15.2}$$

*is a short exact sequence where $S^q(N^*Y)$ is the totally symmetric part of the $q$-fold tensor product of $N^*Y$.*

PROOF. This follows locally by the use of Taylor series and can be globalized using a partition of unity or the uniqueness of the local construction. □

We need to consider below more general structures of this type where some of the functions vanishing on $Y$ are given different weights. Thus in a decreasing sequence

$$\mathcal{I}(Y) = \mathcal{F}^{(1)}(Y) \supset \cdots \supset \mathcal{F}^{(q)}(Y) \subset \mathcal{F}^{(q+1)}(Y) \supset \ldots, \ q \in \mathbb{N} \tag{1.15.3}$$

the elements are to be interpreted as the spaces of functions on $X$ vanishing at least to order $q$ on $Y$. For this to be reasonable we impose various conditions.

Initially suppose that $Y$ is a submanifold of a manifold without boundary. We demand that all the spaces in (1.15.3) be $\mathcal{C}^\infty$ modules and all elements vanish on $Y$, in the usual sense:

$$\mathcal{C}^\infty(X) \cdot \mathcal{F}^{(q)}(X) = \mathcal{F}^{(q)}(X) \subset \mathcal{I}(Y),\ \forall\ q \in \mathbb{N}. \tag{1.15.4}$$

Since we want these to represent orders of vanishing we also require the multiplicative property

$$\mathcal{F}^{(q)}(Y) \cdot \mathcal{F}^{(q')}(Y) \subset \mathcal{F}^{(q+q')}(Y). \tag{1.15.5}$$

In view of this and (1.15.4) it is convenient to extend the definition down to $q = 0$ by defining

$$\mathcal{F}^{(0)}(Y) = \mathcal{C}^\infty(X). \tag{1.15.6}$$

We shall require a partial converse to (1.15.5) in that for any $q$ the sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{F}^{(q+1)}(Y) + \sum_{m_1+\cdots+m_q=q} & \mathcal{F}^{m_1}(Y) \cdot \cdots \cdot \mathcal{F}^{m_q}(Y) \\ & \longrightarrow \mathcal{F}^{(q)}(Y) \xrightarrow{d_{\restriction Y}} \mathcal{C}^\infty(Y; S_q) \longrightarrow 0 \end{aligned} \tag{1.15.7}$$

should be exact for a subbundle $S_q \subset N^*Y$. Furthermore we demand that for some integer $l$

$$\mathcal{I}(Y)^{q+1} \supset \mathcal{F}^{(lq+1)}(Y)\ \forall\ q. \tag{1.15.8}$$

The smallest integer $l$ for which (1.15.8) holds is the largest local homogeneity.

Definition 1.15.11. If $Y \subset X$ is a submanifold of a manifold without boundary a quasi-homogeneity structure at $Y$ is a sequence of ideals in $\mathcal{C}^\infty(X)$ as in (1.15.3) – (1.15.8).

The condition (1.15.8) means in particular that $S_q = \{0\}$ for $q > l$, and $l$ is the smallest integer with this property. Then (1.15.7) implies that the $\mathcal{F}^{(q)}(Y)$ are, for $q > l$, the spans of the products of the $\mathcal{F}^{(t)}(Y)$ for $t < l$ with appropriate multiplicities. Thus all the information in a quasi-homogeneity structure is contained in the first $l$ ideals.

The 'trivial' structure where

$$\mathcal{F}^{(q)}(Y) = \mathcal{I}(Y)^q \tag{1.15.9}$$

is also called the 'radial' quasi-homogeneity structure. The subbundles in (1.15.7) are decreasing with $q$ so give a bundle filtration of $N^*Y$ :

$$N^*Y = S_1 \supset S_2 \supset \cdots \supset S_l = \{0\}. \tag{1.15.10}$$

Thus the existence of non-radial quasi-homogeneous structure is a non-trivial topological restriction on the manifold $Y$.

Any quasi-homogeneous structure is given locally by coordinate functions.

Proposition 1.15.1. *If $\mathcal{F}^{(*)}(Y)$ is a quasi-homogeneous structure for the submanifold $Y \subset X$ (assumed connected) of a manifold without boundary then there is a multi-index $\kappa$ and at any point $p \in Y$ there are local coordinates $x_1, \ldots, x_n$ in terms of which*

$$\mathcal{F}^{(q)}(Y) \cap \mathcal{C}_c^\infty(\Omega) = \sum_{\kappa \cdot \alpha \geq q} x^\alpha \mathcal{C}_c^\infty(\Omega) \tag{1.15.11}$$

*where $\Omega$ is the coordinate neighbourhood.*

PROOF. Consider the decreasing sequence of integers $q(1), \dots, q(r)$ giving the values of $q$ for which $S_q \neq S_{q+1}$ and let $m(j)$ for $j = 1, \dots, r$ be the change in dimension, i.e. $m(j)$ is the dimension of $S_{q(j)}/S_{q(j+1)}$. Near a given point $p \in Y$ we can choose successive functions $x_i \in \mathcal{C}^\infty(X)$ with the property that

$$\begin{gathered} x_i \in \mathcal{F}^{q(j)} \text{ if } i \le M(j) = \sum_{p \ge j} m(j) = \dim S_{q(j)} \\ (dx_i)_{\restriction Y} \text{ span } S_{q(j)} \text{ near } p \text{ for } i \le M(j). \end{gathered} \tag{1.15.12}$$

These functions $x_i$ have independent differentials at $p$ so can be extended to a coordinate system. In terms of these coordinates (1.15.11) holds with

$$\kappa = (q(1), \dots, q(1), q(2), \dots, \dots, q(r), 0, \dots 0),$$

where each $q(j)$ is repeated $m(j)$ times. □

A quasi-homogeneous structure does involve more than a filtration (1.15.10) but any such filtration corresponds to a quasi-homogeneous structure, as is shown in §2.12

For an interior p-submanifold of a manifold with corners the definition of a quasi-homgeneity structure can be applied unchanged. However to extend this notion to boundary p-submanifolds we shall add extra conditions so that (1.15.11) holds with the coordinates being product coordinates. Consider the decomposition of the conormal bundle of $\mathrm{Fa}(Y)$ given by its representation (1.8.7) as a component of the intersection of some set of the boundary hypersurfaces:

$$N^* \operatorname{Fa}(Y) = \bigoplus_{H \in \operatorname{Hu}(B)} N^*_B H \tag{1.15.13}$$

where $\operatorname{Hu}(Y) = \operatorname{Hu}(\operatorname{Fa}(Y)) \subset M_1(X)$ consists of the boundary hypersurfaces containing $Y$. We shall demand that the bundles $S_q$ have a decomposition related to this, namely that there is an integer $\gamma(H) \in \mathbb{N}$ for each $H \in \operatorname{Hu}(Y)$ such that

$$\begin{gathered} S_q \cap N^*_Y \operatorname{Fa}(Y) = \bigoplus_{\gamma(H) \le q} N^*_Y H \ \forall\ q \\ \text{and } \rho_H \in \mathcal{F}^{(\gamma(H))} \ \forall\ H \in \operatorname{Hu}(Y). \end{gathered} \tag{1.15.14}$$

DEFINITION 1.15.12. If $Y \subset X$ is a p-submanifold of a manifold with corners then a quasi-homogeneity structure at $Y$ is a sequence of ideals in $\mathcal{C}^\infty(X)$ as in (1.15.3) – (1.15.8) such that (1.15.14) holds for some integers $\gamma(H) \in \mathbb{N}$ for each $H \in \operatorname{Hu}(Y)$.

Our convention that the first $k$ coordinates be local boundary defining functions means that the multiindex $\kappa$ in (1.15.11) may have to be reordered, but apart from that Proposition 1.15.1 continues to hold:

PROPOSITION 1.15.2. *If $\mathcal{F}^{(*)}(Y)$ is a quasi-homogeneous structure at a p-submanifold $Y \subset X$ of a manifold with corners then there is a multi-index $\kappa$ and at any point $p \in Y$ there are local coordinates $x_1, \dots, x_n$ in terms of which*

$$\mathcal{F}^{(q)}(Y) \cap \mathcal{C}^\infty_c(\Omega) = \sum_{\kappa(p)\cdot\alpha \ge q} x^\alpha \mathcal{C}^\infty_c(\Omega) \tag{1.15.15}$$

*where $\Omega$ is the coordinate neighbourhood and $\kappa(p)$ is a reordering of $\kappa$.*

PROOF. The proof of Proposition 1.15.1 can be followed closely, with the maximum number of boundary defining functions always chosen as local coordinates, using (1.15.14). □

As an example of a quasi-homogeneity structure of geometric origin consider the singular curve in the plane

$$C = \{(x, y) \in \mathbb{R}^2; y^3 = x^2\}. \tag{1.15.16}$$

This is a 'cusp' and is a stable curve in the sense of singularity theory (see [?]). Naturally it suggests the quasi-homogeneity structure in which $x$ is given weight 3 and $y$ is given weight 2. Thus with $Y = \{0\}$, $\mathcal{F}^{(q)}(Y)$ consists of those $\mathcal{C}^\infty$ functions on $\mathbb{R}^2$ which have all terms in their Taylor series at 0 of weight at least $q$. It is of interest to note that this definition is independent of the coordinates in which $C$ takes the standard form (1.15.16). Such quasi-homogeneity structures are used extensively if [**6**], [**7**].

EXERCISE 1.15.12. Recall an important result of Arnol'd [?] which in this particular case shows that given any real-valued smooth function $Y \in \mathcal{C}^\infty(\mathbb{R}^2)$ with $dY$ and $dx$ independent at 0 there is a choice of $X \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $X$ and $Y$ give local coordinates based at 0 with respect to which $C = \{Y^3 = X^2\}$. Using this show that the quasi-homogeneity structure at $\{0\}$ defined above is independent of the choice of coordinates in which (1.15.16) holds.

The notion of a quasi-homogeneity structure at a submanifold is clearly a refinement of the $\mathcal{C}^\infty$ structure of the underlying manifold. In Chapter 5 it is shown how a quasi-homogeneity structure defines a new manifold, in which $X$ is blown-up along $Y$.

## 1.16. Parabolic quasi-homogeneity

The simplest type of quasi-homogeneity structure after the 'radial' structure occurs when some of the functions defining the submanifold have parabolic homogeneity, but no higher. Since this case occurs quite often (see for example [**1**]) we discuss it a little more here. For a parabolic structure there are just two different bundles in (1.15.10), namely $S_1 = N^*Y$ and $S_2$, which we denote simply as $S$. Thus $S \subset N^*Y$ is a subbundle with the decomposition property (1.15.14) becoming in this case:

$$S \cap N^*_Y B = \bigoplus_{H \in P \subset \mathrm{Hu}(Y)} N^*_Y H. \tag{1.16.1}$$

Given such a bundle consider

$$\mathcal{F}^{(2)}(Y) = \{u \in \mathcal{I}(Y); (du)_{|Y} \subset \mathcal{C}^\infty(Y; S)\} + \mathcal{I}(Y)^2 \tag{1.16.2}$$

$$\mathcal{F}^{(j)} = \sum_{m_1 + \cdots + m_j \ge j} \mathcal{F}^{(m_l)}, \ j \ge 3 \tag{1.16.3}$$

where (1.16.3) is an inductive definition.

PROPOSITION 1.16.1. *The definition* (1.16.3) *and* (1.16.2) *fixes a* 1-1 *correspondence between the parabolic quasi-homogeneity structures on a p-submanifold $Y$ of a manifold with corners and the subbundles of the conormal bundle $S \subset N^*Y$ satisfying the decomposition condition* (1.16.1).

Proof. The existence of a parabolic quasi-homogeneity structure with a given bundle, $S$, as in (1.16.1) arising as $S_2$ in (1.15.7) follows from Proposition 1.15.2. It suffices therefore to show that the $\mathcal{F}^{(q)}(Y)$ are determined by $S$. Since $\mathcal{F}^{(1)}(Y) = \mathcal{I}(Y)$ and the $\mathcal{F}^{(q)}$ for $q \geq 3$ are determined by $\mathcal{F}^{(1)}(Y)$ and $\mathcal{F}^{(2)}(Y)$ it is only necessary to note the characterization

$$\mathcal{F}^{(2)}(Y) = \{u \in \mathcal{I}(Y); (du)_{|Y} \in \mathcal{C}^\infty(Y; S)\} \tag{1.16.4}$$

which follows from (1.15.7). $\square$