CHAPTER 6

Toeplitz operators[S.Toeplitz.operators]

Most of the distinctive properties of the Heisenberg (or extended Heisenberg) algebra are already expressed by its Hermite ideal

$$\mathcal{I}^0_{\mathrm{H},+}(X;E) \subset \Psi^0_{\mathrm{H}}(X;E). \tag{284}$$ [HHH2.551]

This is identifiable with the Hermite operators introduced by Boutet de Monvel and Guillemin ([**7**]). In particular the Szegő projector on the boundary of a strictly pseudoconvex domain is an element of the Hermite ideal associated with the induced contact structure. In this chapter we first describe the notion of a generalized Szegő projection, S, directly in terms of the non-commutative symbol map

$$\mathcal{I}^{-1}_{\mathrm{H},+}(X;E) \longrightarrow \mathcal{I}^0_{\mathrm{H},+}(X;E) \xrightarrow{\ \sigma\ } \Psi^0_{\mathrm{is}}(W;F). \tag{285}$$ [HHH2.552]

This leads to the Toeplitz algebras examined by Boutet de Monvel and Guillemin

$$\mathcal{T}_S(X;E) = \{A \in \mathcal{I}^0_{\mathrm{H},+}(X;E); SA = AS = A\} \subset \mathcal{I}^0_{\mathrm{H},+}(X;E). \tag{286}$$ [HHH2.553]

Generally we suppress the projection S from the notation and set $\mathcal{T}(X;E) = \mathcal{T}_S(X;E)$.

Elliptic Toeplitz operators are characterized by the existence of a parametrix in $\mathcal{T}_S(X;E)$, that is an element giving an inverse in the quotient

$$\mathcal{T}_S(X;E)/(\mathcal{T}_S(X;E) \cap \Psi^{-\infty}_{\mathrm{H}}(X;E)).$$

For an elliptic element $A \in \mathcal{T}_S(X;E)$ perturbative invertibility, i.e. the existence of $A' \in \mathcal{T}_S(X;E) \cap \Psi^{-\infty}_{\mathrm{H}}(X;E)$ such that $(A+A')^{-1} \in \mathcal{T}_S(X;E)$ is equivalent to the vanishing of the the index, $\mathrm{Ind}(A) \in \mathbb{Z}$ of A as a Fredholm operator acting on the range of S, $SL^2(X;E)$. We derive, and somewhat generalize, the index formula of Boutet de Monvel ([**6**]) for elliptic Toeplitz operators by reducing it to the index theorem for Dirac operators of Atiyah and Singer. This is to be contrasted with the original (and closely related) proof by Boutet de Monvel which reduces the Toeplitz index formula to the more sophisticated index formula, of Atiyah and Singer, for pseudodifferential operators. This is significant in that the pseudodifferential index formula can be deduced from the Toeplitz formula.

Our derivation of the index formula proceeds through the construction of a resolution corresponding to a generalized Szegő projection (following Boutet de Monvel) mimicking the $\overline{\partial}_b$ complex of a CR manifold. The index of an elliptic Toeplitz operator is expressed in terms of the spectral flow arising from a one-paramater family of such resolutions. Then we show that the curve of resolutions can be 'wrapped up' and deformed to a curve of Dirac operators. The spectral flow of such a family is the odd form of the Atiyah-Singer theorem, so can be computed by suspending to a Dirac operator in one additional variable.

The index theorem for Toeplitz operators is extended to Toeplitz operators on vector bundles and then further to 'higher' Toeplitz operators. This leads directly in Chapter 9 to the reduced index theorem for the Heisenberg calculus itself, that is to the computation of the index of elliptic operators of the form $\mathrm{Id} + A$ with A an element of the Hermite ideal. This in turn is the starting point for our general index theorem for Heisenberg operators.

1. Generalized Szegő projections[SS.Generalized.Szego.projections]

In section 4.7 it is shown that the selection of a compatible complex structure on the symplectic bundle $W = H' \otimes L^{\frac{1}{2}}$ of a contact manifold leads to the family of ground state projections $s \in \Psi^{-\infty}_{\mathrm{is}}(W)$. Using the compression construction of Boutet de Monvel we may then construct a generalized Szegő projection, $S \in \Psi^0_{\mathrm{H}}(X)$ with Heisenberg symbol, s. This projection defines, in the terminology of Boutet de Monvel and Guillemin [**7**], a Toeplitz structure on the contact manifold, or a quantization of the contact structure.

For our applications we consider more general projections in two regards. First we consider projections acting on sections of an auxilliary complex bundle $E \longrightarrow X$ and secondly we may replace s by the more general idempotent symbols s_k and $s_{(N)}$ defined as projections onto the eigenspaces of the family of harmonic oscillators. These symbols are defined in §§ 4.3 - 4.7.

Definition 3. A generalized Szegő projection at level k on sections of a complex vector bundle E over an oriented compact contact manifold X with a given positive almost complex structure is an element $S_k \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ satisfying

$$\sigma_{\mathrm{RW}}(S_k) = s_k \otimes \mathrm{Id}_E,\ S_k^2 = S_k,\ S_k^* = S_k,$$

where the adjoint is defined by some choice of positive smooth density on X. If s_k is replaced by $s_{(N)}$ then the projection is *to* level N. Let $\mathcal{S}_k(X;E)$ and $\mathcal{S}_{(N)}(X;E)$ be the sets of these projections for all admissible choices of J and $\mathcal{S}_k(X;E,\nu)$ and $\mathcal{S}_{(N)}(X;E,\nu)$ the set corresponding to a particular choice of smooth density ν.

If $k = 0$ then $s_0 = s$ and $S = S_0$ is a generalized Szegő projection in the sense of Boutet de Monvel and Guillemin, but generalized in that it acts on sections of the Hermitian vector bundle E.

[HHH2.249]

Proposition 9. *For any Hermitian vector bundle E over a compact contact manifold X and each integer k there is a generalized Szegő projection at level k, that is, a projection $S_k \in \mathcal{I}^0_{H,+}(X;E)$ which is self-adjoint for a given smooth density on X and has Heisenberg symbol $s_k \otimes \mathrm{Id}_E$.*

Proof. The symbolic and completeness properties of the algebra $\mathcal{I}^0_{\mathrm{H},+}(X;E)$ allow S_k to be readily constructed. The surjectivity of the symbol map shows the existence of S'_k such that $\sigma_{\mathrm{H}}(S'_k) = s_k \otimes \mathrm{Id}_E$, $S'_k \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$. Taking the adjoint with respect to a Hermitian structure on E and positive smooth density on X, $\sigma^0_{\mathrm{H}}((S'_k)^*) = s_k \otimes \mathrm{Id}_E$ since $s_k^* = s_k$ in the symbol algebra. Replacing S'_k by $\frac{1}{2}(S'_k + (S'_k)^*)$ gives a self-adjoint element with the same Heisenberg symbol.

Since s_k is a projection, the spectrum of S'_k as a bounded operator on $L^2(X;E)$ is discrete, and of finite multiplicity, ouside 0 and 1. The resolvent family $(S'_k - \tau)^{-1}$ is meromorphic with values in $\Psi^0_{\mathrm{H}}(X;E)$ and $(S'_k - \tau)^{-1} + \tau^{-1} \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$, for $\tau \in \mathbb{C} \setminus \{0,1\}$. The self-adjointness of S means that the poles of this operator lie on the real axis. Choose $0 < T < 1$ such that $1 \pm T \notin \operatorname{Spec}(S'_k)$ and define

$$S_k = -\frac{1}{2\pi i}\oint_{|1-\tau|=T} \frac{d\tau}{(S'_k - \tau)}. \tag{287}$$ [HHH2.271]

Since $\mathcal{I}^0_{\mathrm{H},+}(X;E)$ is closed under the functional calculus, $S_k \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$. It follows directly from (287) and contour manipulation that $S_k^2 = S_k = S_k^*$ and $\sigma^0_{\mathrm{H}}(S_k) = s_k \otimes \mathrm{Id}_E$. □ More detail here.

Thus we have shown that $\mathcal{S}_k(X;E) \neq \emptyset$ as is $\tilde{\mathcal{S}}_k(X;E,\nu) \subset \mathcal{S}_k(X;E)$ the subset consisting of the elements self-adjoint with respect to a given smooth density ν and Hermitian structure on E.

Similarly there exist self-adjoint projections with Heisenberg symbol $s_{(N)}$; if we let

$$S'_{(N)} = \sum_{k\le N} S_k \text{ so that } \sigma(S'_{(N)}) = \sum_{k=0}^{N} s_j \otimes \mathrm{Id}_E$$ [HHH2.261]

then $(S'_{(N)})^2 - S'_{(N)} \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$. We can apply the argument given in the proof of Proposition 9 to obtain a self adjoint projection, $S_{(N)} \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ with $S_{(N)} - S'_{(N)} \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$. These were defined above as the generalized Szegő projections *to level* N; this set is denoted $\mathcal{S}_{(N)}(X;E)$.

2. Toeplitz algebras[SS.Toeplitz.algebras]

Toeplitz operators on a compact contact manifold were defined by Boutet de Monvel and Guillemin in [**7**], corresponding to a choice of generalized Szegő projection S, as the operators of the form SAS where $A \in \Psi^*(X)$. These can be thought of either as acting on $\mathcal{C}^\infty(X)$ or on the range of S.

More generally if $S \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ is a generalized Szegő projection acting on sections of E and $A \in \Psi^0(X;E)$ then we may call SAS a matrix Toeplitz operator. For our purposes it is better to use a somewhat broader definition, corresponding to our broader class of Szegő projections.

[HHH.40.0.2]

Definition 4. Let $S_{(N)} \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ be a generalized Szegő projection to level N on the complex vector bundle E. The space of Toeplitz operators of order 0, at level N, associated to $S_{(N)}$ is

$$\mathcal{T}_{(N)}(X;E) = \left\{ S_{(N)} A S_{(N)}; A \in \Psi^{0,0}_{\mathrm{eH}}(X;E) \right\}. \tag{288}$$ [HHH2.252]

It follows from the definition and the fact that $\mathcal{I}^0_{\mathrm{H},+}(X;E) \subset \Psi^{0,0}_{\mathrm{eH}}(X;E)$ is an ideal that the Toeplitz operators to level N form an algebra.

Although we allow the operator A to be an extended Heisenberg operator of double order 0 this is somewhat illusory. Indeed, if $A \subset \Psi^{0,0}_{\mathrm{eH}}(X;E)$ it follows that

$$S_{(N)} A S_{(N)} = S_{(N)} A' S_{(N)} \text{ with } A' = S_{(N)} A S_{(N)} \in \mathcal{I}^0_{\mathrm{H},+}(X;E) \subset \Psi^0_{\mathrm{H}}(X;E). \tag{289}$$ [HHH2.253]

Thus restricting the Toeplitz multplier, A, to be a a Heisenberg operator or even to an element of the Hermite ideal, leads to the same algebra of operators. The

Toeplitz algebra can be characterized as

$$\mathcal{T}_{(N)}(X;E) = \left\{ A \in \Psi^{0,0}_{\mathrm{eH}}(X;E); S_{(N)}A = AS_{(N)} = A \right\}.$$

For 'true' (generalized) Toeplitz operators, meaning at level 0, we may use standard pseudodifferential operators instead.

[HHH.40.0.3]

Proposition 10. *Let $S \in \mathcal{I}^0_{H,+}(X;E)$ be a generalized Szegő projection (at level 0) then for any $A \in \Psi^{0,0}_{eH}(X;E)$ there exists an operator $A' \in \Psi^0(X;E) + \Psi^{-\frac{1}{2}}(X;E)$ such that $SAS = SA'S$.*

Proof. The Heisenberg symbol of S is the projection

$$s \otimes \mathrm{Id}_E \in \Psi^{-\infty}_{\mathrm{is}}(X; W \otimes \hom E).$$

The Heisenberg symbol of SAS is therefore sas where $a \in \mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}^*X; \hom E)$ is the Heisenberg symbol of A. As s is a rank 1 idempotent symbol it follows that $sas = s \otimes a'$ where $a' \in \mathcal{C}^\infty(X; \hom E)$. Choosing a classical pseudodifferential operator, $A'_0 \in \Psi^0(X;E)$ with $\sigma_0(A'_0) = a'$ ensures that $SA'_0S \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ has Heisenberg symbol $sa's$. Thus

$$S(A - A'_0)A = SAS - SA'_0S \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E).$$

Considering the composition on the left in the extended Heisenberg calculus it is clear that it is only necessary that $\sigma_0(A'_0) = a'$ along the positive contact direction for this identity to hold. Since $SB_1S = S(A - A'_0)S$ if $B_1 = S(A - A'_0)S \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$ we may proceed inductively, assuming that $B_k \in \mathcal{I}^{-k}_{\mathrm{H},+}(X;E)$ and constructing $A'_k \in \Psi^{-\frac{1}{2}k}(X;E)$ so that $SB_kS - SA'_kS \in \mathcal{I}^{-k-1}_{\mathrm{H},+}(X;E)$. This is the same symbolic argument, as the Heisenberg symbol of SB_kS is $s \otimes b'_k \otimes \alpha^{\frac{1}{2}k}$ where $b_k \in \mathcal{C}^\infty(X; \hom E)$ and α is a non-vanishing section of L'. If $A'_k \in \Psi^{-\frac{1}{2}k}(X;E)$ is chosen to have symbol equal to $b'_k\alpha^{\frac{1}{2}k}$ on $L \subset T^*X$, which is always possible, then we get the inductive assumption. Taking the asymptotic sum of these operators A'_k gives an element $\tilde{A}' \in \Psi^0(X;E) + \Psi^{-\frac{1}{2}}(X;E)$ such that $A'' = SAS - S\tilde{A}'S \in \Psi^{-\infty}(X;E)$. Then $A' = \tilde{A}' + A''$ satisfies $SAS = SA'S$. □

Thus, up to a compact perturbation, every matrix Toeplitz operator at level 0 is defined by a classical multiplier.

[HHH2.282]

Corollary 2. *If $A \in \Psi^{0,0}_{eH}(X;E)$ then the Toeplitz operator at level 0, SAS, is of the form $S(a + B)S$, with $a \in \mathcal{C}^\infty(X; \hom E)$ uniquely determined and $B \in \mathcal{I}^{-1}_{H,+}(X;E)$.*

3. Relative index cocycle

The choice of a generalized Szegő projection on a contact manifold is far from unique. If we consider only *formal* projections of this type

$$\mathcal{FS}_{(N)}(X;E) = \{S'_{(N)} \in \mathcal{I}^0_{\mathrm{H},+}(X;E); \sigma_{\mathrm{H}}(S') = s_{(N)}\,, \\ (S')^2 - S' \in \mathcal{I}^{-\infty}_{\mathrm{H},+}(X;E)\}$$

then the arbitrariness is not very significant.

LEMMA 33. *For each N the space of formal Szegő projections to order N is path connnected and simply connected.*

REMARK 3. In fact these spaces are contractible. As we do not need such a strong result for our applications we prove only the simpler statement.

PROOF. Let S_0, $S_1 \in \mathcal{FS}_{(N)}(X;E)$ have principal symbols s_0, s_1 respectively. The principal symbol is determined by a choice of almost complex structure on H and the space of these is contractible. We can find a smooth path s_t of principal symbols of level N Szegő projections acting on sections of E joining s_0 to s_1. Let $S''_t \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ be operators with these principal symbols. It is immediate that $(S''_t)^2 - S''_t \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$ for $t \in [0,1]$. Let $C_i = S_i - S''_i \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$ for $i = 0,1$. Then we define a new family setting

$$S'_t = S''_t + (1-t)C_0 + tC_1.$$

For this family we have

$$(S'_t)^2 - S'_t = 0 \text{ for } t = 0,1 \text{ and } (S'_t)^2 - S'_t \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E). \tag{290}$$ [HHH.11.6.1998.1]

We now proceed by iteration to replace S'_t by elements of $\mathcal{FS}_{(N)}(X;E)$. Let $S_{t0} = S'_t$ and $C_{t0} = (S^2_{t0} - S_{t0})(\mathrm{Id} - 2S_{t0}) \in \mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$ then we set

$$S_{t1} = S_{t0} + C_{t0}$$

A simple calculation shows that

$$S^2_{t1} - S_{t1} = C^2_{t0} - 4(S^2_{t0} - S_{t0})^2 \in \mathcal{I}^{-2}_{\mathrm{H},+}(X;E).$$

Observe that C_{t0} is a smooth function of t with values in $\mathcal{I}^{-1}_{\mathrm{H},+}(X;E)$ and $C_{00} = C_{01} = 0$. Proceeding by induction we may assume that we have constructed $S_{tj} \in \mathcal{I}^0_{\mathrm{H},+}(X;E)$ such that

$$S_{ij} = S_i \text{ for } i = 0,1 \text{ and } S^2_{tj} - S_{tj} \in \mathcal{I}^{-2^j}_{\mathrm{H},+}(X;E).$$

Let $C_{tj} = (S^2_{tj} - S_{tj})(\mathrm{Id} - 2S_{tj})$ and $S_{t(j+1)} = S_{tj} + C_{tj}$. A simple calculation shows that

$$S_{i(j+1)} = S_i \text{ for } i = 0,1 \text{ and}$$
$$S^2_{t(j+1)} - S_{t(j+1)} = C^2_{tj} - 4(S^2_{tj} - S_{tj})^2 \in \mathcal{I}^{-2^{j+1}}_{\mathrm{H},+}(X;E).$$

The families $C_{tj}, j = 0,1,\ldots$ satisfy uniform bounds in the symbolic semi-norms, so we can asymptotically sum the series to obtain a smooth family

$$\tilde{S}_t \sim S_{t0} + \sum_{j=0}^{\infty} C_{tj}.$$

By the inductive hypothesis the operators $E_i = \tilde{S}_i - S_i, i = 0,1$ are in $\Psi^{-\infty}_{\mathrm{H}}(X;E)$. We obtain the desired homotopy of formal projections by setting

$$S_t = \tilde{S}_t + (1-t)E_0 + tE_1.$$

To show that $\pi_1 = 0$ it suffices to show that if S^1_t and S^2_t are two smooth curves in $\mathcal{FS}_{(N)}(X;E)$ with $S^1_0 = S^2_0$, $S^1_1 = S^2_1$ then there is a two parameter family, S^r_t such that

$$S^r_0 = S^1_0, \quad S^r_1 = S^1_1 \text{ for all } r \text{ and } S^r_t\big|_{r=1} = S^1_t, \quad S^r_t\big|_{r=2} = S^2_t.$$

The above algorithm is easily adapted to show this. Indeed, this algorithm can be adapted to show that $\pi_n = 0$ for all n. □

In particular any two generalized Szegő projections are connected by a curve of formal generalized Szegő projections, moreover any two such paths are isotopic by a family of paths with fixed endpoints.

LEMMA 34. *If S and $\tilde{S}$ are generalized Szegő projections (at level 0), then $\tilde{S}$ is Fredholm as an operator from $SL^2(X;E)$ to $\tilde{S}L^2(X;E)$.*

PROOF. It suffices to show that for any two generalized Szegő projections $S\tilde{S}S$ is Fredholm on $SL^2(X;E)$. Indeed this shows directly that the null space of $\tilde{S}S$ is finite dimensional and applied to $\tilde{S}S\tilde{S}$ it also shows that the range of $\tilde{S}S$ is closed and has finite codimension.

To see that $S\tilde{S}S$ is Fredholm on $SL^2(X;E)$ it suffices to show that $A = (\mathrm{Id} - S) + S\tilde{S}S$ is Fredholm on $L^2(X;E)$. The Fredholm criterion on $A \in \Psi^0_{\mathrm{H}}(X;E)$ is that $\sigma^0_{\mathrm{H}}(A) = \mathrm{Id} - s + s\tilde{s}s$ be invertible at each point. Since s is a projection of rank 1 this just means that $s\tilde{s}s \neq 0$, which is shown above. □

Thus we may define the relative index of two generalized Szegő projections by

$$\text{R-Ind}(S,\tilde{S}) = \mathrm{Ind}(\tilde{S}S : SL^2(X;E) \longrightarrow \tilde{S}L^2(X;E))\,.$$

PROPOSITION 11. *The relative index satisfies the cocycle condition*

$$\text{R-Ind}(S,\hat{S}) + \text{R-Ind}(\hat{S},\tilde{S}) = \text{R-Ind}(S,\tilde{S}),\quad \text{[HHH2.273]} \tag{291}$$

it is surjective as a map

$$\text{R-Ind} : \mathcal{S}_0(X;E) \times \mathcal{S}_0(X,E) \longrightarrow \mathbb{Z} \quad \text{[HHH2.274]} \tag{292}$$

and it vanishes if and only if S and $\tilde{S}$ lie in the same component of $\mathcal{S}_0(X;E)$.

PROOF. The additivity of the index shows that the left side of (291) is equal to the index of $\tilde{S}\hat{S}S$ as an operator from $SL^2(X;E)$ to $\tilde{S}L^2(X;E)$. The preceding lemma shows that $\hat{S}$ and S are connected by a curve S_t, $S_0 = S$, $S_1 = \hat{S}$, in $\mathcal{F}\mathcal{S}_0(X)$. The composite $\tilde{S}S_tS$ remains Fredholm for all $t \in [0,1]$ since S_t always differs from some element of $\mathcal{S}_0(X;E)$ by a smoothing operator. Thus the index of $\tilde{S}S_tS$ is independent of t, hence equal to the index of $\tilde{S}S_0S = \tilde{S}S$, i.e. is equal to the right side of (291).

The surjectivity of the map (292) follows from the fact that both S and $\mathrm{Id} - S$ are infinite rank and preserve $\mathcal{C}^\infty(X;E)$. Thus the range of each contains the range of a smoothing projection of any finite rank. Increasing or decreasing S in this way gives another generalized Szegő projection with relative index any preassigned integer.

If S is a fixed Szegő projection then there exists $\epsilon > 0$, depending only on S, such that if $\|\tilde{S} - S\| < \epsilon$ then $S\tilde{S}S$ is invertible on $SL^2(X;E)$. Indeed this follows from the invertibility of

$$\mathrm{Id} - S + S\tilde{S}S = \mathrm{Id} + S(\tilde{S} - S)S\,.$$

This implies that $\text{R-Ind}(S,\tilde{S}) = 0$, so by the cocycle condition it is constant on components in either variable.

It remains to show that $\text{R-Ind}(S,\tilde{S}) = 0$ implies that S and $\tilde{S}$ lie in the same component of $\mathcal{S}_0(X;E)$. To do this we use the following.

[HHH2.554]

LEMMA 35. *If* $[0,1] \ni t \longmapsto S'_t \in \mathcal{FS}_0(X;E)$ *is continuous, i.e. is a curve in* $\mathcal{FS}_0(X;E)$*, then there is a curve* S_t *in* $\mathcal{S}_0(X;E)$ *such that* $S'_t - S_t \in \mathcal{I}^{-\infty}_{H,+}(X;E)$*.*

PROOF. The contour integral (287) constructs a projection $S_{\bar t}$ for each $\bar t$, and this clearly extends smoothly in t to an open interval around $\bar t$. Moreover, different choices of contour give projections differing by finite rank smoothing operators. Thus $[0,1]$ may be covered by a finite number of open intervals on which such families of projections exist, with finite rank differences on intersections. Let M be the minimum number of intervals for which this is possible. Since finite rank subprojections on closed subintervals can always be extended to any open interval on which S_t is defined it follows that were $M > 1$ it could always be reduced. Thus $M = 1$ and the family S_t always exists globally. □

Now, suppose $\text{R-Ind}(S,\tilde S) = 0$. There is always a curve $S'_t \in \mathcal{FS}_0(X;E)$, $t \in [0,1]$, joining S and $\tilde S$. Using Lemma 35 we may lift S'_t to a curve of projections S_t, on which $\text{R-Ind}(S,S_t)$ is constant. Although S_0 is not in general equal to S it does follow that $S - S_0$ and $\tilde S - S_1 \in \mathcal{I}^{-\infty}_{\mathrm{H},+}(X;E)$. Perturbing S_0 we may assume that $S - S_0$ is of finite rank. Modifying S_t further by a curve of finite rank projections, either within the range of S_t or of $\mathrm{Id} - S_t$ as necessary, we can arrange that $\tilde S - S_1$ has finite rank and $\text{R-Ind}(\tilde S, S_1) = 0$. As

$$\text{R-Ind}(S,S_t) = \text{R-Ind}(S,\tilde S) + \text{R-Ind}(\tilde S, S_t)$$

and the right hand side vanishes for all $t \in [0,1]$ we conclude that $\text{R-Ind}(S,S_0) = 0$.

Thus we are reduced to the case that S and S_0 differ by a finite rank operator. Then $G_- = SL^2(X;E) \cap S'L^2(X;E)$ is a closed subspace of finite codimension in both the range of S and of S'. Similarly the span $G_+ = SL^2(X;E) + S'L^2(X;E)$ contains both ranges as subspaces of finite codimension. It follows that the relative index of S and S' is the difference of the ranks of the projections induced by S' and S on the finite dimensional space $G_+ \ominus G_- \subset \mathcal{C}^\infty(X;E)$. Thus the vanishing of the relative index implies that these two projections on $G_+ \ominus G_-$ are homotopic through (self-adjoint) smoothing projections. This completes the proof that $\tilde S$ and S_1 lie in the same component of $\mathcal{S}_0(X;E)$. □

4. Toeplitz index formula[SS.Toeplitz.index.formula]

If $A \in \Psi^0(X;E)$ is a pseudodifferential operator acting on sections a complex vector bundle over a compact oriented, contact manifold X and S is a generalized Szegő projector on sections of E then the associated Toeplitz operator, SAS, is Fredholm on $\operatorname{ran} S$ if and only if $a = \sigma(A) \in \mathcal{C}^\infty(\mathbb{S}^*X; \hom(E))$ is invertible on the positive contact direction. Boutet de Monvel derived a formula for the index, in terms of the symbol. We rederive his formula here, by a slightly different method. The characteristic classes in this formula are discussed in Chapter 8 below.

[HHH2.319]

THEOREM 5. *[L. Boutet de Monvel* [**6**]*] The index of a Fredholm Toeplitz operator is*

$$\operatorname{Ind}(SAS) = \int_X \chi_+(X)\,\mathrm{Ch}^{\mathrm{odd}}(a) \tag{293}$$ [HHH2.320]

where $\chi_+(X)$ is the positive 'contact' characteristic class of X, identified as the Todd class of the bundle H with a positive almost complex structure, $H^{1,0}$,

$$\chi_+(X) = \mathrm{Td}(H^{1,0}) \in H^{\mathrm{even}}(X)^{[HHH2.321]} \tag{294}$$

and $\mathrm{Ch}^{\mathrm{odd}}(a) \in H^{\mathrm{odd}}(X)$ is the odd Chern character of the bundle isomorphism defined by $a = \sigma(A)_{L^+}$.

Proof. The proof of this result by reduction to the Atiyah-Singer index theorem for Dirac operators is contained in the next few sections. The steps are as follows.

1. The index of SAS is the index of SaS where $a = \sigma(A)_{L^+}$ is the bundle isomorphism as described in Corollary 2.
2. The index of SaS is the relative index R-Ind(S, S_a), where $S_a = a^{-1}Sa$ is also a generalized Szegő projection on sections of E.
3. Any generalized Szegő projection has a '$\overline{\eth}_b$-like' resolution and the relative index R-Ind(S, S_a) is equal to the spectral flow of a curve of operators, (308), constructed from these resolutions.
4. By a deformation argument, this spectral flow is in turn shown to be equal to the spectral flow of a curve of Dirac operators, between one such, associated to a positive almost complex structure on H, and its a-conjugate.
5. By suspension, the spectral flow of this curve of Dirac operators is equal to the index of the Spin-$\mathbb{C}$ Dirac operator associated to the induced almost complex structure on $\pi : X \times \mathbb{S}^1 \to X$ and twisted by the bundle defined by suspension of a.

These arguments finally lead to the equality

$$\mathrm{Ind}(SAS) = \mathrm{Ind}(\eth_{+,a})^{[HHH2.322]} \tag{295}$$

where $\eth_{+,a}$ is the positive Spin-$\mathbb{C}$ Dirac operator on $Z = X \times \mathbb{S}^1$

$$\eth_{+,a} : \mathcal{C}^\infty(Z; \Lambda^{0,\mathrm{even}} \otimes E_a) \longrightarrow \mathcal{C}^\infty(Z; \Lambda^{0,\mathrm{odd}} \otimes E_a)^{[HHH2.323]} \tag{296}$$

with E_a is the bundle over Z obtained by lifting E to $X \times [0, 2\pi]$ and using a to glue the ends.

Let $T^{1,0}Z$ denote the holomorphic tangent space for the induced almost complex structure. The index formula for a Spin-$\mathbb{C}$ Dirac operator twisted by a bundle F given by Atiyah and Singer is

$$\mathrm{Ind}(\eth_{+,F}) = \int_Z \mathrm{Td}(T^{1,0}Z)\,\mathrm{Ch}^{\mathrm{even}}(F).^{[HHH2.324]} \tag{297}$$

Here $\mathrm{Ch}^{\mathrm{even}}(F) \in H^{\mathrm{even}}(Z)$ is the standard 'even' Chern character of the bundle F and $\mathrm{Td}(T^{1,0}Z) \in H^{\mathrm{even}}(Z)$ is the Todd class of the tangent bundle with its almost complex structure. This is a generalization of the Riemann-Roch-Hirzebruch theorem, see [**3**]. The Todd class is multiplicative and since, as a complex bundle $T^{1,0}Z = H^{1,0} \oplus \mathbb{C}$ is the pullback of $H^{1,0}$ plus a trivial factor, we see that

$$\mathrm{Td}(T^{1,0}Z) = \pi^* \,\mathrm{Td}(H^{1,0}),\ \mathrm{Td}(H^{1,0}) \in H^{\mathrm{even}}(X).^{[HHH2.325]} \tag{298}$$

The odd Chern character, $\mathrm{Ch}^{\mathrm{odd}}(a)$ is defined in (470); (293) then follows from (295) and (297). □

The first ste in the proof of Theorem 5 has already been discussed. The second step in the argument, reducing the index of a Toeplitz operator to the relative index of as pair of generalized Szegő projections is similar to a result in [**11**].

LEMMA 36. *If $a \in \mathcal{C}^\infty(X;\hom(E))$ is invertible and S is any generalized Szegő projection then*

$$\operatorname{Ind}(SaS) = \text{R-Ind}(S, S_a), \quad S_a = a^{-1}Sa\,. \tag{299}$$ [HHH2.272]

PROOF. Since a is an isomorphism, the index of SaS as an operator on $SL^2(X;E)$ is the same as the index of $a^{-1}SaS = S_aS$ as an operator from $SL^2(X;E)$ to $a^{-1}SL^2(X;E) = S_aL^2(X;E)$. Then (299) follows from the definition of the relative index. □

Suppose that S' is another generalized Szegő projector for E. The cocycle condition shows that

$$\text{R-Ind}(S'aS') = \text{R-Ind}(S', S) + \text{R-Ind}(S, S_a) + \text{R-Ind}(S_a, S'_a).$$

Since $\text{R-Ind}(S_a, S'_a) = \text{R-Ind}(S, S')$ by conjugation invariance of the index, it follows that the Toeplitz index of SaS is independent of the choice of $S \in \mathcal{S}_0(X;E)$. It is therefore determined by a alone.

5. Spectral flow

We now review and expand some well known results about spectral flows, showing that the relative index can be always identified as the spectral flow of a family of approximate reflections.

Let A_t, $t \in [0,1]$ be a norm continuous family of self-adjoint Fredholm operators on a Hilbert space $\mathcal{H}$. Then there exists $\epsilon > 0$ such that the spectrum of each A_t is discrete and of finite multiplicity in $(-\epsilon,\epsilon)$. If $I \subset (-\epsilon,\epsilon)$ is a subinterval with $\bar{I} \subset (-\epsilon,\epsilon)$, let $m_t(I)$ denote the sum of the multiplicities of the eigenvalues of A_t in I. For each $t \in [0,1]$ this is a finite measure on compact subintervals of $(-\epsilon,\epsilon)$. If I is open, $\bar{I} \subset (-\epsilon,\epsilon)$ and $m_{\bar{t}}(I) = m_{\bar{t}}(\bar{I})$ then $m_t(I)$ is locally constant near $\bar{t}$, since eigenvalues persist.

This 'collective continuity' of the eigenvalues implies that there is a partition of $[0,1]$,

$$0 = t_0 \le t_1 \le t_2 \le \cdots \le t_N = 1$$

and associated $\lambda_j \in (-\epsilon,\epsilon)$ such that $\lambda_j \notin \operatorname{Spec}(A_t)$ for $t \in [t_{j-1}, t_j]$, $j = 1, \dots, N$. Indeed, we may choose λ_1 and λ_N arbitrarily in $(-\epsilon,\epsilon) \setminus \operatorname{Spec}(A_0)$ and $(-\epsilon,\epsilon) \setminus \operatorname{Spec}(A_N)$ respectively and still find such a partition. We define the spectral flow at $\lambda \in (-\epsilon,\epsilon) \setminus (\operatorname{Spec}(A_0) \cup \operatorname{Spec}(A_1))$ to be

$$\operatorname{SF}(A_\bullet;\lambda) = \sum_{j=1}^{N-1} \operatorname{sgn}(\lambda_{j-1} - \lambda_j) m_{t_j}((\lambda_{j-1},\lambda_j))\,,\ \lambda = \lambda_0 = \lambda_N\,. \tag{300}$$ [HHH2.275]

That the spectral flow is independent of the partition follows from its constancy under subdivision and the local constancy of $m_t(I)$ mentioned above.

In general the spectral flow of the family does depend on λ. The difference of the spectral flow at two points $\lambda_1,\ \lambda_2 \in (-\epsilon,\epsilon)\setminus(\operatorname{Spec}(A_0) \cup \operatorname{Spec}(A_1))$ is easily computed to be

$$\mathrm{SF}(A_\bullet;\lambda_2)-\mathrm{SF}(A_\bullet;\lambda_1)=\operatorname{sgn}(\lambda_2-\lambda_1)[m_1(\lambda_1,\lambda_2)-m_0(\lambda_1,\lambda_2)].\tag{301}$$ [HHH2.555]

In particular if $[\lambda_1,\lambda_2]\cap(\mathrm{Spec}(A_0)\cup\mathrm{Spec}(A_1))=\emptyset$ then $\mathrm{SF}(A_\bullet,\lambda)$ is constant for $\lambda\in[\lambda_1,\lambda_2]$. Moreover if A_0 and A_1 are isospectral then it follows from (301) that $\mathrm{SF}(A_\bullet;\lambda)$ is independent of λ as it varies in any interval on which the family is Fredholm.

Let A_t be a norm continuous family of self adjoint operators such that for a fixed real number μ, the operators $A_t-\mu$ are Fredholm. As above this implies that there is a $\epsilon>0$ so that the spectrum of each A_t is dicrete with finite multiplicity in $[\mu-\epsilon,\mu+\epsilon]$. The discussion above applies without change to the family $A_t-\mu$. If $\lambda\in(\mu-\epsilon,\mu+\epsilon)$ is not in the spectrum of A_0 or A_1 then we define

$$\mathrm{SF}(A_\bullet;\lambda)=\mathrm{SF}(A_\bullet-\mu;\lambda-\mu).$$

[HHH2.305]

LEMMA 37. *If $\mathcal{H}$ is finite dimensional then*

$$\mathrm{SF}(A_\bullet;\lambda)=m_1(\lambda,\infty)-m_0(\lambda,\infty),\ \lambda\notin\mathrm{Spec}(A_0)\cup\mathrm{Spec}(A_1).\tag{302}$$ [HHH2.306]

PROOF. Since $m_t(\lambda_{j-1},\lambda_j)=\operatorname{sgn}(\lambda_{j-1}-\lambda_j)[m_t(\lambda_{j-1},\infty)-m_t(\lambda_j,\infty)]$ the sum in (300) telescopes in the finite-dimensional case. □

Another standard property of the spectral flow which follows from (300), is its homotopy invariance. That is, if $A_t(s)$ is a norm continuous family of self-adjoint operators for $(t,s)\in[0,1]\times[a,b]$ such that $A_t(s)-\lambda$ is always Fredholm and $A_0(s)-\lambda$ and $A_1(s)-\lambda$ are invertible for $s\in[a,b]$ then $\mathrm{SF}(A_\bullet(a),\lambda)=\mathrm{SF}(A_\bullet(b),\lambda)$. Using such homotopy invariance it follows that if B_t is a continuous family of positive (hence self-adjoint and invertible) operators then

$$\mathrm{SF}(A_\bullet,0)=\mathrm{SF}((BAB)_\bullet,0).\tag{303}$$ [HHH2.283]

[HHH2.307]

LEMMA 38. *If $\mathcal{H}$ is finite dimensional and $0\notin\mathrm{Spec}(A_0)\cup\mathrm{Spec}(A_1)$ then the spectral flow around 0, $\mathrm{SF}(A_\bullet;0)=0$ if and only if A_t is homotopic, through curves of self-adjoint matrices with the same end points to a curve of invertible matrices.*

PROOF. The homotopy invariance of the spectral flow shows that the existence of such a deformation implies that $\mathrm{SF}(\bullet;0)=0$. Conversely, supposing that the spectral flow vanishes we construct a curve from A_0 to A_1 through invertible self-adjoint matrices. The linear isotopy between this and the given curve then provides a deformation with the desired properties.

First, the linear curve from A_0 to $\pi_0^+-\pi_0^-$ where $\pi_0^\pm$ are the projections onto the spans of the positive and negative eigenspaces, consists of invertible self-adjoint matrices and similarly for A_1. By Lemma 37 the spectral flow is the difference of the ranks of π_0^+ and π_1^+. Thus we may find a unitary matrix α such that $\pi_1^\pm=\alpha^*\pi_0^\pm\alpha$. Since the unitary group is connected there is a curve α_t of unitary matrices with $\alpha_0=\mathrm{Id}$ and $\alpha_1=\alpha$. Then $\alpha_t^*(\pi_0^+-\pi_0^-)\alpha_t$ is a curve of invertible self-adjoint matrices connecting $\pi_0^+-\pi_0^-$ and $\pi_1^+-\pi_1^-$. Thus A_0 and A_1 lie in the same component of the invertible self-adjoint matrices. □

The relevance of the notion of spectral flow for the computation of the relative index can be seen from the following Proposition.

PROPOSITION 12. *If S, $\tilde{S}$ are two generalized Szegő projections on sections of E and $S'_t \in \mathcal{FS}_0(X;E)$ is a curve of formal projections from S_0 to S_1 with S'_t self-adjoint with respect to a continuous family of smooth measures ν_t then*

$$\text{R-Ind}(S,\tilde{S}) = \mathrm{SF}(S'_\bullet;\frac{1}{2}).^{[HHH2.276]} \tag{304}$$

Here we define the spectral flow by setting $a_t^2 = \nu_t/\nu_0$, so $S''_t = a_t S'_t a_t^{-1}$ is self-adjoint for the fixed measure ν_0 and we set

$$\mathrm{SF}(S'_\bullet;\lambda) \stackrel{\text{def}}{=} \mathrm{SF}(S''_\bullet;\lambda)\,.^{[HHH2.277]} \tag{305}$$

PROOF. If $S_t(0)$ and $S_t(1)$ are two curves of self-adjoint Fredholm operators with the same end points which are homotopic in the sense that there is a two-operator family $S_t(r)$ with $S_0(r)$ and $S_1(r)$ fixed, then

$$\mathrm{SF}(S_\bullet(0);\lambda) = \mathrm{SF}(S_\bullet(1),\lambda)\,,$$

since the spectral flow is clearly locally constant.

Along any curve of self-adjoint projections the spectral flow at $\frac{1}{2}$ vanishes. As shown above, we may deform $\tilde{S}$ through self adjoint projections to a projection, $\hat{S}$ which differs from S by a finite rank operator which is either a subprojection of S or of $\hat{S}$. As $\text{R-Ind}(\tilde{S},\hat{S}) = 0$ the cocycle formula shows that $\text{R-Ind}(S,\tilde{S}) = \text{R-Ind}(S,\hat{S})$. It is evident that the spectral flow at $\frac{1}{2}$ of the curve $S_t = S + t(\hat{S} - S)$ equals $\text{R-Ind}(S,\hat{S})$. ☐

REMARK 4. This result could be rephrased as

$$\text{R-Ind}(S,\tilde{S}) = \mathrm{SF}(\mathrm{Id} - 2S'_\bullet;0).$$

The operators $\mathrm{Id} - 2S_0, \mathrm{Id} - 2S_1$ are reflections with spectrum contained in $\{\pm 1\}$. In this form the result is well known in the context of Grassmanians defined by classical pseudodifferential operators, see [**4**].

For the proof of the index formula we do not directly employ (304). Instead we embed this family of formal projections into a family of Fredholm Heisenberg complexes. After considerable deformation of this family of complexes we arrive at a Fredholm family of classical pseudodifferential operators. The spectral flows of these families all compute the relative index $\text{R-Ind}(S,S_a)$. We compute the index using a suspension argument which in turn is based on the following deformation lemma.

[HHH2.303]

PROPOSITION 13. *Let $A \in \Psi^1(X;V)$ be an invertible elliptic, self-adjoint pseudodifferential operator acting on sections of a bundle V over a compact manifold X, suppose that $a \in \mathcal{C}^\infty(X;\hom V)$ is unitary and commutes with the symbol of A then the linear family $A_t = (1-t)A + ta^{-1}Aa$ has spectral flow 0 if and only if there is a self-adjoint family $A_t(s) \in \Psi^1(X,V)$, which is norm continuous in $(t,s) \in [0,1]^2$, as unbounded self-adjoint operators with domain $H^1(X;V)$, has constant symbol, $A_t(0) \equiv A_t$, constant endpoints $A_0(s) \equiv A_0$, $A_1(s) \equiv A_1$ and for which $A_t(1)$ is invertible for all $t \in [0,1]$.*

PROOF. Since pseudodifferential operators of order 0 are compact operators from H^1 to L^2, they are approximable, in these norms, by finite rank smoothing operators. Choose a finite rank smoothing approximation B' to $B = A_1 - A_0$ such that the linear homotopy from A_1 to $A_1' = A_0 + B'$ is a curve of invertible operators. Thus, for computing the spectral flow around 0, we may replace A_1 by A_1' (although it may not be conjugate to $A = A_0$). If π_T is the projection onto the spectrum of A in $[-T, T]$ then, as $T \to \infty$, $\pi_T B' \pi_T$ converges to B' as a smoothing operator. We may therefore assume that B' has its Schwartz kernel spanned by a finite linear combination of eigenvectors of A.

The spectral flow around 0 of the modified family $A_t' = A + tB'$ is, by (303), the same as that of the curve of bounded self-adjoint operators on $L^2(X; V)$,

$$P_t = |A|^{-\frac{1}{2}} A_t' |A|^{-\frac{1}{2}} = \Pi^+ - \Pi^- + S(t).$$

Here $\Pi^\pm$ are the projections onto the spans of the positive and negative eigenspaces of A and $S(t)$ is a smoothing operator with fixed finite rank, i.e. acting on some finite dimensional subspace independent of t. Thus $\mathrm{SF}(P_t, 0) = \mathrm{SF}(P_t'; 0)$ where $P_t' = \Pi P_t \Pi$ for a finite rank projection Π which commutes with $\Pi^\pm$ and has range containing the range of $S(t)$. Lemma 38 applies to this finite rank family giving a deformation $P_t'(s)$, acting on the range of Π, so that $P_t'(1)$ is invertible. It follows that $P_t(s) = (\Pi^+ - \Pi^-)(\mathrm{Id} - \Pi) + P_t'(s)\Pi$ is a similar deformation of P_t and then $A_t'(s) = |A|^{\frac{1}{2}} P_t(s) |A|^{\frac{1}{2}}$ is a deformation of A_t' with the desired properties. □

We use a slight extension of this result to simplify the index computation. Let X and X' be compact manifolds and $V \to X, W \to X'$ be vector bundles over each. Let $P : X \times X' \to X$ and $P' : X \times X' \to X'$ denote the canonical projections. We let $V \boxplus W$ denote the exterior sum, i.e. the Whitney sum $P^*(V) \oplus P'^*(W)$. If A is an operator acting on $\mathcal{C}^\infty(X; V)$ and A' an operator acting on $\mathcal{C}^\infty(X'; W)$ then $A \boxplus A'$ is the canonically defined operator acting on $\mathcal{C}^\infty(X \times X'; V \boxplus W)$. If A and A' are Fredholm then so is $A \boxplus A'$. However, even if both operators are elliptic the sum will generally not be a pseudodifferential operator. Nonetheless the following extension of Proposition 13 holds.

$[HHH2.308]$

PROPOSITION 14. *Let $A_t \in \Psi^1(X; V)$ and $A_t' \in \Psi^1(X'; W)$, $t \in [0, 1]$ be two curves of self-adjoint elliptic pseudodifferential operators with constant symbols and with invertible end points acting on sections of Hermitian bundles over compact manifolds and suppose that*

$$\mathrm{SF}(A_\bullet; 0) + \mathrm{SF}(A_\bullet'; 0) = 0.^{[HHH2.309]} \tag{306}$$

Then there is a smooth deformation $C_t(s) = A_t(s) \boxplus A_t'(s) + F_t(s)$ where $F_t(s) = G_t(s) + G_t(s)^*$ with $G_t(s)$ a smooth family of operators with kernels in $\mathcal{C}^\infty(X \times X'; \hom(V, W))$ and $A_t(s)$ and $A_t'(s)$ are deformations as in Proposition 13 except that the invertibility condition is replaced by the invertibility of the operator $A_t(1) + A_t'(1) + F_t(1)$.

PROOF. The linearity of the family is not used in the proof of Proposition 13. Thus as in the proof of that proposition, the curves of pseudodifferential operators can be reduced to finite rank perturbations of the positive and negative spectral projections. Then (306) shows that the direct sum family has vanishing spectral flow, so Lemma 38 again applies to give the finite rank smoothing deformation to an invertible family. □

6. Resolution

If H is a real vector space with a complex structure there is a natural 'Koszul' complex

$$\kappa \in \mathcal{C}^\infty(H'; \hom(\Lambda^{0,*}H'))$$

given by the linear map

$$\Lambda^{0,1}H' \ni \zeta \mapsto (\beta \mapsto \zeta \wedge \beta) \in \hom(\Lambda^{0,*}H')$$

which then can be interpreted as a function on H'. In case $H = T_pY$ where Y is a complex manifold then κ is the symbol of $\bar\partial$.

For a contact manifold with positive almost complex structure we take $H = L^\circ$ and also write $\Lambda^{0,k} = \Lambda^{0,k}X = \Lambda^{0,k}H'$. Then $W = H' \otimes L^{-1/2}$ is the natural symplectic bundle and we may reinterpret the Koszul complex as an element

$$\kappa_+ \in \mathcal{C}^\infty(W_+; \hom(\Lambda^{0,*}X) \otimes L^{-1/2}) .$$

Together with the unique compatible linear element κ_- on W_-, this gives a well-defined element of the symbol space for Heisenberg operators of order 1 which we denote by κ.

[HHH2.278]

Proposition 15. *On any compact contact manifold with positive almost complex structure and generalized Szegő projection S there is a complex $D \in \Psi^1_H(X; \Lambda^{0,*})$ with Heisenberg symbol κ such that*

$$\begin{aligned}(307)\quad & S\mathcal{C}^\infty(X) \xrightarrow{S} \mathcal{C}^\infty(X) \xrightarrow{D_0} \mathcal{C}^\infty(X; \Lambda^{0,1}) \longrightarrow \cdots \\ & \cdots \longrightarrow \mathcal{C}^\infty(X; \Lambda^{0,n-1}) \xrightarrow{D_{n-1}} \mathcal{C}^\infty(X; \Lambda^{0,n}) \xrightarrow{\tilde S} \tilde S\mathcal{C}^\infty(X; \Lambda^{0,n})\end{aligned}$$ [HHH2.279]

is acyclic for a negative generalized Szegő projection $\tilde S$ on $\Lambda^{0,n}$.

Remark 5. Recall that the 'negative' here refers to the switch of the orientation, that is $\tilde S \in \mathcal{I}^0_{\mathrm{H},-}(X; \Lambda^{0,n})$.

Proof. By the surjectivity of the Heisenberg symbol map we may choose a graded operator $\tilde D \in \Psi^1_{\mathrm{H}}(X; \Lambda^{0,n})$ such that $\sigma^1_{\mathrm{H}}(\tilde D) = \kappa$. In fact we may take $\tilde D$ to be a differential operator, in which case its symbol in the usual sense is κ interpreted as a linear function on T^*X.

The components of $\tilde D_0$ have symbols which are the annihilation operators in the symbol algebra. It follows that $\sigma(\tilde D_0 S) = 0$, so $\tilde D_0 S \in \Psi^0_{\mathrm{H}}(X; \Lambda^{0,1})$. Replacing $\tilde D_0$ by $\tilde D_0 \circ (\mathrm{Id} - S)$ does not change the symbol and ensures that $\tilde D_0 S \equiv 0$. Let θ be a choice of contact form and let T be a corresponding Reeb vector field, so in particular $\theta(T) = 1$. As a vector field the symbol of $-iT$ is T interpreted as a linear function on the cotangent bundle; it is strictly positive on the positive contact line. Set $M = -iSTS \in \Psi^2_{\mathrm{H}}(X)$. The operator $\tilde D_0^* \tilde D_0$ has symbol $\kappa_0^*\kappa_0$ which is a multiple of the harmonic oscillator on the positive side and invertible on the negative side of the equator. It follows that $M + \tilde D_0^* \tilde D_0$ has globally invertible Heisenberg symbol and hence is Fredholm. The intersection of the, finite dimensional, null space with the range of $\mathrm{Id} - S$ is the null space of $\tilde D_0$ which is therefore also finite dimensional. Furthermore it follows from the Fredholm property that the range of $\tilde D_0$ is closed. Adding a smoothing operator of the form $A_0(\mathrm{Id} - S)$ to $\tilde D_0$ we can arrange that $D_0 = (\tilde D + A_0)(\mathrm{Id} - S)$ is injective on $(\mathrm{Id} - S)\mathcal{C}^\infty(X)$ and it still has closed range.

Now, the self-adjoint projection onto the range of D_0 is an element $S_1 \in \Psi^0_{\mathrm{H}}(X;\Lambda^{0,1})$, note that $S_1 D_0 = D_0$. Furthermore, the symbolic properties of κ show that $\tilde{D}_1 \circ S_1 \in \Psi^1_{\mathrm{H}}(X;E)$. Proceeding as above replace $\tilde{D}_1$ by $D_1 = (\tilde{D}_1 + A_1)\circ(\mathrm{Id} - S_1)$ with A_1 a finite rank smoothing operator so chosen that it is injective on the range of $\mathrm{Id} - S_1$.

We may then proceed step by step in the construction of a complex of Heisenberg pseudodifferential operators without cohomology in dimensions 1 to $n-1$. At the last step the range of the modified operator D_n is the range of a projection $(\mathrm{Id} - S_n) \in \Psi^0_{\mathrm{H}}(X;\Lambda^{0,n})$. As S_n is the projection onto the nullspace of $D_n D_n^*$ it follows from symbolic considerations that S_n is a negative generalized Szegő projection for the bundle $\Lambda^{0,n}$ belonging to $\mathcal{I}^0_{\mathrm{H},-}(X;\Lambda^{0,n})$. □

In case X is a CR manifold of dimension $2n+1 \ge 5$ then D may be obtained from the $\bar{\partial}_b$ complex by finite rank modification to remove the cohomology outside dimensions 0 and n.

[HHH2.280]

Proposition 16. *Any two resolutions of a fixed Szegő projection S in the sense of* (307) *are homotopic through such resolutions.*

Proof. Let $D(0)$ and $D(1)$ be two such resolutions of S. Since $D(0)$ and $D(1)$ have the same Heisenberg symbols there is a homotopy $\tilde{D}(t) \in \Psi^0_{\mathrm{H}}(X;\Lambda^{0,*})$ with fixed symbols. This may be modified to a homotopy of complexes following the proof of Proposition 15. Thus $\tilde{D}_0(t)(\mathrm{Id} - S)$ may be modified by addition of a smoothing operator of finite rank $A(t)(\mathrm{Id} - S)$, with $A(0) = A(1) = 0$, so that $(\tilde{D}_0(t) + A(t))(\mathrm{Id} - S) = D_0(t)$ is injective on $(\mathrm{Id} - S)\mathcal{C}^\infty(X)$. It follows that the projection on the range $S_1(t)$ is a homotopy from $S_1(0)$ to $S_1(1)$, and so on. □

[HHH2.281]

Corollary 3. *A generalized Szegő projection for any bundle E has a resolution $D \in \Psi^1_H(X;\Lambda^{0,*} \otimes E)$ as in* (307) *with symbol $\kappa \otimes \mathrm{Id}_E$ and with $\tilde{S} \in \mathcal{I}^0_{H,-}(X;\Lambda^{0,n} \otimes E)$ a negative generalized Szegő projection; any pair of resolutions for homotopic projections are homotopic through such resolutions.*

Proof. The inclusion of the bundle E or a homotopy of projections does not materially change the constructions above. □

7. Deformation of resolutions$^{[SS.Deformation.of.resolutions]}$

If A_t, $t \in [0,1]$ is a curve of unbounded self-adjoint operators with a fixed domain $\mathcal{D} \subset \mathcal{H}$ such that $A_t - \lambda$ is Fredholm for all λ in an open interval $I \subset \mathbb{R}$ and $(A_0^2+1)^{-\frac14} A_t (A_0^2+1)^{-\frac14}$ is norm continuous then the spectral flow $\mathrm{SF}(A_\bullet, \lambda')$ is still well-defined for any $\lambda' \in I$ such that $A_0 - \lambda'$ and $A_1 - \lambda'$ are both invertible. Translating by λ' we may suppose that $\lambda' = 0$. Then $B = (A_0^2)^{\frac14}$ is such that $A_t = B A'_t B$ where A'_t is a bounded family of self-adjoint operators which are Fredholm on the Hilbert space $\mathcal{B} = \mathrm{Dom}(B)$ which is dense in $\mathcal{H}$. It is unchanged if another positive isomorphism $\tilde{B} : \mathcal{B} \longrightarrow \mathcal{H}$ is chosen in place of B. Furthermore $\mathrm{SF}(A_\bullet, \lambda')$ satisfies the same addition law ** as in the bounded case.

Let R be a resolution, in the sense described above, of a generalized Szegő projection S. Consider the operator in the Heisenberg calculus acting on sections of $\Lambda^{0,*} \otimes E$

$$G = S + R + R^* + (-1)^{n+1}\tilde{S},\ \dim X = 2n+1 \tag{308}$$ [HHH2.263]

Here $\tilde{S}$ is the generalized Szegő projection on sections of $\Lambda^{0,n} \otimes E$ which occurs in the resolution. Clearly G is a formally self-adjoint element of the Heisenberg calculus of order 1. It is an isomorphism on $\mathcal{C}^\infty(X; \Lambda^{0,*} \otimes E)$, essentially by the definition of a resolution.

It is not the case that G is elliptic as a Heisenberg operator, since S and $\tilde{S}$ are of order 0 while R is of order 1.

[HHH2.264]

Proposition 17. *The operator G in* (308) *is a self-adjoint isomorphism onto $L^2(X; \Lambda^{0,*} \otimes E)$ from the dense subspace of $L^2(X; \Lambda^{0,*} \otimes E)$*

$$\operatorname{Dom}(G) = SL^2(X;E) + \tilde{S}L^2(X; \Lambda^{0,n} \otimes E) + H^1_H(X; \Lambda^{0,*} \otimes E). \tag{309}$$ [HHH2.265]

If $a \in \mathcal{C}^\infty(X;E)$ is a unitary isomorphism and G_t is the linear homotopy between $G_0 = G$ and $G_1 = a^{-1}Ga$ then

$$\text{R-Ind}(S, S_a) = -\operatorname{SF}(G_\bullet, 0). \tag{310}$$ [HHH2.266]

Proof. The space $\operatorname{Dom}(G)$ in (309) can be written as the orthogonal direct sum of Hilbert spaces

$$\operatorname{Dom}(G) = SL^2(X;E) + (\operatorname{Id} - S)(\operatorname{Id} - \tilde{S})H^1_{\mathrm{H}}(X; \Lambda^{0,*} \otimes E) + \tilde{S}L^2(X; \Lambda^{0,n} \otimes E). \tag{311}$$ [HHH2.267]

It follows that G is self-adjoint since $R + R^*$ is an unbounded self-adjoint operator with domain the middle summand in (311).

The generalized Szegő projection S_a differs from S by a compact operator, so it follows that the linear homotopy

$$G_t = (1-t)G + ta^{-1}Ga \tag{312}$$ [HHH2.268]

is a curve of self-adjoint operators with the fixed domain $\operatorname{Dom}(G)$. Moreover, G_0 and G_1 are both invertible so the spectral flow $\operatorname{SF}(G_\bullet, 0)$ does exist. Note that $\operatorname{Dom}(G)$ is determined by the contact structure and the principal symbols of S and $\tilde{S}$.

We compute the spectral flow of the family G_t by replacing it with a simpler family which is homotopic through a norm continuous family of operators each of which is defined and Fredholm on $\operatorname{Dom}(G)$. Briefly, we first deform S_a, through generalized Szegő projections, all with the same principal symbol to a projection, S' which is either a sub or super-projection of S. Applying Corollary 3 we obtain a homotopy of resolutions and therefore operators G'_t, each with domain $\operatorname{Dom}(G)$, corresponding to this homotopy of projections. By definition $\operatorname{SF}(G'_\bullet; 0) = 0$. Having replaced S_a by S' we now construct a resolution, R' for S' which is a finite rank perturbation of R itself. That the spectral flow of the linear family joining G to G' is $\text{R-Ind}(S, S') = \text{R-Ind}(S, S_a)$ is immediate from the construction of G'. The statement of the proposition then follows from the homotopy invariance of the spectral flow.

CLE: We have not really proved or even stated a theorem to cover this exact situation.

For the moment suppose that the underlying compact contact manifold has dimension $2n+1$ with n even. For simplicity we assume that $k = \text{R-Ind}(S, S_a) \geq 0$, otherwise we simply reverse the rôles of S and S_a. We deform S_a through generalized Szegő projections until it is of the form $S' = S - P$ where P is a self-adjoint smoothing projection of finite rank onto a subspace of the range of S, that is $PS = SP = P$. All the projections in this family have the same principal symbol. If G'_t is a family of operators corresponding, as described above to this family of projections then, by definition $\mathrm{SF}(G'_\bullet;0) = 0$. We now replace S_a by S'.

Consider how to modify the resolution R of S to obtain a resolution of S'. For each form dimension $i = 1,2,\dots n$ select a finite dimensional subspace of $\mathcal{C}^\infty(X;\Lambda^{0,i}\otimes E)$ with orthogonal projection P_i. For i odd these should be chosen to be subspaces of the orthocomplement of the range of D_{i-1} with dimension equal to the rank of $P = P_0$. Without loss of generality we can assume that these ranges lie in $\mathcal{C}^\infty$. For even $i \ge 2$ the range of P_i is defined to be the range of $D_{i-1}P_{i-1}$. As the range of P_{i-1} is contained in the orthocomplement of the nullspace of D_i we see that the rank of P_i equals the rank of P_{i-1}. For each even i choose an isomorphism, A_i, of the ranges of P_i and P_{i+1}. These are evidently smoothing operators of finite rank. The operators

$$D'_i = D_i + P_{i+1}A_iP_i,\ i \text{ even },\ D'_i = (\mathrm{Id}-P_{i+1})D_i(\mathrm{Id}-P_i),\ i \text{ odd} \quad {}^{[HHH2.270]} \tag{313}$$

then give a resolution of $S-P$, $P=P_0$. The dual Szegő projector is $\tilde{S}' = \tilde{S} + P_n$ (recall that we have assumed n to be even.)

For these resolutions of S and S', the two operators G and G' differ by an operator of finite rank. Let Q be the projection onto the sum of the ranges of the P_i. Then Q commutes with both resolutions and

$$\begin{gathered}(\mathrm{Id}-Q)G = (\mathrm{Id}-Q)G',\\ QGQ = Q^2G = QG = P_0 + \sum_{i\text{ odd}} D_iP_i + P_iD_i^*,{}_{[HHH2.284]}\\ QG' = \sum_{i\text{ even}} A_i + A_i^* - P_n\ (n\text{ even}).\end{gathered} \tag{314}$$

For even i, the eigenvalues of $A_i + A_i^*$ on the range of $P_i + P_{i+1}$ are non-zero and occur in pairs with opposite signs. Similarly for odd i the eigenvalues of $D_iP_i + P_iD_i^*$ are non-zero and occur in pairs with opposite signs. The ranks of the spaces on which the two sums act are the same and the spectral flow comes from the k $+1$ eigenvalues of P_0 which become -1 eigenvalues for $-P_n$. Thus the spectral flow is $-k$ and (310) holds in case n is even.

If n is odd the argument is similar except that (314) is replaced by

$$\begin{gathered}(\mathrm{Id}-Q)G = (\mathrm{Id}-Q)G',\\ QG = P_0 + \sum_{i\text{ odd}} D_iP_i + P_iD_i^* + P_n,{}_{[HHH2.286]}\\ QG' = \sum_{i\text{ even}} A_i + A_i^*\ (n\text{ odd}).\end{gathered} \tag{315}$$

The the spectral flow from G to G', around 0, corresponds to the change of $2k$ unit eigenvalues of $P_0 + P_n$ to pairs with opposite signs. The net flow is therefore $-k$. This proves (310) in general and the Proposition is proved. □

A moment's consideration of the proof of Proposition 17 reveals that relative indices come in pairs and in the process of proving (310) we have also proved a similar formula for $\text{R-Ind}(\tilde{S},\tilde{S}_{\tilde{a}})$. Here

$$\tilde{a} = \mathrm{Id}_{\Lambda^{0,n}} \otimes a$$

and $\tilde{S}$ is the generalized Szegő projector acting on $\mathcal{C}^\infty(X;\Lambda^{0,n}\otimes E)$ with symbol supported in the lower hemisphere. We call such projectors generalized 'conjugate Szegő projectors.' Following our previous notation we let

$$\tilde{S}_{\tilde{a}} = \tilde{a}^{-1}\tilde{S}\tilde{a}.$$

Indeed the discussion of such relative indices runs exactly parallel to that presented for Szegő projectors with symbol supported in the upper hemisphere.

[HHH2.2641]

PROPOSITION 18. *The operator G in* (308) *is a self-adjoint isomorphism from* $\mathrm{Dom}(G)$ *onto* $L^2(X;\Lambda^{0,*}\otimes E)$. *Let $\bar{S}$ be a generalized conjugate Szegő projector acting on $\mathcal{C}^\infty(X;\Lambda^{0,n}\otimes E)$. If $a\in\mathcal{C}^\infty(X;E)$ is a unitary isomorphism and G_t is the linear homotopy between $G_0=G$ and $G_1=a^{-1}Ga$ then*

$$\text{R-Ind}(\bar{S},\bar{S}_{\tilde{a}}) = (-1)^n\,\mathrm{SF}(G_\bullet,0). \tag{316}$$ [HHH2.2661]

Here $\tilde{a}=\mathrm{Id}_{\Lambda^{0,n}}\otimes a$.

PROOF. It follows from the cocycle formula for relative indices that if $\tilde{S}$ is another generalized conjugate Szegő projector then

$$\text{R-Ind}(\bar{S},\bar{S}_{\tilde{a}}) = \text{R-Ind}(\tilde{S},\tilde{S}_{\tilde{a}}).$$

Thus we can replace $\bar{S}$ in our considerations with $\tilde{S}$, the conjugate Szegő projector introduced in Proposition 15. In the proof of Proposition 17 we construct a projection P such that S_a is homotopic to $S-P$ and a resolution G' of this projector. If n is even then $\tilde{S}$ is replaced by $\tilde{S}+P_n$ where the rank of P_n equals the rank of P. From Corollary 3 it follows that this resolution, G' is homotopic to $a^{-1}Ga$ and so in particular $\tilde{S}+P_n$ is homotopic through generalized conjugate Szegő projectors to $\tilde{S}_{\tilde{a}}$. From these considerations it is immediate that

$$\text{R-Ind}(\tilde{S},\tilde{S}_{\tilde{a}}) = -\operatorname{rank}P_n = -\text{R-Ind}(S,S_a).$$

If n is odd then $\tilde{S}$ is replaced in G' by $\tilde{S}-P_n$ and so an identical argument shows that

$$\text{R-Ind}(\tilde{S},\tilde{S}_{\tilde{a}}) = \operatorname{rank}P_n = \text{R-Ind}(S,S_a).$$

Formula (316) now follows from (310). □

Of particular interest is the case where $E=\Lambda^{n+1,0}\otimes F$ for some complex vector bundle F and $a=\mathrm{Id}_{\Lambda^{n+1,0}}\otimes a'$. In this case $\Lambda^{0,n}\otimes E=\Lambda^{n+1,n}\otimes F$. As $\Lambda^{n+1,n}$ is a trivial line bundle, the relative indicies satisfy $\text{R-Ind}(\bar{S},\bar{S}_{\tilde{a}})=\text{R-Ind}(\bar{S},a'^{-1}\bar{S}a')$. We have the following corollary

[HHH12.15.98.1]

COROLLARY 4. *Let S be a generalized Szegő projector for $\Lambda^{n+1,0}\otimes F$, $\bar{S}$ be a generalized conjugate Szegő projector acting for F and suppose $a'\in\mathcal{C}^\infty(X;\hom(F))$. The relative indices satisfy the following relation*

$$\text{R-Ind}(\bar{S},a'^{-1}\bar{S}a') = (-1)^{n+1}\,\text{R-Ind}(S,a^{-1}Sa) \tag{317}$$ [HHH12.16.98.2]

where $a=\mathrm{Id}_{\Lambda^{n+1,0}}\otimes a'$.

8. Self-adjoint Dirac operators[SS.Self−adjoint.Dirac]

Let $\kappa = \kappa_{\text{even}} + \kappa_{\text{odd}}$ be the decomposition of the symbol of a resolution D into its restriction to $\Lambda^{0,\text{even}}$ and $\Lambda^{0,\text{odd}}$. Then $\kappa + \kappa^*$ is a graded Clifford action of the Clifford algebra of H' on $\Lambda^{0,*}$. Thus it can be written

$$\kappa + \kappa^* = \begin{pmatrix} 0 & \kappa_{\text{odd}} + \kappa^*_{\text{even}} \\ \kappa_{\text{even}} + \kappa^*_{\text{odd}} & 0 \end{pmatrix} : \\ H' \ni \eta \longrightarrow \hom(\Lambda^{0,*}),\ (\kappa+\kappa^*)^2 = \mathrm{Id}\,|\eta|^2. \tag{318}$$ [HHH2.287]

The parity matrix

$$\kappa_0 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$$

anti-commutes with $\kappa + \kappa^*$ and so extends the Clifford action to the whole of $T^*X = H' \oplus L$. In terms of this metric decomposition for an associated metric the Clifford action is

$$\mathrm{cl}(\eta + s\alpha) = \kappa(\eta) + \kappa^*(\eta) + s\kappa_0. \tag{319}$$ [HHH2.288]

In fact the Clifford module structure on $\Lambda^{0,*}$ so defined corresponds to the Spin-$\mathbb{C}$ reduction of the structure group provided by the choice of a positive almost complex structure on H. The associated Dirac operator defined by a choice of compatible (Hermitian Clifford) connection

$$\eth = \mathrm{cl} \circ \frac{1}{i}\nabla :\ \mathcal{C}^\infty(X;\Lambda^{0,*}) \xrightarrow{-i\nabla} \mathcal{C}^\infty(X; T^*X \otimes \Lambda^{0,*}) \xrightarrow{\mathrm{cl}} \mathcal{C}^\infty(X;\Lambda^{0,*}) \tag{320}$$ [HHH2.289]

is an elliptic self-adjoint operator on X.

More generally if E is any Hermitian vector bundle with unitary connection ∇_E then replacing ∇ by the the tensor product connection on $\Lambda^{0,*} \otimes E$ the same formula (320) defines an elliptic self-adjoint Dirac operator on $\Lambda^{0,*} \otimes E$. We shall denote the operator defined by these choices of data $\eth_{E,\nabla}$.

If $a \in \mathcal{C}^\infty(X; \hom E)$ is a unitary bundle isomorphism then $\nabla_a = a^{-1} \circ \nabla_E \circ a$ is also a unitary connection on E. The Dirac operator defined by this new connection is

$$\eth_{E,\nabla_a} = a^{-1} \eth_{E,\nabla} a. \tag{321}$$ [HHH2.290]

These two differential operators are self-adjoint and Fredholm with the same domain, the standard Sobolev space $H^1(X; \Lambda^{0,*} \otimes E)$; they are conjugate and hence isospectral. Since they have the same principal symbol the linear homotopy between them,

$$\eth_{E,\nabla(t)} = (1-t)\eth_{E,\nabla} + t\eth_{E,\nabla_a}, \tag{322}$$ [HHH2.292]

also consists of self-adjoint elliptic differential operators. Thus the spectral flow of this family is well-defined around any real point not in the spectrum of $\eth_{E,\nabla}$ and is independent of that point.

[HHH2.291]

Proposition 19. *The index of the Toeplitz operator SaS for $a \in \mathcal{C}^\infty(X; \hom E)$, a unitary isomorphism of an Hermitian vector bundle, is equal to minus the spectral flow of the family $\eth_{E,\nabla(t)}$ in* (322) *for any choice of unitary Clifford connection on $\Lambda^{0,*}$ and any choice of base unitary connection on E.*

Proof. As a result of Proposition 17 we know that the index of the Toeplitz operator SaS is minus the spectral flow (around 0) of the family given by the linear homotopy, G_t between the two operators G_0 and G_1 each given by (308), with G_0 corresponding to S and a resolution D and G_1 the conjugate resolution, so $G_1 = a^{-1}Ga$.

Thus provided we deform this family of operators, keeping them self-adjoint and Fredholm on a fixed domain (although the domain will vary with the deformation parameter) and with conjugate endpoints the spectral flow will stay unchanged. Let T be the 'Reeb vector field' fixed by the choice of compatible metric, it is the unit vector field orthogonal to H. Thus the symbol $l = -i\sigma(T)$ restricted to the contact line L is the linear coordinate fixed by the global section α. We may choose a smooth 1-parameter family $M_s \in \Psi^s(X; \Lambda^{0,*} \otimes E)$ which is elliptic near the positive direction L with symbol $l^s \otimes \mathrm{Id}$ and elliptic near $-L$ with symbol $-|l|^s$; we may even requite that $M_1 = -i\nabla_T$. Then we replace S and $\tilde{S}$ by the Toeplitz operators of order s, SM_sS and $\tilde{S}M_s\tilde{S}$. This leads us to the deformation of G_t to

$$(323)\quad \begin{aligned} G_t(s) &= (1-t)G_0(s) + ta^{-1}G_0(s)a,\\ G_0(s) &= SM_sS + D + D^* + (-1)^n\tilde{S}M_s\tilde{S},\ s \in [0, \frac{1}{2}].^{[HHH2.294]} \end{aligned}$$

These operators are self-adjoint with the domain similar to that in (309), namely

$$(324)\qquad \mathrm{Dom}(G_t(s)) = SH^s(X; E) + \tilde{S}H^s(X; \Lambda^{0,n} \otimes E) + H^1_{\mathrm{H}}(X; \Lambda^{0,*} \otimes E).^{[HHH2.295]}$$

Since $0 \le s \le \frac{1}{2}$, the projections of the third term, $H^1(X; \Lambda^{0,*} \otimes E)$ under S or $\tilde{S}$ is contained in the first and second terms respectively. In fact these are equal for $s = \frac{1}{2}$. Thus (324) is really the obvious orthogonal direct sum of Hilbert spaces on which $G_t(s)$ is Fredholm and self-adjoint. The spectral flow therefore remains constant in s so we may replace $G_t = G_t(0)$ by $G_t(\frac{1}{2})$.

As already noted, the domain of $G_t(\frac{1}{2})$ is just the Heisenberg Sobolev space $H^1_{\mathrm{H}}(X; \Lambda^{0,*} \otimes E)$ and indeed $G_t(\frac{1}{2})$ is elliptic as a Heisenberg operator of order 1. We may further deform it, now with this fixed domain, but within the space $\Psi^{1,1}_{\mathrm{eH}}(X; \Lambda^{0,*} \otimes E) \supset \Psi^1_{\mathrm{H}}(X; \Lambda^{0,*} \otimes E)$ provided ellipticity and the conjugacy of the endpoints is maintained. In this step we deform S and $\tilde{S}$ to the identity. Thus set

$$(325)\quad \begin{aligned} G'_t(r) &= (1-t)G_0(r) + ta^{-1}G_0(r)a,\\ G_0(r) &= T_rM_{\frac{1}{2}}T_r + D + D^* + (-1)^n\tilde{T}_rM_{\frac{1}{2}}\tilde{T}_r,\\ T_r &= (1-r)S + r\,\mathrm{Id},\ \tilde{T}_r = (1-r)\tilde{S} + r\,\mathrm{Id}; r \in [0,1].^{[HHH2.296]} \end{aligned}$$

Ellipticity has been established at $r = 0$ and the r-dependent terms are all in the space $\Psi^{\frac{1}{2},1}_{\mathrm{eH}}(X; \Lambda^{0,*} \otimes E)$ so radial ellipticity is unchanged. For the Heisenberg symbol the r-derivative is a fibre-constant multiple of $T_r(\mathrm{Id} - s) + (-1)^n\tilde{T}_r(\mathrm{Id} - \tilde{s})$. This vanishes on the range of s and the range of $\tilde{s}$ on which the symbol therefore remains invertible. On the complement the symbol of $G'_t(r)$ is equal to the symbol of

$$\begin{pmatrix} r^2M_{\frac{1}{2}} & D_{\mathrm{odd}} + D^*_{\mathrm{even}} \\ D_{\mathrm{even}} + D^*_{\mathrm{odd}} & -r^2M_{\frac{1}{2}} \end{pmatrix}$$

independent of the parity of the dimension. This is already known to be elliptic and is invertible since the diagonal terms are fibre constant so the eigenvalues are

$\pm(r^4\sigma(M_{\frac{1}{2}})^2+\lambda^2)^{\frac{1}{2}}$ where λ runs over the eigenvalues of $\kappa+\kappa^*$. On this subspace $\kappa+\kappa^*$ is invertible and thus $\lambda\neq 0$.

Thus we are reduced to computing the spectral flow of the family $G_t'(1)$. This is the linear homotopy between the operator

$$\begin{pmatrix} M_s & D_{\mathrm{odd}}+D^*_{\mathrm{even}} \\ D_{\mathrm{even}}+D^*_{\mathrm{odd}} & -M_s \end{pmatrix} \tag{326}$$ [HHH2.299]

for $s=\frac{1}{2}$ and its conjugate by a (which does not affect the symbol). Recall that the pseudodifferential operator $M_s\in\Psi^s(X;\Lambda^{0,*}\otimes E)\subset\Psi^{s,2s}_{\mathrm{eH}}(X;\Lambda^{0,*}\otimes E)$ is elliptic near L and $-L$, as sections of $\mathbb{S}^*X$, where its symbol is $|l|^s$ and $-|l|^s$ for the induced norm on L. For $s\in[\frac{1}{2},1]$ we may consider the homotopy between the operator in (326) and its a-conjugate. This is self-adjoint and elliptic as a family in $\Psi^{1,2s}_{\mathrm{eH}}(X;\Lambda^{0,*}\otimes E)$.

Recall that we had chosen $M_1=-i\nabla_T$ for the Reeb vector field T. The resulting elliptic homotopy in $\Psi^{1,2}_{\mathrm{eH}}(X;\Lambda^{0,*}\otimes E)$ is between an operator, and its a-conjugate.These operators have the same symbol as the differential operator $\eth_{\nabla,E}$. Thus we may make a final deformation, keeping the symbol fixed and the endpoints a-conjugate to this family of Dirac operators. This completes the proof of Proposition 19. □

9. Suspension[SS.Suspension]

The final step in the reduction of the index of a Toeplitz operator to the index of a Dirac operator is to suspend the family of Dirac operators in (322).

Let $Z=X\times\mathbb{S}^1$ with $\pi:Z\longrightarrow X$ the natural projection. This manifold has a natural class of positive almost complex structures. Namely the hyperplane bundle of the contact manifold gives a decomposition of the tangent bundle of Z as

$$TZ=TX\oplus T\mathbb{S}^1=H+(H^\perp)+\mathbb{R}. \tag{327}$$ [HHH2.300]

Here $H^\perp$ is spanned by a Reeb vector field and the fibre complex structure on H is extended to TZ by demanding that $JT=U$, the tangent vector to the circle. This almost complex structure fixes a natural class of Dirac operators, corresponding to a choice of unitary connection. The Dirac operator $\eth_Z$ acts on $\Lambda^{0,*}$ and corresponds to $\bar\partial+\bar\partial^*$ in the Kähler case.

The unitary isomorphism a of the Hermitian bundle E over X leads to a bundle E_a over Z. Namely if we lift E to $X\times\mathbb{R}$ and consider the global sections which satisfy the periodicity condition

$$\zeta(x,t+2\pi)=a(x)\zeta(x,t)\ \forall\ (x,t)\in X\times\mathbb{R} \tag{328}$$ [HHH2.301]

we obtain all the global sections of the bundle E_a over Z. Choosing a unitary connection on E_a we can consider the twisted Dirac operators $\eth_{Z,a}$ acting on sections of $\Lambda^{0,*}\otimes E_a$.

The operator $\eth_{Z,a}$ is self-adjoint and $\mathbb{Z}_2$ graded for the parity grading of $\Lambda^{0,*}$. The restricted operator $\eth^+_{Z,a}:\mathcal{C}^\infty(Z;\Lambda^{0,\mathrm{even}}\otimes E_a)\longrightarrow\mathcal{C}^\infty(Z;\Lambda^{0,\mathrm{odd}}\otimes E_a)$ is elliptic and hence Fredholm. As is well-known (see for example [**4**]) its index is computed in term of the spectral flow of the associated family on X.

[HHH2.302]

Proposition 20. *The spectral flow of the family of self-adjoint Dirac operators (322) is equal to the index of* $\eth^+_{Z,a}$.

Proof. The Clifford action, cl' of T^*Z on $\Lambda^{0,*}Z$ for an admissible Hermitian metric on Z is defined as usual through the complex structure. The covector dt is the real part of a form $\mu_0 = dt + iJdt$ which gives a decomposition of the anti-holomorphic exterior bundle of Z :

$$\Lambda^{0,*}Z = \pi^*\Lambda^{0,*}X + \mu_0 \wedge \pi^*\Lambda^{0,*}.^{[HHH2.310]} \tag{329}$$

Identifying π^*T^*X with the orthocomplement of dt gives the standard 'suspension' relationship between the Clifford action cl' and the Clifford action cl of T^*X on $\Lambda^{0,*}$. Namely if $\eta \in T^*X$ then

$$\mathrm{cl}(\eta) = i\,\mathrm{cl}'(dt)\,\mathrm{cl}'(\eta).^{[HHH2.311]} \tag{330}$$

If ∇ is a 'unitary' connection on T^*X extended to a unitary connection on T^*Z then the positive Dirac operator on Z is related to the Dirac operator on X through

$$\eth^+_Z = -i\,\mathrm{cl}'(dt)(\partial_t + i\eth_X).^{[HHH2.312]} \tag{331}$$

In the general case of the bundle E_a constructed from the Hermitian bundle E over X we must choose a unitary connection on E to define the Dirac operator. If ∇_E is any unitary connection on E then in terms of the presentation (328) of sections of E the family of connections $\nabla_t = (1-\rho(t))\nabla_E + \rho(t)a^*\nabla_E a$, $t \in [0,1]$, extended trivially in the direction of ∂_t, gives such a connection on E_a. Here $\rho(t) \in \mathcal{C}^\infty([0,1])$ is a non-negative function which vanishes near $t = 0$ and is identically 1 near $t = 1$. Of course the definition should be extended globally using the transformation law (328).

Thus in the general case, the positive Dirac operator associated to E_a may be written in a similar form to (331)

$$\eth^+_{Z,a} = -i\,\mathrm{cl}'(dt)(\partial_t + i\eth_{X,t})^{[HHH2.313]} \tag{332}$$

where $\eth_{X,t}$ is the 1-parameter family of self-adjoint Dirac operators on X associated to the connections ∇_t. Thus the spectral flow of the family $\eth_{X,t}$ is equal to the spectral flow of the family (322), since they have the same endpoints and symbols.

Thus the index of $\eth^+_{Z,a}$ is the index of the operator

$$\begin{aligned}\partial_t + \eth_{X,t} : \big\{u \in H^1(X \times [0,2\pi]; \Lambda^{0,*}X \otimes E); u(2\pi, z) = a(z)u(0,z)\big\}\\ \longrightarrow L^2(X \times [0,2\pi]; \Lambda^{0,*}X \otimes E).^{[HHH2.314]}\end{aligned} \tag{333}$$

The index remains constant under smooth deformations of the family $\eth_{X,t}$ as elliptic, self-adjoint operators which are constant near the end points and remain elliptic. It follows from Proposition 13 that if the spectral flow of $\eth_{X,t}$ vanishes then we may deform this family to be invertible. Then index of the resulting operator

$$\partial_t + A(t)^{[HHH2.315]} \tag{334}$$

acting as in (333) is zero. Indeed, let $\Pi^\pm(t)$ be the projections onto the positive and negative parts of $A(t)$. These are constant near the end points, and

$$\partial_t = \partial_t(\Pi^+ + \Pi^-) = \Pi^+(t)\partial_t\Pi^+(t) + \Pi^-(t)\partial_t\Pi^-(t) + H(t)^{[HHH2.316]} \tag{335}$$

where $H(t)$ is compact as an operator (333) (and vanishes near the endpoints).

Thus the index is the same with term $H(t)$ dropped. Then it follows directly that

$$\begin{aligned}(336)\quad &\Pi^+(t)\partial_t\Pi^+(t)+\Pi^-(t)\partial_t\Pi^-(t)+A(t)\\ &=\Pi^+(t)\partial_t\Pi^+(t)+A^+(t)+\Pi^-(t)\partial_t\Pi^-(t)-A^-(t)\end{aligned}$$ [HHH2.317]

is an isomorphism as an operator (333). Here $A(t)=A^+(t)-A^-(t)$ is the (smooth) division into positive and negative parts. Clearly, it is Fredholm, let us consider the null space. Working in the equivariant model described in (333) we can suppose that the inner product on $\Lambda^{0,*}\otimes E$ and volume form on X_t are independent of the parameter t. Suppose that $u(t)$ is in the nullspace of the operator in (336) and satisfies $u(2\pi)=au(0)$. Let $u_\pm(t)=\Pi^\pm(t)u(t)$ and $\langle\cdot,\cdot\rangle$ denote the (t-independent) L^2-pairing on sections. We deduce that for each t we have

$$\frac{1}{2}\partial_t\langle u_\pm(t),u_\pm(t)\rangle\pm\langle A^\pm(t)u_\pm(t),u_\pm(t)\rangle=0.$$

If for some t, $u_\pm(t)\neq 0$ then in light of the fact that $A^\pm(t)$ are definite operators on the range of $\Pi^\pm(t)$ we would conclude that

$$\begin{aligned}(337)\quad &\langle u^+(2\pi),u^+(2\pi)\rangle<\langle u^+(0),u^+(0)\rangle \text{ and}\\ &\langle u^-(2\pi),u^-(2\pi)\rangle>\langle u^-(0),u^-(0)\rangle \text{ respectively.}\end{aligned}$$

This contradicts the unitary character of the boundary condition and therefore shows that the nullspace of the operator in (336) is trivial. This in turn proves the theorem in case the spectral flow vanishes.

In the general case we may use Proposition 14 instead. The preceeding argument applies to the direct sum of two families with opposite spectral flows and shows that the index is determined by the spectral flow. Thus to complete the proof of the proposition it suffices to examine one case of (334) with $A(t)$ of spectral flow one. It does not need to come from a Dirac family.

Such an example is given by the trivial bundle on the circle, $X=\mathbb{S}_\theta$. Take $a=e^{i\theta}$. Then the operator $A(t)$ in (334) may be taken to be

$$(338)\qquad A(t)=D_\theta+s(t),\ s\in\mathcal{C}^\infty(0,1),\ s(t)=0 \text{ in } t<\frac{1}{4},\ s(t)=1 \text{ in } t>\frac{3}{4}.$$ [HHH2.318]

In this case it is easy to see that both the index and the spectral flow are 1. □

10. The index formula for conjugate Toeplitz operators

Consider a conjugate Szegő projector, $\bar{S}\in\mathcal{I}^0_{\mathrm{H},-}(X;E)$, acting on sections of the complex bundle E. If $a\in\mathcal{C}^\infty(X;\hom(E))$ then we define $\bar{S}a\bar{S}$ to be a conjugate Toeplitz operator. Using Corollary 4 and Theorem 5 we can obtain a cohomological formula for

$$(339)\qquad \operatorname{Ind}(\bar{S}a\bar{S})=\operatorname{R-Ind}(\bar{S},a^{-1}\bar{S}a).$$ [12.17.98.1]

Of course we can equally well consider operators of the form $\bar{S}A\bar{S}$ where $A\in\Psi^0(X;E)$ or $A\in\Psi^0_{\mathrm{H}}(X;E)$ or even the extended Heisenberg algebra. If $A\in\Psi^0(X;E)$ then a is simply $\sigma(A)$ restricted to the negative contact direction. If A is a Heisenberg, or extended Heisenberg, pseudodifferential operator then a is determined by the projection of the Heisenberg symbol onto the ground state. As in the previous case this operator is Fredholm if and only if a is invertible. With these prelimaries we may now easily establish the following formula for the index.

[HHH2.3191]

THEOREM 6. *[L. Boutet de Monvel [**6**]] The index of a Fredholm conjugate Toeplitz operator, defined by $A \in \Psi^0(X; E)$ is*

$$\operatorname{Ind}(\bar{S}A\bar{S}) = \int_X \chi_-(X)\operatorname{Ch}^{\mathrm{odd}}(a) \tag{340}$$ [HHH2.3201]

where $\chi_-(X)$ is the negative 'contact' characteristic class of X, defined in terms of the Todd class of the bundle H with a negative almost complex structure by,

$$\chi_-(X) = (-1)^{n+1}\operatorname{Td}(H^{0,1}) \in H^{\mathrm{even}}(X) \tag{341}$$ [HHH2.3211]

and $\operatorname{Ch}^{\mathrm{odd}}(a) \in H^{\mathrm{odd}}(X)$ is the odd Chern character of the bundle isomorphism defined by $a = \sigma(A)_{L^-}$.

REMARK 6. The appearance of $(-1)^{n+1}$ in this formula has a simple interpretation: In formula (293) we use the contact form θ to define a volume form $\theta \wedge (d\theta)^n$ and therefore an orientation on X. For the case of comjugate Toeplitz operators the "positive" contact direction becomes $-\theta$ and therefore the orientation is defined by $(-1)^{n+1}\theta \wedge (d\theta)^n$.

PROOF. Let $E' = \Lambda^{n+1,0} \otimes E$ and set $b = \operatorname{Id}_{\Lambda^{n+1,0}} \otimes a$. If S denotes a generalized Szegő projector acting on sections of E' then from Corollary 4 we obtain that

$$\operatorname{Ind}(\bar{S}A\bar{S}) = (-1)^{n+1}\operatorname{R-Ind}(S, b^{-1}Sb).$$

From (3) we see that

$$\operatorname{R-Ind}(S, b^{-1}Sb) = \int_Z \operatorname{Td}(T^{1,0}Z)\operatorname{Ch}^{\mathrm{even}}(F'). \tag{342}$$

It follows from the form of b that $F' = E_a \otimes \pi^*(\Lambda^{n+1,0})$. Here E_a is the bundle over $X \times \mathbb{S}$ defined from the lift of E to $X \times [0, 2\pi]$ with ends glued via a and therefore

$$\operatorname{Ch}^{\mathrm{even}}(F') = \pi^*(e^{c_1(H^{1,0})})\operatorname{Ch}^{\mathrm{even}}(F). \tag{343}$$

It is well known that $e^{c_1(H^{1,0})}\operatorname{Td}(H^{1,0}) = \operatorname{Td}(H^{0,1})$, combining these formulæ gives (340) and (341). □

There is another approach to conjugate Toeplitz which leads to a slightly different formula. Suppose that $E \to X$ is a complex vector bundle and let $\bar{E}$ denote the conjugate bundle. Let C denote the complex conjugation map which carries sections of E to sections $\bar{E}$ and vice versa. If $S_{\bar{E}}$ is a generalized Szegő projector acting on sections of $\bar{E}$ then setting

$$\bar{S}_E = CS_{\bar{E}}C$$

defines a generalized conjugate Szegő projector acting on sections of E. If $a \in \mathcal{C}^\infty(X; \operatorname{Hom}(E))$ then $\bar{a} \in \mathcal{C}^\infty(X; \operatorname{hom}(\bar{E}))$ and

$$\overline{T}_a = \bar{S}_E a \bar{S}_E = CS_{\bar{E}}CaCS_{\bar{E}}C$$

is a conjugate Toeplitz operator acting on $\mathcal{C}^\infty(X; E)$ defined by this section. A simple calculation shows that

$$S_{\bar{E}}CaCS_{\bar{E}} = S_{\bar{E}}\bar{a}S_{\bar{E}} = T_{\bar{a}}.$$

If a is an invertible section so that $\overline{T}_a$ is Fredholm then, as $C : \mathcal{C}^\infty(X;E) \to \mathcal{C}^\infty(X;\bar{E})$ is an isomorphism we see that

$$\operatorname{Ind}(\overline{T}_a) = \operatorname{Ind}(T_{\bar{a}}).$$

Combining these observations leads to a second formula for $\operatorname{Ind}(\overline{T}_a)$: If $A \in \Psi^0(X;E)$ is such that $s = \sigma_0(A)_{L^-}$ is invertible then

$$\operatorname{Ind}(\bar{S}_E A \bar{S}_E) = \int_X \operatorname{Td}(H^{1,0}) \operatorname{Ch}^{\text{odd}}(\bar{a}). \tag{344}$$ [8.5.99.20]

CLE: Can we divide to conclude that $\operatorname{Ch}^{\text{odd}}(\bar{a}) = (-1)^{n+1} e^{c_1} \operatorname{Ch}^{\text{odd}}(a)$?

Comparing these two formulæ we see that

$$[\operatorname{Td}(H^{1,0}) \operatorname{Ch}^{\text{odd}}(\bar{a})]_{2n+1} = (-1)^{n+1} e^{c_1(H^{1,0})} \operatorname{Td}(H^{1,0}) \operatorname{Ch}^{\text{odd}}(a). \tag{345}$$

11. Higher level Toeplitz index formula [S.Higher.level.index]

The index theorem of Boutet de Monvel, Theorem 5, applies equally well to an operator written SBS where $B \in \Psi^0_{\mathrm{H}}(X;E)$, since this is an equivalent representation of Toeplitz operators. The 'symbol' a is then defined by

$$\sigma_{\mathrm{H}}(SBS) = s\#\sigma_{\mathrm{H}}(B)\#s = as,\ a \in \mathcal{C}^\infty(X; \hom E). \tag{346}$$ [HHH2.327]

We shall now generalize this formula directly to higher level Toeplitz operators. Namely if $S_{(N)}$ is a generalized Toeplitz projection to level N the Heisenberg symbol of higher level Toeplitz operator $S_{(N)}BS_{(N)}$ is just

$$\sigma_{\mathrm{H}}(S_{(N)}BS_{(N)}) = s_{(N)}\#\sigma_{\mathrm{H}}(B)\#s_{(N)} \in t_{(N)}. \tag{347}$$ [HHH2.328]

Here $t_{(N)}$ is defined in (141). Recall the isotropic operators constructed in (153). Since the construction is natural, in the case of a contact manifold with compatible almost complex structure these become sections of the corresponding symbol bundles. They give explicit isomorphisms between $t_{(N)}$ and $\hom(F_{(N)})$ where $F_{(N)}$ is the sum up to order N of the symmetric powers of $W^{1,0}$ in (151). Thus we may identify the symbol in (347) with such a homomorphism

$$a_{(N)} = q_{(N)}\#\sigma_{\mathrm{H}}(S_{(N)}BS_{(N)})\#q^*_{(N)} \in \mathcal{C}^\infty(X; \hom(F_{(N)} \otimes E)). \tag{348}$$ [HHH2.329]

[HHH2.259]

Proposition 21. *If $S_{(N)}BS_{(N)}$ is a Toeplitz operator to level N acting on sections of a bundle E then as an operator on the range of $S_{(N)}$ it is Fredholm if and only if the homomorphism* (348) *is invertible and its index is equal to that of the corresponding Toeplitz operator at level* 0

$$\operatorname{Ind}(S_{(N)}BS_{(N)}) = \operatorname{Ind}(S^{(N)}a_{(N)}S^{(N)}) = \int_X \operatorname{Td}(H^{1,0}) \operatorname{Ch}^{\text{odd}}(a_{(N)}), \tag{349}$$ [HHH2.260]

acting on sections of the bundle $E \otimes F_{(N)}$, defined by $a_{(N)}$. Here $S^{(N)}$ is a Szegő projection at level 0 *acting on sections of $F_{(N)} \otimes E$.*

Proof. The surjectivity of the Heisenberg symbol map allows us 'quantize' elements of the symbol calculus. Thus we may find

$$Q'_{(N)} \in \mathcal{I}^0_{\mathrm{H},+}(X; F_{(N)} \otimes E, E),\ \text{with } \sigma(Q'_{(N)}) = \tilde{q}_{(N)} \otimes \operatorname{Id}_E. \tag{350}$$ [HHH2.256]

Now $S_{(N)}$ is a generalized Szegő projection to level N acting on E and $S^{(N)}$ is a generalized Szegő projector, at level 0, for the bundle $E \otimes F_{(N)}$ then

$$Q_{(N)} = S^{(N)} \cdot Q'_{(N)} \cdot S_{(N)} \in \mathcal{I}^0_{\mathrm{H},+}(X; E, F_{(N)} \otimes E) \tag{351}$$ [HHH2.258]

is Fredholm as a map from the range of $S_{(N)}$ to the range of $S^{(N)}$. It follows that $S_{(N)}BS_{(N)}$ is Fredholm on the range of $S_{(N)}$ if and only if

$$Q^*_{(N)}S_{(N)}BS_{(N)}Q_{(N)} = S^{(N)}B'S^{(N)} \quad [HHH2.332] \tag{352}$$

is Fredholm on the range of $S^{(N)}$; these two operators then have the same index. Here $B' \in \Psi^0_{\mathrm{H}}(X; E \otimes F_{(N)})$ has Heisenberg symbol $q^*_{(N)}\#\sigma_{\mathrm{H}}(B)\#q_{(N)} = a_{(N)}s$. This gives both the characterization of Fredholm operators in the Toeplitz algebra to level N and the equality of indexes in (349). Now the index formula, (349), for a Fredholm Toeplitz operator at level N directly from (293). □

The remarks at the end of the previous section can be repeated almost verbatim for higher level conjugate Toeplitz operators, to wit, if $S_{(N,\bar{E})}$ is a Szegő projector to level N acting on $\mathcal{C}^\infty(X; \bar{E})$ and C is complex conjugation then

$$\bar{S}_{(N,E)} = CS_{(N,\bar{E})}C$$

is a conjugate Szegő projector to level N acting on $\mathcal{C}^\infty(X; E)$. If a is a residual symbol with values in $\hom(E)$ then we regard it as an element of $\mathcal{I}^0_{\mathrm{H},-}(X; E)$ and $\bar{A}$ as an element of $\mathcal{I}^0_{\mathrm{H},+}(X; \bar{E})$. The symbol $\mathrm{Id}_E + a$ is invertible if and only if $\mathrm{Id}_{\bar{E}} + \bar{a}$ is. Let A be an operator with principal symbol a and $\bar{A}$ an operator with principal symbol $\bar{a}$. Since

$$\mathrm{Ind}(\bar{S}_{(N,E)}A\bar{S}_{(N,E)}) = \mathrm{Ind}\, S_{(N,\bar{E})}\bar{A}S_{(N,\bar{E})}$$

we obtain the formula

$$\mathrm{Ind}(\bar{S}_{(N,E)}A\bar{S}_{(N,E)}) = \int_X \mathrm{Td}(H^{1,0}\,\mathrm{Ch}^{\mathrm{odd}}(\bar{a}_{(N)}). \quad [8.5.99.21] \tag{353}$$

In this formula $\bar{a}$ defines an section of $\hom(\bar{E} \otimes F_{(N)}$ according to the composition from the "upper" isotropic algebra. If the fiber dimension of E is 1 then a is a scalar symbol and we can apply (174) and (483) to obtain the more explicit formula

$$\mathrm{Ind}(\bar{S}_{(N,E)}A\bar{S}_{(N,E)}) = -\int_X \mathrm{Td}(H^{1,0}\,\mathrm{Ch}^{\mathrm{odd}}(a_{(N)}). \quad [8.5.99.22] \tag{354}$$

In the last formula we are regarding a as an element $\mathcal{I}^0_{\mathrm{H},+}(X; E)$.