CHAPTER 5

Heisenberg and extended Heisenberg algebras $^{[S.Heisenberg.algebras]}$

In this chapter we show that the spaces of Heisenberg and extended Heisenberg operators defined in Chapter 3 form order-filtered $*$-algebras. The symbol homomorphism for the Heisenberg algebra, into a bundle of isotropic algebras as discussed in Chapter 4, is also described in detail as is its analogue for the extended algebra. These results for the Heisenberg algebra were obtained by Beals and Griener ([**2**]) and Taylor ([**27**]); the latter also has some results in the direction of those obtained here for the extended Heisenberg algebra. We recall, and in so doing slightly modify, their treatment. Although our main interest is in the non-degenerate, or maximally non-commutative, case corresponding to a contact manifold, after treating this important special case we also discuss the composition in the general case of an oriented line subbundle of the cotangent bundle.

1. Localization of the product $^{[S.Localization.product]}$

Initially we consider a compact manifold X with oriented contact line bundle S. The hyperplane bundle which is the annihilator of S, $H \subset TX$ is thought of as defining the corresponding tangency structure. As already shown in Chapter 3 the space $\Psi^k_{\mathrm{H}}(X)$, as a space of kernels on X^2, is a two-sided $\mathcal{C}^\infty(X^2)$-module. Acting as operators they are also quasilocal, the kernels all having singular support within the diagonal. Since the smoothing operators form an ideal this effectively localizes the computation of composites as we shall now show.

If U_i is any finite open cover of X then we may choose a finer cover, Ω_i, such that

$$\Omega_i \cap \Omega_j \neq \emptyset \Longrightarrow \Omega_i \cup \Omega_j \subset U_l.^{[HHH2.544]} \tag{183}$$

In terms of the open cover Ω_i we decompose $A \in \Psi^k_{\mathrm{H}}(X)$ as a sum

$$A = \sum_i A_i + A',\ A' \in \Psi^{-\infty}(X),\ \operatorname{supp}(A_i) \subset \Omega_i \times \Omega_i.^{[HHH2.543]} \tag{184}$$

Let $B \in \Psi^{k'}_{\mathrm{H}}(X)$ be a second operator with B_i and B' the corresponding terms in a decomposition. Then the only terms in

$$A \circ B = (\sum_i A_i + A') \circ (\sum_i B_i + B')^{[HHH2.542]} \tag{185}$$

which are not already known to be smoothing are of the form

$$A_i \circ B_j,\ \operatorname{supp}(A_i) \cup \operatorname{supp}(B_j) \subset U_l \times U_l$$

for some coordinate patch U_l

Since the class of operators $\Psi^k_{\mathrm{H}}(X)$ is independent of the choice of metric used to give the Riemann-Weyl quantization we may assume that each U_l is a coordinate patch and, in case L is a contact bundle, we may also assume the a contact form on

X is given by the Darboux form in the local coordinates. In any case, we may use Weyl quantization in these coordinates; we may also assume that the symbol has compact support in the base. This is of course, Riemann-Weyl quantization with respect to the flat metric.

The Schwartz kernel of an operator, $A \in \Psi^k_{\mathrm{H}}(X)$ is then of the form

$$A(x,x') = \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^{2n+1}} a(\frac{x+x'}{2},\xi) e^{i(x-x')\cdot\xi} d\xi, \ N = \dim X, \tag{186}$$

where $a \in \rho^{-k}\mathcal{C}^\infty({}^L\overline{T}^* X)$.

In the contact case, where $N = 2n+1$, and in Darboux coordinates $z_0, \hat{z}, \check{z}$ the line bundle is spanned by the form

$$\alpha = dz_0 + \frac{1}{2}\sum_{j=1}^n (\check{z}_j d\hat{z}_j - \hat{z}_j d\check{z}_j).^{[HHH2.113]} \tag{187}$$

We then 'straighten the contact field' by using the symbols of the associated basis of vector fields

$$\begin{gathered} Z_0 = \partial_{z_0}, \ \hat{Z}_j = \partial_{\hat{z}_j} - \frac{1}{2}\check{z}_j \partial_{z_0}, \ \check{Z}_j = \partial_{\check{z}_j} + \frac{1}{2}\hat{z}_j \partial_{z_0}, {}_{[2.1.1998.1a]} \\ \alpha(\hat{Z}_j) = \alpha(\check{Z}_j) = 0, \ j = 1, \dots, n \end{gathered} \tag{188}$$

to define local fiber coordinates in the cotangent space. Let

$$\begin{gathered} \sigma_0 = \zeta_0, \ \hat{\sigma}_j(z,\zeta) = \sigma(\hat{Z}_j)/i = \hat{\zeta}_j - \frac{1}{2}\check{z}_j\zeta_0 \text{ and} \\ \check{\sigma}_j(z,\zeta) = \sigma(\check{Z}_j)/i = \check{\zeta}_j + \frac{1}{2}\hat{z}_j\zeta_0, \ j = 1, \dots, n, \end{gathered}$$

be the corresponding functions on the cotangent bundle in terms of the canonical coordinates z_0, ζ_0, $\hat{z}, \hat{\zeta}$ and $\check{z}, \check{\zeta}$ dual to $z_0, \hat{z}, \check{z}$. Set $\sigma = (\sigma_0, \hat{\sigma}, \check{\sigma}) = (\sigma_0, \tilde{\sigma})$. These local coordinates z, σ, in place of z, ζ, have the desirable property that they each have fixed homoneneity, *viz.* $0, 1$ or 2, in the parabolic sense arising from α; in particular they are parabolic symbols of orders 0, 1 or 2. It is in this sense that these coordinates 'straighten' the subbundle L^0.

For the purposes of manipulation of integral formulæ we may assume that all symbols are in $\dot{\mathcal{C}}^\infty({}^H\overline{T}^* X)$ and then subsequently use continuity in symbol topologies. Using similar notation to that above for the second, 'primed', coordinates and changing variable from ξ to $\sigma(\frac{1}{2}(z+z'),\zeta)$ gives

$$A(z,z') = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{i\Phi(z-z',z',\sigma)} a^\#(\frac{z+z'}{2},\sigma) d\sigma^{[HHH2.115]} \tag{189}$$

$$a^\#(z,\sigma) = \int_{\mathbb{R}^N} e^{-i\Phi(y,2z,\sigma)} A(z+\frac{1}{2}y, z-\frac{1}{2}y) dy,^{[HHH2.114]} \tag{190}$$

where

$$\Phi(z,z',\sigma) = (z_0 + \frac{1}{2}\omega(\tilde{z},\tilde{z}'))\sigma_0 + \tilde{z}\cdot\tilde{\sigma}.^{[HHH2.550]} \tag{191}$$

Here $z = (z_0, \tilde{z})$, $a^\#(z, \sigma(z,\zeta)) = a(z,\zeta)$ and $\omega = -d\alpha$, so

$$\omega(\tilde{z} - \tilde{z}', \tilde{z} + \tilde{z}') = 2\omega(\tilde{z},\tilde{z}') = 2(\hat{z}\cdot\check{z}' - \check{z}\cdot\hat{z}') \text{ in the contact case.}^{[HHH2.546]} \tag{192}$$

Lemma 5 shows that the parabolic symbol estimates on the amplitude then become

$$|D_z^\alpha D_\sigma^\beta a^\#(z,\sigma)| \le C_{\alpha\beta}\langle\sigma\rangle_L^{m-|\beta|_L}. \quad (193)$$ [HHH2.175]

Let B be a second operator associated to an amplitude b. Essentially the same computation as led to (107) shows that the Schwartz kernel, $C(z,z')$, of $C = A\circ B$ is given, modulo a smoothing term, by (189) with amplitude $c^\#$ where

$$\begin{gathered} c^\#(z,\sigma) = Q(a^\#,b^\#)(z,\sigma) = \phi(z)\pi^{-2N}\int e^{2i(u\cdot\mu - v\cdot\nu + 2\omega(\tilde u,\tilde v)\sigma_0)} \\ a^\#(u_0+z_0+\omega(\tilde u,\tilde v+\tilde z),\tilde u+\tilde z,\nu+\sigma)b^\#(v_0+z_0+\omega(\tilde v,\tilde u+\tilde z),\tilde v+\tilde z,\mu+\sigma) \\ du\,dv\,d\nu d\mu. \end{gathered} \quad (194)$$ [HHH2.116]

Here we have inserted a cutoff function to keep the support compact in z. We wish to show that $c^\#$ satisfies the estimates (193) of order $m+m'$ if $a^\#$ and $b^\#$ satisfy these estimates with orders m and m', respectively.

The phase function 2ψ, in (194) is linear in u and v and satisfies

$$\begin{gathered} (1-\Delta_u-\Delta_v)e^{2i\psi} = he^{2i\psi},\ h = 1+4|\mu+2\omega(\cdot,\tilde v)\sigma_0|^2 + 4|\nu+2\omega(\cdot,\tilde u)\sigma_0|^2 \\ \psi = u\cdot\mu - v\cdot\nu + 2\omega(\tilde u,\tilde v)\sigma_0). \end{gathered} \quad (195)$$ [HHH2.160]

The symbols $a^\#$ and $b^\#$ have compact support in the base variables, u,v and z. Indeed we may assume that these supports are as small as desired, so that $|u|$, $|v| < \epsilon$. Then, for $\delta > 0$ small with ϵ,

$$1+|\mu|+|\nu| \ge \frac{1}{2}\delta|\sigma| \Longrightarrow Ch > 1+(|\mu|+|\nu|+|\sigma|)^2.$$

Furthermore all u and v derivatives are relatively bounded, $|\partial_u^\alpha\partial_v^\beta h| \le Ch$. Repeated use of (195) in (194) introduces a factor of h^{-N}, at the expense of differentiation in u and v. This shows that we may insert a cuf-off factor keeping supports in

$$|\mu|+|\nu|+1 \le \delta|\sigma|,\ \delta > 0, \quad (196)$$ [HHH2.169]

and in so doing make an error which is a bilinear map from the symbol spaces to $\dot{\mathcal{C}}^\infty({}^L\overline{T}^*X)$. Such a factor leaves a compactly supported integrand ensuring the absolute convergence of the integral:

$$\begin{gathered} Q'(a^\#,b^\#)(z,\sigma) = \phi(z)\pi^{-2N}\int e^{2i(u\cdot\mu - v\cdot\nu + 2\omega(\tilde u,\tilde v)\sigma_0)}\phi(\frac{\langle\mu,\nu\rangle}{\langle\sigma\rangle}) \\ a^\#(u_0+z_0+\omega(\tilde u,\tilde v+\tilde z),\tilde u+\tilde z,\nu+\sigma)b^\#(v_0+z_0+\omega(\tilde v,\tilde u+\tilde z),\tilde v+\tilde z,\mu+\sigma) \\ du\,dv\,d\nu d\mu. \end{gathered} \quad (197)$$ [HHH2.161]

We split the fibers of the cotangent space into two types of regions,

$$|\tilde\sigma| < c|\sigma_0| \text{ and } |\tilde\sigma| > c|\sigma_0|,\ c > 0. \quad (198)$$ [HHH2.545]

The first of these is the *classical* region. The second includes the whole of the parabolic region and a small part of the classical region, depending on the size of the constant c. This is the principal difference between the treatment here and earlier discussions where the fiber was split into regions which reflect more closely the parabolic homogeneity, e.g. $|\tilde\sigma| < c|\sigma_0|^{\frac{1}{2}}$.

2. The classical region [S.Classical.region]

First we consider the composite symbol in the 'classical' region, given by the first inequality in (198). This we insert a symbolic cutoff keeping

$$|\sigma_0| \le C|\tilde{\sigma}| \tag{199}$$ [HHH2.162]

for some positive constant C; it can be made large as $\epsilon \downarrow 0$. We denote the resulting bilinear form by Q_1'. In such a region the weights $\langle\sigma\rangle$ and $\langle\sigma\rangle_L$ are both bounded above and below by positive multiples of $\langle\tilde{\sigma}\rangle$. The symbol estimates on Q_1' therefore essentially amount to the composition theorem for standard pseudifferential operators. Thus, if ϵC is small enough then for some $c > 0$, $c|\tilde{\mu} + \tilde{\sigma}| \ge |\tilde{\sigma}| + |\tilde{\mu}|$ so the symbol estimates can be written

$$|\partial_w^\alpha \partial_\mu^\beta a^\#(w, \mu + \sigma)| \le C\langle\mu + \sigma\rangle_L^{m-|\beta|_L} \le C'(\langle\mu\rangle_L + \langle\sigma\rangle)^{m-|\beta|_L} \tag{200}$$ [HHH2.176]

The 'classical' rescaling,

$$M = \langle\sigma\rangle^{-1}\mu,\ N = \langle\sigma\rangle^{-1}\nu,\ (\langle\sigma\rangle \sim \langle\sigma\rangle_L) \tag{201}$$ [HHH2.123]

reduces the phase function in (197) to

$$\psi_1(u, v, M, N; \sigma) = (u \cdot M - v \cdot N + 2\omega(\tilde{u}, \tilde{v})\frac{\sigma_0}{\langle\sigma\rangle})\langle\sigma\rangle. \tag{202}$$ [HHH2.158]

The coefficient of $\langle\sigma\rangle$ is a quadratic form on $\mathbb{R}^{4N}$ which is uniformly non-degenerate in the support of the integrand. The rest of the integrand becomes

$$\begin{aligned} g(z, u, v, M, N, \sigma) = a^\#(&u_0 + z_0 + \omega(\tilde{u}, \tilde{v} + \tilde{z}), \tilde{u} + \tilde{z}, N\langle\sigma\rangle + \sigma) \\ &b^\#(v_0 + z_0 + \omega(\tilde{v}, \tilde{u} + \tilde{z}), \tilde{v} + \tilde{z}, M\langle\sigma\rangle + \sigma)\phi(\langle\sigma\rangle\langle(M, N)\rangle) \end{aligned}$$

The Jacobian of this transformation is $\langle\sigma\rangle^{2N}$ so

$$Q_1'(a^\#, b^\#)(z, \sigma) = \pi^{-2N}\langle\sigma\rangle^{2N} \int e^{2i\psi_1} g(z, u, v, M, N, \sigma)\, du\, dv\, dN\, dM. \tag{203}$$ [HHH2.124]

The transformed integrand in (203) has compact support and it satisfies the uniform bounds

$$|D_u^\alpha D_v^\beta D_M^\gamma D_N^\delta g| \le \|a\|_m \|b\|_{m'} \langle\sigma\rangle^{m+m'} \tag{204}$$ [HHH2.125]

where the norms (which depend on the exponents) on a and b are L-parabolic symbol norms.

Recall the lemma of stationary phase,

[HHH2.147]

Lemma 29. *If q is a real, non-degenerate, quadratic form on a vector space V and $\mathcal{C}_\infty^\infty(V)$ is the space of functions will all derivatives (uniformly) bounded then the integral extends by continuity (for the inductive limit topology) from the dense subspace, $\mathcal{C}_c^\infty(V)$ to a functional*

$$\langle v\rangle^\infty \mathcal{C}_\infty^\infty(V) = \bigcup_{m\in\mathbb{R}} \langle v\rangle^m \mathcal{C}_\infty^\infty(V) \ni g \longrightarrow \int e^{iq} g\, dv$$

which depends smoothly on the parameters of q and is such that

$$\left|\int e^{iq} g\, dv\right| \le C_N \int |g(v)|(1 + |v|)^{-N} dv,\ N \ge m + \dim V,\ g \in \langle v\rangle^m \mathcal{C}_\infty^\infty(V). \tag{205}$$ [HHH2.159]

Proof. Integration by parts. ☐

This lemma has an elegant reinterpretation in terms of the quadratic scattering wavefront set of Wunsch [**30**].

It applies directly to (203) to give a complete asymptotic expansion of the integral. Here we simply note the leading estimate

$$\|\langle\sigma\rangle_L^{-m-m'}Q(a^\#,b^\#)\|_{L^\infty}\le\|a\|_m\|b\|_{m'}\text{ in region (199)}^{[HHH2.126]}\tag{206}$$

for some norms on the parabolic symbol spaces.

3. The parabolic region$^{[S.parabolic.region]}$

It remains to consider the bilinear form, Q_2' which is given by (197) with an additional cutoff keeping

$$|\sigma_0|\ge c\langle\sigma\rangle,\ \ c>0.^{[HHH2.171]}\tag{207}$$

This region has two connected components, for simplicity of notation we shall assume that $\sigma_0>0$, the other case is not essentially different. We are interested in the behaviour as $\sigma_0\to\infty$. Now we may insert an additional cutoff to keep $|\mu_0|+|\nu_0|\le\frac12\sigma_0$, since in the complementary region integration by parts in u_0 and v_0 shows rapid decrease in σ_0, and hence $|\sigma|$. Combining (196) and (207) shows that, in the support of the integrand we have the estimates

$$\langle\mu+\sigma\rangle_L^2\ge\langle\mu_0+\sigma_0\rangle\ge c(\langle\mu_0\rangle+\langle\sigma_0\rangle),\ \ c>0$$

with a similar estimate for $\langle\nu+\sigma\rangle_L$. It follows that, as a function of μ_0, $a^\#$ satisfies the symbol estimates

$$|D_w^\alpha D_{\mu_0}^k a^\#(w,\mu+\sigma)|\le C\langle\mu+\sigma\rangle_L^m(\langle\mu_0\rangle+\langle\sigma_0\rangle)^{-k};^{[HHH2.177]}\tag{208}$$

$b^\#$ behaves similarly. We may use the rescaling, $\mu=\sigma_0M$, $\nu=\sigma_0N$ and then perform a stationary phase analysis in the (compactly supported) variables u,v,M_0,N_0 in which $\tilde M$ and $\tilde N$ are simply parameters. As a function of u,v,M_0,N_0, the phase function,

$$2(u_0M_0-v_0N_0+2\omega(\tilde u,\tilde v)+\tilde u\cdot\tilde M-\tilde v\cdot\tilde N)\sigma_0$$

has its critical point at

$$u_0=M_0=0,\quad v_0=N_0=0,\quad \hat u=\frac12\check N,\quad \check u=-\frac12\hat N,\quad \hat v=\frac12\check M\text{ and }\check v=-\frac12\hat M.$$

Applying stationary phase replaces Q_2' by an asymptotic sum of bilinear forms

(209)

$$\begin{gathered}Q_2'\sim\sum_{j\ge0}\sigma_0^{-j}\widetilde Q(G_j(a^\#\otimes b^\#))\text{ where,}\\ \widetilde Q(a^\#\otimes b^\#)=\pi^{-N+1}\int_{\mathbb R^{4n}}d\xi\,d\eta\,e^{i\omega(\xi,\eta)}H\text{ with}\end{gathered}$$

$$\begin{aligned}&H(a^\#\otimes b^\#)\\&=a^\#(z_0+\frac12\xi\cdot\tilde z\sigma_0^{-\frac12}+\frac14\omega(\eta,\xi)\sigma_0^{-1},\tilde z+\frac12\tilde\omega(\xi)\sigma_0^{-\frac12};\sigma_0,\sigma_0^{\frac12}(\xi+\tilde\sigma\sigma_0^{-\frac12}))\\&\quad b^\#(z_0+\frac12\eta\cdot\tilde z\sigma_0^{-\frac12}-\frac14\omega(\eta,\xi)\sigma_0^{-1},\tilde z+\frac12\tilde\omega(\eta)\sigma_0^{-\frac12};\sigma_0,\sigma_0^{\frac12}(\eta+\tilde\sigma\sigma_0^{-\frac12})).\end{aligned}$$

$^{[HHH2.163]}$

Here $\tilde{\omega}(\tilde{M}) = (\check{M}, -\hat{M})$ is the Hamilton map of the symplectic form ω and the variables have been again rescaled to $\xi = 2\tilde{M}\sigma_0^{\frac{1}{2}}$, $\eta = 2\tilde{N}\sigma_0^{\frac{1}{2}}$. The G_k are the bilinear differential operators

$$G_k = \frac{1}{k!}q^k; \tag{210}$$

[HHH2.170]

each G_k at most preserves the sum of the orders of $a^\#$ and $b^\#$. Thus $G_j(a^\# \otimes b^\#)$ is a sum of symbols evaluated at the same arguments as H in (209).

The bilinear form, $\widetilde{Q}(a^\# \otimes b^\#)$ can be interpreted as an isotropic product evaluated at $\tilde{\sigma}\sigma_0^{-\frac{1}{2}}$. To do that we need only check that, in terms of the frequency variables, $\xi' = \xi + \tilde{\sigma}\sigma_0^{-\frac{1}{2}}$, $\eta' = \eta + \tilde{\sigma}\sigma_0^{-\frac{1}{2}}$ the integrand satisfies the appropriate symbolic estimates, (100). We may rewrite the arguments in the base variables of $a^\#$ in (209) in terms of ξ' and η' as

$$\begin{aligned}
z_0 + \frac{1}{2}\xi \cdot \tilde{z}\sigma_0^{-\frac{1}{2}} + \frac{1}{4}\omega(\eta, \xi)\sigma_0^{-1} &= z_0 + \frac{1}{2}\xi' \cdot \tilde{z}\sigma_0^{-\frac{1}{2}} - \frac{1}{2}\tilde{\sigma} \cdot \tilde{z}\sigma_0^{-1} \\
&+ \frac{1}{4}\omega(\eta', \xi')\sigma_0^{-1} - \frac{1}{4}\omega(\tilde{\sigma}\sigma_0^{-\frac{1}{2}}, \xi')\sigma_0^{-1} - \frac{1}{4}\omega(\eta', \tilde{\sigma}\sigma_0^{-\frac{1}{2}})\sigma_0^{-1}, \\
\tilde{z} + \frac{1}{2}\tilde{\omega}(\xi)\sigma_0^{-\frac{1}{2}} &= \tilde{z} + \frac{1}{2}\tilde{\omega}(\xi')\sigma_0^{-\frac{1}{2}} - \frac{1}{2}\tilde{\omega}(\tilde{\sigma})\sigma_0^{-1}
\end{aligned}$$

and similarly for $b^\#$. It then follows from (193), recalling that $|\xi'|$ and $|\eta'|$ are bounded by multiples of $\sigma_0^{\frac{1}{2}}$, that H satisfies the symbolic estimates

$$|D_{\xi'}^\alpha D_{\eta'}^\beta H| \le C_{\alpha\beta}\sigma_0^{\frac{1}{2}(m+m')}\langle\xi'\rangle^{m-|\alpha|}\langle\eta'\rangle^{m'-|\beta|} \tag{211}$$

[HHH2.180]

where the constants are uniform in σ Thus (209) may be interpreted as the product formula in the isotropic calculus evaluated at $\tilde{\sigma}\sigma_0^{-\frac{1}{2}}$. The result is therefore a symbol of order $m + m'$ in these variables and so, given the estimate on the symbol norm above, satisfies

$$\begin{aligned}
|Q_2'(a^\#, b^\#)| &\le \|a\|_m \|b\|_{m'} \sigma_0^{(m+m')/2} \langle\tilde{\sigma}\sigma_0^{-\frac{1}{2}}\rangle^{m+m'} \\
&\le \|a\|_m \|b\|_{m'} \langle\sigma\rangle_L^{m+m'} \text{ in (207).}
\end{aligned}$$

Combined with (206) this is the leading parabolic symbol estimate on the amplitude of the composite.

To obtain the symbolic composition formula we expand the integrands, $G_j(a^\# \otimes b^\#)$ in Taylor series in the base variables about $(z_0, \tilde{z})$. For example

$$\begin{aligned}
H \sim \sum_{\mu,\nu} C_{\mu\nu} P_\mu a^\#(z; \sigma_0, \sigma_0^{\frac{1}{2}}(\xi + \tilde{\sigma}\sigma_0^{-\frac{1}{2}})) \times \\
P_\nu b^\#(z; \sigma_0, \sigma_0^{\frac{1}{2}}(\eta + \tilde{\sigma}\sigma_0^{-\frac{1}{2}}))\xi^\mu \eta^\nu \sigma_0^{-\frac{1}{2}(|\mu|+|\nu|)}
\end{aligned} \tag{212}$$

[c10.23.1998.2]

where the $\{P_\mu\}$ are constant coefficient polynomials in the vector fields defined in (188). Thus, for example the term $\widetilde{Q}(G_0(a^\# \otimes b^\#))$ in (209) can be replaced by the asymptotic sum:

$$\widetilde{Q}(G_0(a^{\#}\otimes b^{\#}))\sim\sum_{\mu,\nu}C_{\mu\nu}(-1)^{|\mu|}\pi^{-N+1}\int_{\mathbb{R}^{4n}}\tilde{\omega}(D_{\tilde{\sigma}})^{\nu}P_{\mu}a^{\#}(z;\sigma_0,\sigma_0^{\frac12}(\xi+\tilde{\sigma}\sigma_0^{-\frac12}))\tilde{\omega}(D_{\tilde{\sigma}})^{\mu}P_{\nu}b^{\#}(z;\sigma_0,\sigma_0^{\frac12}(\eta+\tilde{\sigma}\sigma_0^{-\frac12}))e^{i\omega(\xi,\eta)}d\xi d\eta \tag{213}$$ [c10.23.1998.3]

It follows from the symbolic estimates, (193) that this is well defined as an asymptotic sum. Indeed each term in the sum in (209) has a similar expansion. In light of the factors, σ_0^{-j} which appear in (209) there will arise only finitely many terms of any given degree and thus the double sum is well defined as an asymptotic expansion.

4. Heisenberg composition[S.Heisenberg.composition]

The estimates in the two preceeding sections lead directly to the basic composition result.

[HHH2.172]

PROPOSITION 8. *The bilinear form* (194) *is jointly continuous as map*

$$Q:\rho^{-m}\mathcal{A}_c({}^L\overline{T}^*\mathbb{R}^{2n+1})\times\rho^{-m'}\mathcal{A}_c({}^L\overline{T}^*\mathbb{R}^{2n+1})\longrightarrow\rho^{-m-m'}\mathcal{A}_c({}^L\overline{T}^*\mathbb{R}^{2n+1}) \tag{214}$$ [HHH2.173]

where the subscript 'c' denotes compactness of supports in the base variables. If a and b are symbols in these spaces then $c=\sigma_W(q_W(a)\circ q_W(b))$ has an asymptotic expansion, in terms of the coordinates, (z,σ), of the form

$$c^{\#}(z,\sigma)=\sum_{\mu\nu}C_{\mu\nu}\int_{\mathbb{R}^{4n}}e^{i\omega(\xi,\eta)}Q_{\nu}P_{\mu}a^{\#}(z;\sigma_0,\sigma_0^{\frac12}(\xi+\tilde{\sigma}\sigma_0^{-\frac12}))Q'_{\mu}P'_{\nu}b^{\#}(z;\sigma_0,\sigma_0^{\frac12}(\eta+\tilde{\sigma}\sigma_0^{-\frac12})d\xi d\eta. \tag{215}$$ [c10.23.98.4]

Here $C_{\mu\nu}$ are universal constants and Q_μ Q'_μ (respectively P_μ P'_μ) are universal differential operators in the z-variables (resp. in the σ-variables) which do not depend on m and m'.

PROOF. In the discussion leading to the statement of the proposition we have shown the leading estimate on the amplitude of the product

$$\|\langle\sigma\rangle_L^{-m-m'}Q(a^{\#},b^{\#})\|_{L^\infty}\le\|a\|_m\|b\|_{m'}.$$

To get the higher symbol estimates on the composite operator we differentiate (194) with respect to z and σ and observe that $D_z^\alpha D_\sigma^\beta c^{\#}(z,\sigma)$ is given by such a regularized integral with total symbol order $m+m'-|\beta|_L$. This is immediate for the z and $\tilde{\sigma}$ variables. Let D_0 and $\tilde{D}$ denote differentiation with respect to the first and the other fibre variables. Then differentiation with respect to the first fibre variable, σ_0, gives after an integration by parts

$$D_0c(z,\sigma)=Q(D_0a\otimes b+a\otimes D_0b+\frac12\omega(\tilde{D}a,\tilde{D}b)).$$

Again this is a linear combination of products with total order $m+m'-2$. The general case follows by induction, so (214) is proved.

By the argument above the asymptotic expansions for the symbol of the composite are valid in regions of the form $|\tilde{\sigma}| < \pm c\sigma_0$ and in fact determine the symbol of the composite operator up to a rapidly vanishing error in the complement of $\sigma_0 = 0$. On the other hand we have shown that the symbol of the composite belongs to $\rho^{-m-m'}\mathcal{A}_{\mathrm{c}}({}^L\overline{T}^*\mathbb{R}^{2n+1})$ and therefore the asymptotic expansions must match up along the equitorial subbundle and therefore completely describe the symbol in the full cotangent space. □

Since this is the local form of the composition where we already know the coordinate invariance we immediately conclude that for any compact contact manifold

$$\Psi^m_{\mathrm{H},\infty}(X) \circ \Psi^{m'}_{\mathrm{H},\infty}(X) \subset \Psi^{m+m'}_{\mathrm{H},\infty}{}^{[HHH2.150]} \tag{216}$$

is a filtered algebra.

5. The subalgebra $\Psi^*_{\mathbf{H}}(X)$ [S.classical.subalgebra]

To prove that the classical subspace is a subalgebra we only need to investigate the effect of applying the radial vector field R_L appearing in Lemma 6 in the formula (194). Let $R_u = 2u_0\partial_{u_0} + \tilde{u}\cdot\partial_{\tilde{u}}$ denote the parabolic vector field in the variables u. Then the phase function in (194) satisfies the identity

$$R(u\cdot\mu - v\cdot\nu + 2\omega(\tilde{u},\tilde{v})\sigma_0) = 0 \text{ where } R = (R_\mu + R_\nu + R_\sigma - R_u - R_v).^{[HHH2.181]} \tag{217}$$

The divergence of R is $2n+2$, so $R^t = -R - 2n - 2$. Applying R_σ to (194) and integrating by parts then gives

$$Rc^\# = Q_{\mathrm{H}}(Ra^\# \otimes b^\# + a^\# \otimes Rb^\# + E_1(a^\# \otimes b^\#))^{[HHH2.151]} \tag{218}$$

where R denotes the application of R_L in the last $2n+1$ variables and E_1 is a smooth bilinear differential operator (so preserves classical expansions) which lowers the total order by at least 1. Explicitly,

$$\begin{aligned} &E_1(a^\# \otimes b^\#)(u,\nu;v,\mu) = \\ &[(D_{\mu_0} - \omega(D_{\tilde{\mu}}, D_{\tilde{\nu}}) - \frac{1}{2}\omega(D_{\tilde{\mu}},\tilde{z}))\partial_{u_0} + (D_{\nu_0} - \omega(D_{\tilde{\mu}}, D_{\tilde{\nu}}) + \frac{1}{2}\omega(D_{\tilde{\nu}},\tilde{z}))\partial_{v_0} \\ &\qquad - \frac{1}{2}D_{\tilde{\mu}}\cdot\partial_{\tilde{u}} + \frac{1}{2}D_{\tilde{\nu}}\cdot\partial_{\tilde{v}}]a(u,\nu)b(v,\mu).^{[HHH2.178]} \end{aligned} \tag{219}$$

It follows that if $R-m$ reduces the order of a by 1 and $R-m'$ reduces the order of b by 1 then $R-m-m'$ reduces the order of c by 1. Iteration of the argument now shows that

$$\Psi^m_{\mathrm{H}}(X) \circ \Psi^{m'}_{\mathrm{H}}(X) \subset \Psi^{m+m'}_{\mathrm{H}}(X),^{[HHH2.152]} \tag{220}$$

as anticipated. We have now, in essence, proved the basic results on composition of Heisenberg pseudodifferential operators.

[HHH2.174]

Theorem 1. *(Beals-Greiner* [**2**]*, Taylor* [**27**]*; see also Dynin* [**8**], [**9**]*) On any compact contact manifold the space of Heisenberg operators $\Psi^m_{H,\infty}(X;E)$ acting on sections of a vector bundle E form a $*$-closed order-filtered algebra; the subspaces of 'classical' elements, $\Psi^m_H(X;E)$, are preserved under composition and in particular the subspaces $\Psi^{\mathbb{Z}}_H(X;E)$ of classical elements of integral order is also a filtered $*$-algebra.*

PROOF. Only the addition of the bundle coefficients needs to be examined and this is minor since we already know that the kernels form a $\mathcal{C}^\infty(X^2)$ module we may define the general class of operators from one bundle to another as

$$\Psi^m_{\mathrm{H}}(X;E,F)=\Psi^m_{\mathrm{H}}(X)\otimes_{\mathcal{C}^\infty(X^2)}\mathcal{C}^\infty(X;\mathrm{Hom}(E;F)).^{[HHH2.212]} \tag{221}$$

Composition results are immediate. □

6. Composition for a general line bundle[S.General.composition]

In the general case of a trivial line subbundle of T^*X we cannot arrange such a uniform expression as (187) for a section of S. However, we may easily arrange this 'up to second order'. That is we may choose local coordinates z_0, $\hat z$, $\check z$ and $\dot z$, consisting respectively of 1, n', n' and n'' individual variables, such that a local section is of the form

$$\begin{gathered}\alpha=dz_0+\frac12\sum_{j=1}^{n'}(\check\alpha_j(z)d\hat z_j-\hat\alpha_j(z)d\check z_j)+\sum_{l=1}^{n''}\beta_l(z)d\dot z_l,\\ \check\alpha_j(z)=\check z_j+)(|z|^2),\ \hat\alpha_j=\hat z_j+O(|z|^2),\ \beta_l=O(|z|^2).^{[HHH2.547]}\end{gathered}\tag{222}$$

As corresponding basis of vector fields we take, in place of (188)

$$\begin{gathered}Z_0=\partial_{z_0},\ \hat Z_j=\partial_{\hat z_j}-\frac12\check\alpha_j(z)\partial_{z_0},\ \check Z_j=\partial_{\check z_j}+\frac12\hat\alpha_j(z)\partial_{z_0},\ \dot Z_l=\partial_{\dot z_j}-\beta_l(z)\partial_{z_0\,[HHH2.548]}\\ \alpha(\hat Z_j)=\alpha(\check Z_j)=\alpha(\dot Z_l)0,\ j=1,\dots,n',\ l=1,\dots,n''.\end{gathered}\tag{223}$$

This yields the fibre coordinates on the cotangent bundle

$$\begin{gathered}\sigma_0=\zeta_0,\ \hat\sigma_j(z,\zeta)=\sigma(\hat Z_j)/i=\hat\zeta_j-\frac12\check\alpha_j(z)\zeta_0,\\ \check\sigma_j(z,\zeta)=\sigma(\check Z_j)/i=\check\zeta_j+\frac12\hat\alpha_j(z)\zeta_0,\ j=1,\dots,n\text{ and}\\ \dot\sigma_l=\sigma(\dot Z_l)/i=\dot\zeta_l-\beta_(z)\zeta_0,\ l=1,\dots,n''.\end{gathered}$$

Now with $\sigma=(\zeta_0,\hat\sigma,\check\sigma,\dot\sigma)$ the formula for Weyl quantization becomes (189), (190) but in place of (191) and (192)

$$\begin{gathered}\Phi(z-z',z',\sigma)=((z_0-z_0')+\gamma(\tilde z-\tilde z',\frac12(z+z')))\sigma_0+(\tilde z-\tilde z')\cdot\tilde\sigma,\\ \gamma(\tilde z,z')=\frac12\sum_{j=1}^{n'}[\hat z_j\hat\alpha_j(z')-\check z_j\check\alpha_j(z')]+\sum_{l=1}^{n''}\dot z_l\beta_l(z')\text{ so}\\ \gamma(\tilde z-\tilde z',\frac12(z+z'))=\frac12(\hat z\cdot\check z'-\check z\cdot\hat z')+O(|z-z'||z+z'|^2).^{[HHH2.549]}\end{gathered}\tag{224}$$

7. Extended Heisenberg composition[S.Extended.algebra]

Composition of elements of the extended Heisenberg calculus can be discussed starting from the same formula, (194), which is valid for rapidly decreasing symbols. [HHH2.213]

THEOREM 2. *On any compact contact manifold the spaces of extended Heisenberg operators $\Psi^{m,k}_{eH,\infty}(X;E)$, acting on sections of a vector bundle E, form a $*$-closed order bifiltered algebra; the subspaces of 'classical' elements $\Psi^{m,k}_{eH}(X;E)$ are* *preserved under composition and in particular $\Psi^{\mathbb{Z},\mathbb{Z}}_{eH}(X;E)$ is an algebra. If the contact bundle is oriented then the two Heisenberg orders are separately filtered.*

PROOF. We only discuss the minor changes needed to the argument above. First, the symbol estimates on the 'straightened' amplitudes become

$$|D^\alpha_z D^\beta_\sigma a^\#(z,\sigma)| \le C_{\alpha\beta}\langle\sigma\rangle^{-m+k-\beta_0}\langle\sigma\rangle_L^{2m-k-|\tilde\beta|} \text{[HHH2.179]} \tag{225}$$

as follows from Lemma 12. Assuming that $b^\#$ satisfies the corresponding estimates with multi-order m',k' we wish to show at first the leading estimate for $c^\#$ with multi-order $m+m',k+k'$.

The discussion concerning (195) is unchanged and allows a cut-off function to be inserted in (194) ensuring the support property (196).

Proceeding under the restriction (199) we get in place of (200)

$$|\partial^\alpha_w\partial^\beta_\mu a^\#(w,\mu+\sigma)| \le C(\langle\mu\rangle+\langle\sigma\rangle)^{-m+k-\beta_0}(\langle\mu\rangle_L+\langle\sigma\rangle)^{2m-k-|\tilde\beta|}.$$

Under the same rescaling, (201), this leads to essentially the same estimate (204) but with the appropriate norms. This leads to the estimate corresponding to (206) which can be written

$$\sup_{\text{Region (199)}} \|\langle\sigma\rangle^{-m-m'}Q(a^\#,b^\#)\|_{L^\infty} \le \|a\|_{m,k}\|b\|_{m',k'}.$$

In the complementary region (207) we have in addition $\langle\mu+\sigma\rangle \ge c(\langle\sigma\rangle+\langle\mu\rangle)$, $c>0$, because of (196) and hence in place of (208) we find

$$|D^\alpha_w D^l_{\mu_0} a^\#(w,\mu+\sigma)| \le C\langle\mu+\sigma\rangle_L^{2m-k}\langle\mu+\sigma\rangle^{-m+k}(\langle\mu_0\rangle+\langle\sigma_0\rangle)^{-l};$$

This allows the same rescaling and stationary phase analysis leading to (209) with the estimates (recalling that ξ' and η' are bounded with respect to $\sigma_0^{\frac12}$) (211) altered to

$$|D^\alpha_{\xi'}D^\beta_{\eta'}G| \le C\sigma_0^{\frac12(k+k')}\langle\xi'\rangle^{2m-k-|\alpha|}\langle\eta'\rangle^{2m'-k'-|\beta|}.$$

The result is thefore a symbol of order $2(m+m')-k-k'$, evaluated at $\sigma_0^{-\frac12}\tilde\sigma$ and with an additional factor of $\sigma_0^{\frac12(k+k')}$. This gives, in the region (207) the leading estimate

$$|c^\#(z,\sigma)| \le \|a\|_{m,k}\|b\|_{m',k'}\langle\sigma\rangle^{k+k'-m-m'}\langle\sigma\rangle_L^{2(m+m')-k-k'}.$$

Since the estimates (225) just mean that ∂_z leaves the orders unchanged, $\partial_{\tilde\sigma}$ lowers both orders by 1 whereas ∂_{σ_0} lowers k by 2 and m by 1, the full symbol estimates just follow by differentiation as before.

Similarly the fact that the composite of operators with classical symbols is classical now follows by applying radial vector fields. As noted in §5 the parabolic radial vector field is radial for the parabolic boundary and the standard radial vector field is radial for the other boundary hypersurface of ${}^L\overline{T}^*X$. The discussion of the parabolic radial vector field then carries over essentially unchanged to the extended calculus; the difference is that E_{-1} only lowers the *parabolic order*. The corresponding discussion for the usual radial vector field, R', is similar but simpler (indeed it is essentially the proof of composition properties for classical operators). In particular, (217) may be replaced by

$$(R'_\mu+R'_\nu+R'_\sigma)-R'_u-R'_v)(u\cdot\mu-v\cdot\nu+2\omega(\tilde u,\tilde v)\sigma_0) = -2\omega(\tilde u,\tilde v)\sigma_0.$$

Again integration by parts gives an 'error' term which is lower order on the *classical* face.

The independence of the orders $k_\pm$ at the two components of the Heisenberg boundary of ${}^{\mathrm{eL}}\overline{T}^*X$ is clear from the discussion following (207) since the regions $\sigma_0 > 0$ and $\sigma_0 < 0$ do not interact at finite order for the symbols. □

The symbolic composition properties are again easily deduced. From (215) we obtain the symbol along the parabolic boundary faces of ${}^{L}\overline{T}^*X$. In the previous case this completely determined the symbol as the two parabolic boundary components meet along a hypersurface. In the case at hand we also need a composition formula for the symbol along the radial boundary hypersurface. As the composition reduces to the classical computation along this face, the germ of the symbol c at (z,ζ) is determined by the germs of the symbols of a and b at the same point. In light of this observation the symbol c at such points is unchanged if we cut a and b off in a neighborhood of the parabolic boundary components. Therefore we can simply use the classical Weyl composition formula,

$$c(z,\zeta) \sim \sum_{0\le k} \frac{i^k}{k!}(D_u\cdot D_\eta - D_v\cdot D_\xi)^k[a(u,\xi)b(v,\eta)]\big|_{u=v=z,\xi=\eta=\zeta}.$$

The expansion is uniformly valid for $|\zeta_0| < \delta|\widetilde{\zeta}|$ for any $\delta > 0$. That the expansions must match up in the corners is an immediate consequence on our basic composition theorem and the fact that composition preserves the classical subsapces.

It follows from this composition result that, if L is trivial, then the subspaces $\mathcal{I}^k_{\mathrm{H},\pm}(X;E)$ are indeed filtered ideals in both the extended and the Heisenberg algebras. Even if L is not orientable there is one (filtered) ideal corresponding to elements of the Heisenberg algebra with Riemann-Weyl amplitude vanishing to infinite order at the equatorial sphere in the boundary of ${}^{L}\overline{T}^*X$.

8. Conformal symplectic bundles[SS.conformal.symplectic.bundle]

Symplectic bundles, together with their associated bundles of isotropic algeras, are discussed in SS4.6. More generally, a conformal symplectic bundle H is a vector bundle with a trivial line subbundle $G \subset \Lambda^2 H'$ specified such that any non-vanishing section of G is a symplectic form, i.e. is non-degenerate. Though any conformal symplectic bundle has a symplectic structure, coming from a section of G, there may simply not be a natural choice.

As usual with such structures we may replace a conformal symplectic bundle H by a naturally symplectic vector bundle $\tilde{H}$ which is bundle isomorphic to it. To construct $\tilde{H}$, observe that the absolute real powers of a trivial line bundle, such as G, make invariant sense. A section of $|G|^a$ is simply a smooth absolutely a-homogeneous function on $G'\setminus 0$, $f \in \mathcal{C}^\infty(G'\setminus 0)$ such that $f(sx) = |s|^a f(x)$ for all $s \in \mathbb{R}\setminus 0$.

[HHH2.182]

LEMMA 30. *If H is a conformal symplectic vector bundle with $G \subset \Lambda^2 H'$ the associated trivial line bundle of non-degenerate forms, then $\tilde{H} = H \otimes G^{\frac12}$ is canonically a symplectic vector bundle for each choice of orientation of G.*

PROOF. Since any positive section of $G^{\frac{1}{2}} \otimes G^{\frac{1}{2}} \equiv G$ is a symplectic form on H, the symplectic structure on $\tilde{H}$ is the bilinear form

$$\omega(h \otimes s, h' \otimes s') = (s \otimes s')(h, h').$$

This is clearly independent of choices. □

More generally a vector bundle H may have a projective symplectic structure, in the sense that there is a subbundle $G \subset \Lambda^2 H'$ any local non-vanishing section of which is a local symplectic form on the fibres of H, but G may not be trivial. If G is not trivial then the 'sphere bundle' of G, $\mathbb{S}G \longrightarrow Y$ is an oriented double cover of Y and H lifts to a conformal symplectic vector bundle over $\mathbb{S}G$. Thus the associated symplectic vector bundle $\tilde{H}$ is well-defined as a bundle over $\mathbb{S}G$ rather than Y.

If W is a symplectic bundle then, as shown in SS2.1, the space of 'quadratic' symbols on the fibres of W is a filtered algebra, where the product is the isotropic product on each fibre. In particular $\mathcal{C}^\infty({}^q\overline{W})$ is an algebra. The 'negative' of the symplectic bundle, W_-, which is the same bundle but with the symplectic form $-\omega$ defines an isomorphic algebra $\mathcal{C}^\infty({}^q\overline{W}_-)$. Now the union of these two compactified spaces is a sphere bundle, which can be naturally identified as the parabolic sphere at infinity ${}^{\mathbb{R}}\mathbb{S}(W \times \mathbb{R})$. The Schwartz class functions on $W_\pm$ define two-sided ideals with repsect to the isotropic product. We denote these by $\Psi^{-\infty}_{\text{is}}(W_\pm)$ to emphasize that we are using the isotropic product.

[HHH2.186]

LEMMA 31. *If H is a conformal symplectic vector bundle and L is the dual of the line bundle of conformal symplectic structures on H then the isotropic calculus on $W = H \otimes L^{\frac{1}{2}}$ induces, via pull-back under* (50) *for both signs of orientation, a non-commutative product on* $\mathcal{C}^\infty({}^L\mathbb{S}(H \times L))$.

Proof incomplete

PROOF. By Lemma 10, for each choice of orientation of L, the smooth functions on ${}^q\overline{W}$ lift to be those smooth functions on $\overline{W}$ with only even terms in their expansion at infinity along each radial line. This characterization of the smooth functions on the two half-spheres forming ${}^{\mathrm{L}}\mathbb{S}(H \oplus L)$ combines to characterize the smooth functions on ${}^{\mathrm{L}}\mathbb{S}(H \oplus L)$ as those functions smooth on each half-sphere and with matching Taylor series at the equatorial sphere, i.e. with coefficients of powers r^{-4k} equal and coefficients of r^{-4k-2} of equal after change of sign, for $k = 0, 1, \dots$. The expansion of the isotropic product in (109) shows that the isotropic product on the interior of the two half-spaces preserves this compatibility condition. It therefore induces a natural product on $\mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}(H \times L))$ as claimed. □

A description of the Heisenberg symbol in this form was given by Taylor in [**27**].

9. Heisenberg symbol[SS.Heisenberg.symbol]

If X is a compact manifold with oriented contact line bundle $L \subset T^*X$ then $H = T^*X/L$ is a conformal symplectic bundle with L the dual of the bundle of symplectic forms. By Lemma 10 the boundary ${}^{\mathrm{L}}\mathbb{S}^*X$ of the L-parabolic compactification of T^*X may be smoothly identified with the union of the two sphere bundles ${}^q\overline{W}_+$ and ${}^q\overline{W}_-$ where $W = (T^*X/L) \otimes L^{\frac{1}{2}}$ and the signs arise from the choice of orientation. Thus Lemma 31 shows that the fibres of $\mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}^*X)$ has a non-commutative product which is, on each fibre, two copies of the isotropic calculus. This is the symbol algebra for Heisenberg operators of order 0.

Lemma 11 shows that

$$\mathcal{C}^\infty({}^q\overline{W}_+;B^k)\equiv\rho_q^{-\frac{1}{2}k}\mathcal{C}^\infty({}^q\overline{W}_+;L^{\frac{1}{2}k})^{[HHH2.235]}\tag{226}$$

where L is a bundle over X, so is trivial on the fibres of ${}^q\overline{W}_+$. Thus, recalling the convention on orders which follows from the inclusion $\rho_q^{-\frac{1}{2}k}\mathcal{C}^\infty({}^q\overline{W})\subset\rho^{-k}\mathcal{C}^\infty(\overline{W})$, we may identify this space with the fibre-isotropic space

$$\mathcal{C}^\infty({}^q\overline{W}_\pm;B^k)\equiv\Psi^k_{\mathrm{Is}}(W_+;L^{\frac{1}{2}k}).^{[HHH2.236]}\tag{227}$$

This gives these spaces the structure of a graded algebra with an isotropic product

$$\mathcal{C}^\infty({}^q\overline{W}_\pm;B^k)\#\mathcal{C}^\infty({}^q\overline{W}_\pm;B^{k'})\subset\mathcal{C}^\infty({}^q\overline{W}_\pm;B^{k+k'});^{[HHH2.237]}\tag{228}$$

the negative case follows by reversing the orientation of L.

[HHH2.234]

THEOREM 3. *For any compact contact manifold with oriented contact bundle L the products* (228) *are consistent with the identification in Lemma 10 of ${}^L\mathbb{S}^*X={}^q\overline{W}_+\cup{}^q\overline{W}_-$ and so combine to give $\oplus_k\mathcal{C}^\infty({}^L\mathbb{S}^*X;B^k)$ the structure of a graded algebra; the principal symbol map* (88) *then identifies $\Psi^k_H(X)/\Psi^{k-1}_H(X)$ and $\mathcal{C}^\infty({}^L\mathbb{S}^*X;B^k)$ as graded algebras.*

PROOF. The persistence of the compatibility conditions across the equatorial subsphere of ${}^{\mathrm{L}}\mathbb{S}^*X$ follows as in the proof of Lemma 31. In any case it is a consequence of the second part of the theorem, and is not used in the proof.

The formula for the composition of the symbols of Heisenberg pseudodifferential operators follows from (209). Note that σ_0 is exactly the function f constructed in the proof of Lemma 11. The principal symbol of the composite arises from the $j=0$ term in the sum in (209) and the bilinear form G may be simplified by Taylor expansion in the base variables. Thus G produces the same leading term as

$$G_0=a^\#(z_0,\tilde z;\sigma_0,\sigma_0^{\frac{1}{2}}(\xi+\tilde\sigma\sigma_0^{-\frac{1}{2}}))b^\#(z_0,\tilde z;\sigma_0,\sigma_0^{\frac{1}{2}}(\eta+\tilde\sigma\sigma_0^{-\frac{1}{2}})).$$

This is just the isotropic composition formula for the symplectic vector space $T^*X/L\otimes L^{\frac{1}{2}}$. □

The symbolic composition properties of the extended Heisenberg calculus are very similar. Recall that the boundary of ${}^{\mathrm{eL}}\overline{T}^*X$ has three components, ${}^q\overline{W}_\pm$ and $\mathbb{S}^*_RX$. The last is the radial boundary component, it is diffeomorphic to $\mathbb{S}T\times[-1,1]$, where $\mathbb{S}T=\partial{}^q\overline{W}_+$ is the equatorial sphere bundle of ${}^L\overline{T}^*X$. The boundary of $\mathbb{S}^*_RX$ has two components which we label according to our choice of orientation, $\mathbb{S}T_\pm=\mathbb{S}^*_RX\cap{}^q\overline{W}_\pm$. The normal bundles to the boundary components are slightly different. Writing B_L and B_R for the normal bundles to the Heisenberg and radial boundary components of ${}^{\mathrm{eL}}\overline{T}^*X$ and B for the normal bundle to ${}^q\overline{W}_+$ as part of the boundary of ${}^L\overline{T}^*X$ we find

$$\mathcal{C}^\infty({}^{\mathrm{eL}}\mathbb{S}^*X;B^k_LB^m_R)\big|_{{}^q\overline{W}_+}=\rho^{-m+k}\mathcal{C}^\infty({}^q\overline{W}_+;B^k).^{[HHH2.238]}\tag{229}$$

The further identification (226) then gives the bi-graded algebra structure

$$\mathcal{C}^\infty({}^{\mathrm{eL}}\mathbb{S}^*X;B^k_LB^m_R)\big|_{{}^q\overline{W}_+}=\Psi^{m-\frac{1}{2}k}_{\mathrm{Is}}({}^q\overline{W}_+;L^{\frac{1}{2}k}).^{[HHH2.239]}\tag{230}$$

Here, as before, L is a bundle over X, so is canonically trivial on the fibres of ${}^q\overline{W}_+$ as a ball bundle over X.

The principal symbol map for the (quadratic) isotropic calculus gives a surjective homorphism to the commutative graded algebra

$$\Psi_{\mathrm{Is}}^{m-\frac{1}{2}k}({}^q\overline{W}_+;L^{\frac{1}{2}k}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}T;B_R^{m-\frac{1}{2}k}L^{\frac{1}{2}k}) \equiv \mathcal{C}^\infty(\mathbb{S}T;B_R^mB_L^k)^{[HHH2.240]} \tag{231}$$

B_R is the normal bundle to the boundary of ${}^q\overline{W}_+$, hence also the normal bundle to the radial part of the boundary of ${}^{\mathrm{eL}}\overline{T}^*X$. On the right in (231) we have used (230) to restore the normal bundles to the boundary of ${}^{\mathrm{eL}}\overline{T}^*X$. Similar identifications hold for the reversal of the orientation of L. Thus we see that the full 'symbol algebra' for the extended Heisenberg calculus is

$$\begin{gathered}\mathcal{C}^\infty({}^{\mathrm{eL}}\mathbb{S}^*X;B_L^kB_R^m) = \\ \big\{(a_+,b,a_-)\in\Psi_{\mathrm{Is}}^{m-\frac{1}{2}k}({}^q\overline{W}_+;L^{\frac{1}{2}k})\oplus\mathcal{C}^\infty(\mathbb{S}_R^*X;B_R^mB_L^k)\oplus\Psi_{\mathrm{Is}}^{m-\frac{1}{2}k}({}^q\overline{W}_-;L^{\frac{1}{2}k}); \\ \sigma_{\mathrm{is}}(a_+)=b|_{\mathbb{S}T_+},\ \sigma_{\mathrm{is}}(a_-)=b|_{\mathbb{S}T_-}\big\},^{[HHH2.241]}\end{gathered} \tag{232}$$

with the product being the commutative product on the radial part and the isotropic product on the Heisenberg part.

[HHH2.242]

THEOREM 4. *For any compact contact manifold with oriented contact bundle L the product on* (232) *is consistent with the principal symbol map* (89) *which then identifies* $\Psi_{eH}^{m,k}(X)/\Psi_{eH}^{m-1,k-1}(X)$ *and* $\mathcal{C}^\infty({}^{eL}\mathbb{S}^*X;B_L^kB_R^m)$ *as bi-graded algebras.*

10. Composition in terms of the Heisenberg group [S.Flat]

We give a short calculation of an explicit formula giving the asymptotic expansion of the symbol of a composition of two Weyl quantized, Heisenberg symbols. This calculation closely follows the analogous computation for the Weyl quantization in the Euclidean case. In order to keep track of the homogeneities of the various terms we make extensive usage of the Heisenberg group structure.

Letting $x=(x_0,x'),y=(y_0,y')$, $x',y'\in\mathbb{R}^{2n}$ denote a pair of points in $\mathbb{H}^n$ we express the binary operation in $\mathbb{H}^n$ as:

$$x\cdot y = x+y+\omega(x',y')e_0.^{[HHH.401]} \tag{233}$$

Here ω is a symplectic form on $\mathbb{R}^{2n}$ and $e_0=(1,0\dots,0)$. Observe that $x^{-1}=-x$ and there is a linear transformation, $L_{x'}$ depending linearly on x' such that

$$x\cdot y^{-1} = L_{x'}(x-y).^{[HHH.402]} \tag{234}$$

If $x'=(\tilde{x},\hat{x})$ and we fix

$$\omega(x',y')=\frac{1}{2}[\tilde{x}\cdot\hat{y}-\hat{x}\cdot\tilde{y}]$$

then

$$L_{x'}=\begin{pmatrix}1 & -\frac{\hat{x}}{2} & \frac{\tilde{x}}{2}\\ 0 & \mathrm{Id} & 0\\ 0 & 0 & \mathrm{Id}\end{pmatrix}.$$

The contact form defines a global dilation structure in the fibers of the tangent and cotangent spaces as follows: Let θ denote the contact form and T the characteristic vector field defined by the conditions:

$$\theta(T)=1,\ i_Td\theta=0.^{[HHH.400]} \tag{235}$$

This allows us to globally split the tangent and cotangent bundles as:

$$T\mathbb{H}^n = \mathbb{R}T \oplus \ker\theta, \quad T^*\mathbb{H}^n = \mathbb{R}\theta \oplus T^\perp.$$

Let $p^1 : T^*\mathbb{H}^n \to T^\perp$ and $p^2 : T^*\mathbb{H}^n \to \mathbb{R}\theta$ denote the projections defined by the splitting. Note that in the natural linear coordinates on $T^*\mathbb{H}^n$ these projections depend non–trivially on the base variable. We define the dilation structure by

$$M_t(x,\xi) = t^2 p_x^2(\xi) + t p_x^1(\xi).$$

On the Heisenberg group we can introduce global coordinates on the cotangent space (x,σ) such that the dilation structure on the fibers of the bundle is globally trivialized. We let

$$\sigma(x,\xi) = (L_{x'}^t)^{-1}\xi$$

where $L_{x'}$ is the linear transformation appearing in (234). In these coordinates:

$$M_t(x,\sigma) = (x,(t^2\sigma_0, t\sigma_1,\dots,t\sigma_{2n})).$$

The functions $\sigma_i(x,\xi)$ have an interpretation as the classical symbols of the left invariant vector fields $(-iT, -iX_1,\dots,-iX_{2n})$ on $\mathbb{H}^n$. Let $a(x,\xi)$ denote a globally defined parabolic symbol on $\mathbb{H}^n$. We define $\mathfrak{a}(x,\sigma) = a(x, L_{x'}^t\sigma)$, i.e. $a(x,\xi) = \mathfrak{a}(x,\sigma(x,\xi))$. Define the left, right and Weyl quantizations as follows:

$$\begin{aligned} a^L(X,D)u(x) &= \frac{1}{(2\pi)^{2n+1}}\iint a(x,\xi)e^{i<x-y,\xi>}u(y)dyd\xi, \\ &= \frac{1}{(2\pi)^{2n+1}}\iint \mathfrak{a}(x,\sigma)e^{i<x\cdot y^{-1},\sigma>}u(y)dyd\sigma, \end{aligned} \tag{236}$$

[HHH.403a]

$$\begin{aligned} a^R(X,D)u(x) &= \frac{1}{(2\pi)^{2n+1}}\iint a(y,\xi)e^{i<x-y,\xi>}u(y)dyd\xi, \\ &= \frac{1}{(2\pi)^{2n+1}}\iint \mathfrak{a}(y,\sigma)e^{i<x\cdot y^{-1},\sigma>}u(y)dyd\sigma, \end{aligned} \tag{237}$$

[HHH.403b]

$$\begin{aligned} a^W(X,D)u(x) &= \frac{1}{(2\pi)^{2n+1}}\iint a(\frac{x+y}{2},\xi)e^{i<x-y,\xi>}u(y)dyd\xi, \\ &= \frac{1}{(2\pi)^{2n+1}}\iint \mathfrak{a}(\frac{x+y}{2},\sigma)e^{i<x\cdot y^{-1},\sigma>}u(y)dyd\sigma. \end{aligned} \tag{238}$$

[HHH.403c]

The second set of formulæ follow from the fact that

$$x\cdot y^{-1} = L_{\frac{x'+y'}{2}}(x-y).$$

It is a simple consequence of the fact, $y^{-1} = -y$ that the Weyl quantization has the expected symmetry property: the symbol of the adjoint operator, $a^W(X,D)^*$ is $\bar{a}(x,\xi)$. From these equations we easily derive formulæ for the Schwartz kernels, K^L, K^R, K^W, respectively which in turn lead to the following expressions for the respective symbols as Fourier transforms:

$$\begin{aligned} \mathfrak{a}(x,\sigma) &= \int K^L(x, t^{-1}\cdot x)e^{-i<t,\sigma>}dt, \\ \mathfrak{a}(x,\sigma) &= \int K^R(t\cdot x, x)e^{-i<t,\sigma>}dt, \\ \mathfrak{a}(x,\sigma) &= \int K^W(\left(\frac{t}{2}\right)\cdot x, \left(\frac{t}{2}\right)^{-1}\cdot x)e^{-i<t,\sigma>}dt. \end{aligned} \tag{239}$$

[HHH.404]

In this section we obtain an explicit composition formula for the Weyl quantization.

We begin with $\mathfrak{a}, \mathfrak{b}$ Schwartz class functions on $T^*\mathbb{H}^n$ which we assume have compact support in the base variables.

(240) $$a^W(X,D)b^W(X,D)f(x) = \frac{1}{(2\pi)^{4n+2}} \iiiint \mathfrak{a}(\frac{x+z}{2},\eta)\mathfrak{b}(\frac{z+y}{2},\xi)e^{i[<x\cdot z^{-1},\eta>+<z\cdot y^{-1},\xi>]}f(y)dyd\xi dz d\eta. \quad [HHH.405]$$

Let $\mathfrak{c}(x,\theta)$ denote the Weyl–symbol of the composition, as an oscillatory integral it is given by:

(241) $$\mathfrak{c}(x,\theta) = \frac{1}{(2\pi)^{4n+2}} \iiiint \mathfrak{a}(\frac{\left(\frac{t}{2}\right)\cdot x+z}{2},\eta)\mathfrak{b}(\frac{\left(\frac{t}{2}\right)^{-1}\cdot x+z}{2},\xi) e^{i[<\left(\frac{t}{2}\right)\cdot x\cdot z^{-1},\eta>+<z\cdot x^{-1}\cdot\left(\frac{t}{2}\right),\xi>-<t,\theta>]}d\xi dz d\eta dt. \quad [HHH.406]$$

We carry through the calculations with Schwartz-class symbols as above, using continuity of the final result in the symbolic semi-norms to obtain the formula for an arbitrary pair of Heisenberg symbols.

A computation shows that

(242) $$\left(\frac{t}{2}\right)\cdot x\cdot z^{-1} + z\cdot x^{-1}\cdot\left(\frac{t}{2}\right) = L_{z'-x'}t \quad [HHH.407]$$

and therefore

(243) $$\mathfrak{c}(x,\theta) = c_n' \iiiint \mathfrak{a}(\frac{\left(\frac{t}{2}\right)\cdot x+z}{2},\eta)\mathfrak{b}(\frac{\left(\frac{t}{2}\right)^{-1} x+z}{2},\xi) e^{i[<\left(\frac{t}{2}\right)\cdot x\cdot z^{-1},\eta-(L^t_{z'-x'})^{-1}\theta>+<z\cdot x^{-1}\cdot\left(\frac{t}{2}\right),\xi-(L^t_{z'-x'})^{-1}\theta>]}d\xi dz d\eta dt. \quad [HHH.408]$$

Here and in the sequel c_n' and c_n are dimensional constants. In the final formula we will determine the value of c_n. We change the order of the integrations and change variables

$$(\xi,\eta) \longrightarrow ((L^t_{z'-x'})^{-1}\xi, (L^t_{z'-x'})^{-1}\eta)$$

to obtain:

(244) $$c(x,\theta) = c_n' \iiiint \mathfrak{a}(\frac{\left(\frac{t}{2}\right)\cdot x+z}{2},L^t_{z'-x'}\eta)\mathfrak{b}(\frac{\left(\frac{t}{2}\right)^{-1} x+z}{2},L^t_{z'-x'}\xi) e^{i[<(L_{z'-x'})^{-1}\left(\frac{t}{2}\right)\cdot x\cdot z^{-1},\eta-\theta>+<(L_{z'-x'})^{-1}z\cdot x^{-1}\cdot\left(\frac{t}{2}\right),\xi-\theta>]}d\xi dz d\eta dt,$$

which equals

(245)
$$\mathfrak{c}(x,\theta) = c_n' \iiiint \mathfrak{a}(\frac{\left(\frac{t}{2}\right)\cdot x+z}{2},L^t_{z'-x'}(\eta+\theta))\mathfrak{b}(\frac{\left(\frac{t}{2}\right)^{-1} x+z}{2},L^t_{z'-x'}(\xi+\theta)) e^{i[<(L_{z'-x'})^{-1}\left(\frac{t}{2}\right)\cdot x\cdot z^{-1},\eta>+<(L_{z'-x'})^{-1}z\cdot x^{-1}\cdot\left(\frac{t}{2}\right),\xi>]}d\xi dz d\eta dt$$

[HHH.409]

The next change of variables entails introducing (α,β) defined by

(246) $$\alpha = -\frac{1}{2}L^{-1}_{z'-x'}\left(\frac{t}{2}\right)\cdot x\cdot z^{-1}, \quad \beta = \frac{1}{2}L^{-1}_{z'-x'}z\cdot x^{-1}\cdot\left(\frac{t}{2}\right). \quad [HHH.410]$$

Using the definitions in (246) we easily derive that

$$z = (\alpha + \beta) \cdot x, \quad t = 2(\beta - \alpha).^{[HHH.411]} \tag{247}$$

From (247) we obtain that

$$\frac{\left(\frac{t}{2}\right) \cdot x + z}{2} = \beta \cdot x, \quad \frac{\left(\frac{t}{2}\right)^{-1} \cdot x + z}{2} = \alpha \cdot x.^{[HHH.412]} \tag{248}$$

Using these relations and reordering the integrations appropriately we obtain that

$$\mathfrak{c}(x, \theta) = c_n \iiiint \mathfrak{a}(\beta \cdot x, L^t_{\alpha' + \beta'}(\eta + \theta)) \mathfrak{b}(\alpha \cdot x, L^t_{\alpha' + \beta'}(\xi + \theta)) \\ e^{2i[<\beta, \xi> - <\alpha, \eta>]} d\xi d\eta d\alpha d\beta.^{[HHH.413]} \tag{249}$$

In reexpressing the argument of $L_\bullet$ we have used the fact that $(x \cdot z^{-1})' = x' - z'$ and (247). The value of the constant c_n remains fixed for the remainder of the calculation. Its value is determined in (263).

We obtain a further simplification by setting $\psi = L^t_{\alpha' + \beta'}(\eta + \theta)$ and $\phi = L^t_{\alpha' + \beta'}(\xi + \theta)$. The argument of the exponential in (249) becomes:

$$< \beta, (L^t_{\alpha' + \beta'})^{-1} \phi - \theta > - < \alpha, (L^t_{\alpha' + \beta'})^{-1} \psi - \theta >= \\ < \alpha - \beta, \theta > + < (L_{\alpha' + \beta'})^{-1} \beta, \phi > - < (L_{\alpha' + \beta'})^{-1} \alpha, \psi > .^{[HHH.414]} \tag{250}$$

A simple calculation shows that $L^{-1}_{x'} = L_{-x'}$ and more importantly

$$L_{\alpha' + \beta'} \beta = L_{\alpha'} \beta, \quad L_{\alpha' + \beta'} \alpha = L_{\beta'} \alpha.^{[HHH.415]} \tag{251}$$

The exponential can now be written:

$$< \alpha, \theta - L^t_{-\beta'} \psi > + < \beta, L^t_{-\alpha'} \phi - \theta > .$$

Using (251) and once again changing variables with

$$\tilde{\xi} = L^t_{-\alpha'} \phi - \theta, \quad \tilde{\eta} = L^t_{-\beta'} \psi - \theta$$

$$\mathfrak{c}(x, \theta) = c_n \iiiint \mathfrak{a}(\beta \cdot x, L^t_{\beta'}(\tilde{\eta} + \theta)) \mathfrak{b}(\alpha \cdot x, L^t_{\alpha'}(\tilde{\xi} + \theta)) \\ e^{2i[<\beta, \tilde{\xi}> - <\alpha, \tilde{\eta}>]} d\tilde{\xi} d\tilde{\eta} d\alpha d\beta.^{[HHH.416]} \tag{252}$$

To simplify the notation we now drop the tildes from $\tilde{\xi}$ and $\tilde{\eta}$. The formula in (252) should be regarded as giving the composition formula in a manner analogous to the Gauss transform formalism used in the Euclidean theory. Using the estimates given in section 4 one can show that this formula defines a map $(\mathfrak{a}, \mathfrak{b}) \to \mathfrak{c}$ with the expected continuity properties with respect to the symbolic seminorms. It can therefore be extended to arbitrary pairs of Heisenberg symbols.

We now show that, up to smoothing errors this formula is local in the base variables. We continue with $\mathfrak{a}$ and $\mathfrak{b}$ as above. Let $\{X_0, X_1, \ldots, X_{2n}\}$ denote vector fields which generate the Lie algebra of the Heisenberg group. For each $(2n+1)$–multi-index, μ there is a polynomial, P_μ of degree $|\mu|$ (in non–commuting variables) such that

$$\partial^\mu_\beta [f(\beta \cdot x)] \restriction_{\beta = 0} = P_\mu(X) f(x).^{[HHH.417]} \tag{253}$$

Using these expressions we obtain the formal Taylor expansion

$$f(\beta\cdot x,r)\sim\sum P_\mu(X)f(x,r)\frac{\beta^\mu}{\mu!}. \tag{254}$$ [HHH.418]

Here r denotes additional parameters.

Putting the formal Taylor expansions into (252) we obtain

(255)
$$\mathfrak{c}(x,\theta)\sim c_n\sum\frac{(-1)^{|\nu|}}{2^{|\mu|+|\nu|}\mu!\nu!}\iiiint P_\mu(X)\mathfrak{a}(x,L^t_{\beta'}(\eta+\theta))P_\nu(X)\mathfrak{b}(x,L^t_{\alpha'}(\xi+\theta))$$
$$D^\mu_\xi D^\nu_\eta e^{2i[<\beta,\xi>-<\alpha,\eta>]}d\xi d\eta d\alpha d\beta.$$ [HHH.419]

Note that we do not differentiate the α and β variables appearing in the second arguments of $\mathfrak{a}$, $\mathfrak{b}$ respectively as this would lead to a non–summable series. Indeed this is an essential difference between the Heisenberg case and the Euclidean case. We can integrate this by parts to obtain:

(256) $\mathfrak{c}(x,\theta)\sim$
$$c_n\sum\frac{(-1)^{|\mu|}}{2^{|\mu|+|\nu|}\mu!\nu!}\iiiint P_\mu(X)D^\nu_\eta[\mathfrak{a}(x,L^t_{\beta'}(\eta+\theta))]P_\nu(X)D^\mu_\xi[\mathfrak{b}(x,L^t_{\alpha'}(\xi+\theta))]$$
$$e^{2i[<\beta,\xi>-<\alpha,\eta>]}d\xi d\eta d\alpha d\beta.$$ [HHH.420]

A little care is required at this point as the derivatives in the η_0 and ξ_0 directions bring out factors of β' and α' respectively. Observe that

$$iD_{\eta_0}[f(L^t_{\beta'}(\eta+\theta))]=f_{\sigma_0}(L^t_{\beta'}(\eta+\theta))+\tilde\beta\cdot f_{\hat\sigma}(L^t_{\beta'}(\eta+\theta))-\hat\beta\cdot f_{\tilde\sigma}(L^t_{\beta'}(\eta+\theta)).$$

Using this expression we easily obtain that

$$\begin{aligned}&\iint iD_{\eta_0}[f(L^t_{\beta'}(\eta+\theta))]g(L^t_{\alpha'}(\xi+\theta))e^{2i[<\beta,\xi>-<\alpha,\eta>]}d\eta d\xi\\ &=\iint\big[f_{\sigma_0}(L^t_{\beta'}(\eta+\theta))g(L^t_{\alpha'}(\xi+\theta))+\\ &\frac{i}{2}(f_{\hat\sigma}\cdot g_{\tilde\sigma}-f_{\tilde\sigma}\cdot g_{\hat\sigma})(L^t_{\beta'}(\eta+\theta),L^t_{\alpha'}(\xi+\theta))\big]e^{2i[<\beta,\xi>-<\alpha,\eta>]}d\eta d\xi\end{aligned} \tag{257}$$ [HHH.420.1]

When eventually we take $\mathfrak{a}$ and $\mathfrak{b}$ to be homogeneous, in these terms, the homogeneity drops by 2.

With this understood we can complete the analysis of the individual terms in (256). Letting $IJ;KL$ be a label for a single term we have, for an appropriate constant, that each is of the form

(258)
$$\mathfrak{c}_{IJ;KL}(x,\theta)=c_nC_{IJ;KL}\iiiint\mathfrak{a}_{IJ}(x,L^t_{\beta'}(\eta+\theta))\mathfrak{b}_{KL}(x,L^t_{\alpha'}(\xi+\theta))$$
$$e^{2i[<\beta,\xi>-<\alpha,\eta>]}d\xi d\eta d\alpha d\beta.$$ [HHH.420.2]

For a function, a defined on $\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}$ we let

$$\tilde a(x,\xi)=\int_{\mathbb{R}^{2n+1}}a(x,y)e^{-i<y,\xi>}dy\text{ and }\hat a(x,\xi_0,y')=\int_{\mathbb{R}}a(x,y)e^{-i<y_0,\xi_0>}dy_0.$$

With this notation we see that

$$(259)\quad \mathfrak{c}_{IJ;KL}(x,\theta) = c_n C_{IJ;KL} \iint \tilde{\mathfrak{a}}_{IJ}(x, 2\alpha_0 + 2\omega(\alpha',\beta'), 2\alpha') \\ \tilde{\mathfrak{b}}_{KL}(x, -2\beta_0 - 2\omega(\beta',\alpha'), -2\beta') e^{2i<\alpha-\beta,\theta>} d\alpha d\beta.$$

Changing variables, with

$$s_0 = 2(\alpha_0 + \omega(\alpha',\beta')), \quad t_0 = -2(\beta_0 + \omega(\beta',\alpha')) \\ s' = 2\alpha,' \quad t' = -2\beta'$$

we see that

$$(260)\quad \mathfrak{c}_{IJ;KL}(x,\theta) = c_n' C_{IJ;KL} \iint \tilde{\mathfrak{a}}_{IJ}(x,s)\tilde{\mathfrak{b}}_{KL}(x,t) \\ e^{i[\omega(s',t')\theta_0 + <s'+t',\theta'> + (s_0+t_0)\theta_0]} ds' dt' ds_0 dt_0.^{[HHH.420.3]}$$

We can apply the Parseval formula, see **[19]** to the $ds'dt'$ integral to obtain that

$$(261)\quad \mathfrak{c}_{IJ;KL}(x,\theta) = \frac{c_n' C_{IJ;KL}}{\theta_0^{2n}} \iint \hat{\mathfrak{a}}_{IJ}(x,s_0,\mu'+\theta')\hat{\mathfrak{b}}_{KL}(x,t_0,\nu'+\theta') e^{i\frac{4\omega(\mu',\nu')}{\theta_0}} \\ e^{i(s_0+t_0)\theta_0} d\mu' d\nu' ds_0 dt_0.^{[HHH.420.4]}$$

A little care is required at this stage. The symplectic form $\omega(s',t)$ in the exponent must be regarded as a *symmetric* quadratic form on $\mathbb{R}^{4n}$. The matrix of this form is therefore

$$S = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & \\ 0 & -\mathrm{Id}_n & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 \end{pmatrix}.$$

Performing the $ds_0 dt_0$-integration we see that this gives

$$(262)\qquad \mathfrak{c}_{IJ;KL}(x,\theta_0,\theta') = C_{IJ;KL}\mathfrak{a}_{IJ}\#\mathfrak{b}_{KL}(x,\theta_0,\theta')$$

where

$$\mathfrak{a}\#\mathfrak{b}(x,\theta_0,\theta') = \frac{c_n'}{\theta_0^{2n}} \iint \mathfrak{a}(x,\theta_0,\mu'+\theta')\mathfrak{b}(x,\theta_0,\nu'+\theta') e^{\frac{2i(\tilde{\mu}\cdot\hat{\nu} - \tilde{\nu}\cdot\hat{\mu})}{\theta_0}} d\mu' d\nu'.$$

Using the estimates employed in the proof of Theorem 1 the asymptotic expansions, with corresponding estimates on the remainder terms extend easily to pairs of symbols in the upper ideal. To complete the calculation we determine the value of the dimensional constant: Let $\mathfrak{a} = 1$, the symbol of the identity and $\mathfrak{b}(x,\sigma)$ be an arbitrary element in the upper ideal then as $\mathfrak{a}(X,D)\mathfrak{b}(X,D) = \mathfrak{b}(X,D)$ it clear that

$$\mathfrak{b}(x,\theta_0,\theta') = \frac{c_n'}{\theta_0^{2n}} \iint \mathfrak{b}(x,\theta_0,\mu'+\theta') e^{i\frac{4\omega(\mu',\nu')}{\theta_0}} d\mu' d\nu'.$$

Using the Fourier inversion formula we obtain that $c_n' = \frac{1}{\pi^{2n}}$ and therefore

$$(263)\quad \mathfrak{a}\#\mathfrak{b}(x,\theta_0,\theta') = \left[\frac{1}{\pi\theta_0}\right]^{2n} \iint \mathfrak{a}(x,\theta_0,\mu'+\theta')\mathfrak{b}(x,\theta_0,\nu'+\theta') e^{i\frac{4\omega(\mu',\nu')}{\theta_0}} d\mu' d\nu'^{[HHH.420.10]}$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are Heisenberg homogeneous symbols of orders m and m' respectively then the full symbol of $\mathfrak{a}(X,D)\circ\mathfrak{b}(X,D)$ is given asymtotically as a sum

$$\mathfrak{c}(x,\sigma)\sim\sum_{j=-\infty}^{0}Q_j(\mathfrak{a},\mathfrak{b}).$$

Here $Q_j(\mathfrak{a},\mathfrak{b})$ are the terms in the sum in (256) homogeneous of order $m+m'+j$. From (261) it follows that for each j there is a finite collection of pairs of differential operators, $\{E^j_k,F^j_k,k\in K_j\}$ so that

$$Q_j(\mathfrak{a},\mathfrak{b})=\sum_{k\in K_j}E^j_k\mathfrak{a}\#F^j_k\mathfrak{b}.$$

We now apply formulæ (256), (257) and (261) to compute the first several terms in the asymptotic expansion of $\mathfrak{c}$. Here we assume that $\mathfrak{a}$ and $\mathfrak{b}$ are in the upper ideal, Heisenberg homogeneous of orders m and m' respectively. As a first step we define the differential operators in (254):

$$\begin{aligned}
P_0f(x)=\partial_{\beta_0}f(\beta\cdot x)\big|_{\beta=0}&=\partial_{x_0}f(x),\\
P_if(x)=\partial_{\beta_i}f(\beta\cdot x)\big|_{\beta=0}&=(\partial_{x_i}+\frac{1}{2}x_{i+n}\partial_{x_0})f(x)\text{ for }1\leq i\leq n,\\
P_if(x)=\partial_{\beta_i}f(\beta\cdot x)\big|_{\beta=0}&=(\partial_{x_i}-\frac{1}{2}x_{i-n}\partial_{x_0})f(x)\text{ for }n+1\leq i\leq 2n.
\end{aligned}\tag{264}$$

We let P_{ij} denote the second order operators defined so that

$$\partial_{\beta_i}\partial_{\beta_j}f(\beta\cdot x)\big|_{\beta=0}=P_{ij}f(x).$$

Note that as $[\partial_{\beta_i},\partial_{\beta_j}]=0$ for all pairs i,j but $[P_i,P_{i+n}]\neq 0$ it is in general *not* the case that $P_{ij}=P_iP_j$. With these preliminaries we easily obtain

$$\mathfrak{c}(x,\sigma)=Q_0(\mathfrak{a},\mathfrak{b})+Q_{-1}(\mathfrak{a},\mathfrak{b})+Q_{-2}(\mathfrak{a},\mathfrak{b})+\text{l.o.t. where}\tag{265}$$

$$Q_0(\mathfrak{a},\mathfrak{b})=\mathfrak{a}\#\mathfrak{b},\ Q_{-1}(\mathfrak{a},\mathfrak{b})=-\frac{1}{2i}\sum_{i=1}^{2n}[P_i\mathfrak{a}\#\mathfrak{b}_{\sigma_i}-\mathfrak{a}_{\sigma_i}\#P_i\mathfrak{b}]\text{ and}$$

$$Q_{-2}=\frac{1}{4}\sum_{i,j=1}^{2n}P_j\mathfrak{a}_{\sigma_i}\#P_i\mathfrak{b}_{\sigma_j}-\frac{1}{2i}[P_0\mathfrak{a}\#\mathfrak{b}_{\sigma_0}-\mathfrak{a}_{\sigma_0}\#P_0\mathfrak{b}]-$$

[HHH.420.5]

$$\frac{1}{8}\sum_{i,j=1}^{2n}[\mathfrak{a}_{\sigma_i\sigma_j}\#P_{ij}\mathfrak{b}+P_{ij}\mathfrak{a}\#\mathfrak{b}_{\sigma_i\sigma_j}]-\frac{1}{4}[\omega_{\#}(\nabla_{\sigma'}P_0\mathfrak{a},\nabla_{\sigma'}\mathfrak{b})+\omega_{\#}(\nabla_{\sigma'}P_0\mathfrak{b},\nabla_{\sigma'}\mathfrak{a})],$$

here

$$\omega_{\#}(\nabla_{\sigma'}a,\nabla_{\sigma'}b)=\sum_{i=1}^{n}[a_{\sigma_i}\#b_{\sigma_{i+n}}-a_{\sigma_{i+n}}\#b_{\sigma_i}].$$

The lower order terms are homogeneous of degrees $m+m'-3$ or less.

11. Fiber traces of commutators[S.Flat.trace]

Using the calculations in § 10 we now compute the principal term in the symbolic traces of several types of commutators. Recall that for a Heisenberg homogeneous symbol $\mathfrak{a}(x, \sigma_0, \sigma')$ in the upper ideal we defined

$$\operatorname{Tr}(\mathfrak{a}(x, \sigma_0)) = \left[\frac{1}{2\pi}\right]^n \int_{\mathbb{R}^{2n}} \mathfrak{a}(x, \sigma_0, \sigma') d\sigma'.$$

A simple application of the Plancherel formula establishes the following formula.

[HHH.420.11]

LEMMA 32. *If $\mathfrak{a}$ and $\mathfrak{b}$ are Heisenberg homogeneous symbols in the upper ideal then*

$$\operatorname{Tr}(\mathfrak{a}\#\mathfrak{b})(x, \sigma_0) = \left[\frac{1}{2\pi}\right]^n \int_{\mathbb{R}^{2n}} \mathfrak{a}(x, \sigma_0, \sigma')\mathfrak{b}(x, \sigma_0, \sigma') d\sigma'. \tag{266}$$ [HHH.420.12]

An immediate corollary of (266) is the fact that

$$\operatorname{Tr}(\mathfrak{a}\#\mathfrak{b}) = \operatorname{Tr}(\mathfrak{b}\#\mathfrak{a}). \tag{267}$$ [HHH.420.13]

Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous symbols in the upper ideal then it follows easily from (265), (266) and (267) that

(268)

$$\begin{aligned}\operatorname{Tr}\sigma([\mathfrak{a}(X, D), \mathfrak{b}(X, D)]) &= \operatorname{Tr}(Q_{-1}(\mathfrak{a}, \mathfrak{b}) - Q_{-1}(\mathfrak{b}, \mathfrak{a})) + \text{l. o. t.} \\ &= i\left[\frac{1}{2\pi}\right]^n \sum_{j=1}^{2n} \int (P_j \mathfrak{a}\mathfrak{b}_{\sigma_j} - \mathfrak{a}_{\sigma_j} P_j \mathfrak{b})(x, \sigma_0, \sigma') d\sigma' + \text{l. o. t.}.\end{aligned}$$

For the computation of the Hochschild homology it is very useful to compute the symbol of a commutator of Toeplitz operators. Let S be the Szegő projector defined the standard flat CR-structure on the Heisenberg group. The associated $\Box_b$-operator on functions is

$$\Box_b = -\frac{1}{4}\left[\sum_{j=1}^{2n} P_j^2 - inP_0\right]$$

Its symbol is therefore

$$\sigma_2(\Box_b) = \frac{1}{4}[|\sigma'|^2 - n\sigma_0].$$

The symbol of S is determined by the conditions

$$s\#s = s \text{ and } \sigma_2(\Box_b)\#s = 0.$$

An elementary calculation shows that

$$s = 2^n \exp(\frac{|\sigma'|^2}{\sigma_0}).$$

With these normalizations it is easy to show that

$$\operatorname{Tr}(s) = \operatorname{Tr}(s\#s) = \sigma_0^n.$$

It is easy to show that $T = -iP_0$ commutes with S. Let a, b be smooth functions on $\mathbb{H}^n$. The principal symbol of the commutator of the classical Toeplitz operators,

$[SaT^mS, SbT^{m'}S]$ is the leading term of the fiber trace of $\sigma(S[aT^mS, bT^{m'}S])$ divided by the fiber trace of $\sigma(S)$ itself. The following formula is an easy consequence of the symbol calculus

$$[aT^mS, bT^{m'}S] = [aS, bS]T^{m+m'} + (maTb - m'bTa)ST^{m+m'-1} + \text{l. o. t.}.$$

Here l. o. t. are terms of order less than $m + m' - 2$. As the symbol of T is σ_0 we see that

$$\begin{aligned} \mathrm{Tr}(\sigma(S[aST^m, bST^{m'})) = \mathrm{Tr}(\sigma(S[aS, bS])\sigma_0^{m+m'} + \\ i\sigma_0^{m+m'+n-1}(m'bP_0a - maP_0b) + \text{l. o. t.}. \end{aligned} \tag{269}$$ [HHH.420.14]

We are therefore left with the computation of $\mathrm{Tr}(\sigma(S[aS, bS])$ to leading order.

A calculation using (239) shows that the Weyl symbol of the multiplication operator $M_a : f(x) \to a(x)f(x)$ is given by

$$\sigma(M_a)(x, \sigma) = a(x).$$

Using (265) we obtain the formula

$$\begin{aligned} \sigma(aS) =& a^{(0)} + a^{(-1)} + a^{(-2)} + \text{l. o. t.} \\ =& a(x)s(\sigma) - \frac{1}{2i}\sum_{j=1}^{2n} P_ja(x)s_{\sigma_j}(\sigma) - \frac{1}{2i}P_0a(x)s_{\sigma_0}(\sigma) - \\ & \frac{1}{8}\sum_{i,j=1}^{2n} P_{ij}as_{\sigma_i\sigma_j} + \text{l. o. t.} \end{aligned} \tag{270}$$

Similarly we obtain that

$$\begin{aligned} \sigma(aSb) = abs + \frac{1}{2i}\sum_{j=1}^{2n}[aP_jb - bP_ja]s_{\sigma_j} + \frac{1}{2i}[aP_0b - bP_0a]s_{\sigma_0} - \\ \frac{1}{8}\sum_{i,j=1}^{2n}[bP_{ij}a + aP_{ij}b]s_{\sigma_i\sigma_j} + \frac{1}{4}\sum_{i,j=1}^{2n} P_jaP_ibs_{\sigma_j\sigma_i} + \text{l. o. t.}. \end{aligned} \tag{271}$$

Using this formula and an analogous one for $\sigma(BSA)$ we easily obtain that

$$\sigma(ASB - BSA) = \frac{1}{i}\sum_{j=0}^{2n}[aP_jb - bP_ja]s_{\sigma_j} + \text{l. o. t.}. \tag{272}$$

Note that the sum extends from $j = 0$, the lower order terms are order -3 and less. Using this formula and the composition formula we obtain that

$$\begin{aligned} \sigma(S(ASB - BSA)S) = \frac{1}{i}\sum_{j=0}^{2n}(aP_jb - bP_ja)s\#s_{\sigma_j}\#s - \\ \frac{1}{2}\sum_{i,j=1}^{2n} P_i[aP_jb - bP_ja](s_{\sigma_i}\#s_{\sigma_j}\#s - s\#s_{\sigma_j}\#s_{\sigma_i}) + \text{l. o. t.} \end{aligned} \tag{273}$$

From (266) we see that for $1 \leq j \leq 2n$ we have

$$\mathrm{Tr}(s_{\sigma_j}\#s) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \partial_{\sigma_j}s^2 d\sigma' = 0,$$

thus these terms contribute nothing to the fiber trace. A simple calculation shows that

$$\operatorname{Tr}(s\#s_{\sigma_0}) = \frac{n\sigma_0^{n-1}}{2}.$$

If we differentiate the identity $s\#s = s$ we obtain

$$s_{\sigma_j}\#s + s\#s_{\sigma_j} = s_{\sigma_j}.$$

Using this identity we easily obtain that

$$s_{\sigma_i}\#s_{\sigma_j}\#s = s\#s_{\sigma_i}\#s_{\sigma_j}.$$

Therefore

$$s_{\sigma_i}\#s_{\sigma_j}\#s = \sigma_0^{-n}\operatorname{Tr}(s\#s_{\sigma_i}\#s_{\sigma_j})s. \tag{274}$$

Using these observations we can simplify the formula above to obtain

$$\sigma(S(ASB-BSA)S) = \frac{n\sigma_0^{-1}}{2i}[aP_0b - bP_0a]s - \tag{275}$$
$$\frac{1}{2}\sum_{i,j=1}^{2n}[(P_iaP_jb - P_ibP_ja) + (aP_iP_jb - bP_iP_ja)](s\#s_{\sigma_i}\#s_{\sigma_j} - s\#s_{\sigma_j}\#s_{\sigma_i}) + \text{l. o. t.}^{[HHH.5.27.99.1]}$$

A final simplification which shows that the formula has the expected form arises by observing that the second summation can be interpreted as the trace of a product of two matrices, the second of which is skew symmetric. The first term in the bracket is also skew symmetric, we replace the second with its skew part to obtain

$$\sigma(S(ASB-BSA)S) = \frac{n\sigma_0^{-1}}{2i}[aP_0b - bP_0a]s - \tag{276}$$
$$\frac{1}{2}\sum_{i,j=1}^{2n}[(P_iaP_jb - P_ibP_ja) + \frac{1}{2}(a[P_i,P_j]b - b[P_i,P_j]a)](s\#s_{\sigma_i}\#s_{\sigma_j} - s\#s_{\sigma_j}\#s_{\sigma_i}) + \text{l. o. t.}$$

To complete this calculation we need to compute the matrix

$$\gamma_{ij} = \operatorname{Tr}(s\#[s_{\sigma_i}\#s_{\sigma_j} - s_{\sigma_j}\#s_{\sigma_i}]).$$

Using the formula for $s(\sigma)$ we see that

$$\gamma_{ij} = \frac{2^{2n+2}}{\pi^{3n}\sigma_0^{2n+2}}\iiint[(\sigma_i+\mu_i)(\sigma_j+\nu_j) - (\sigma_j+\mu_j)(\sigma_i+\nu_i)]\times \tag{277}$$
$$e^{-\frac{|\mu'+\sigma'|^2}{\sigma_0}}e^{-\frac{|\nu'+\sigma'|^2}{\sigma_0}}e^{\frac{2i\omega(\mu',\nu')}{\sigma_0}}e^{-\frac{|\sigma'|^2}{\sigma_0}}d\mu'd\nu'd\sigma'.$$

Letting $x = \mu' + \sigma'$ and $y = \nu' + \sigma'$ we obtain

$$\gamma_{ij} = \frac{2^{2n+2}}{\pi^{3n}\sigma_0^{2n+2}}\iiint[x_iy_j - x_jy_i]\times \tag{278}$$
$$e^{-\frac{|\sigma'|^2}{\sigma_0}}e^{\frac{2i\omega(y-x,\sigma')}{\sigma_0}}d\sigma' e^{-\frac{|x|^2+|y|^2}{\sigma_0}}e^{\frac{2i\omega(x,y)}{\sigma_0}}dxdy.$$

Doing the σ'-integral

$$\gamma_{ij} = \left[\frac{2^{2n+2}}{\pi^{2n}\sigma_0^{n+2}}\right]\iint[x_iy_j - x_jy_i]e^{-\frac{2(|x|^2+|y|^2-<x,y>)}{\sigma_0}}e^{\frac{2i\omega(x,y)}{\sigma_0}}dxdy. \tag{279}$$

Setting $s = x - y/2$ this becomes

$$\gamma_{ij} = \left[\frac{2^{2n+2}}{\pi^{2n}\sigma_0^{n+2}}\right] \iint [s_i y_j - s_j y_i] e^{-\frac{2|s|^2}{\sigma_0}} e^{\frac{2i\omega(s,y)}{\sigma_0}} ds e^{-\frac{3|y|^2}{2\sigma_0}} dy. \tag{280}$$

From this form it is clear that $\gamma_{ij} = 0$ unless $j = i \pm n$ and that

$$\gamma_{j(j+n)} = -\gamma_{(j+n)j} \text{ for } j = 1, \ldots, n.$$

Doing the integral in the remaining cases we obtain that

$$\gamma_{j(j+n)} = i\sigma_0^{n-1}. \tag{281}$$

Using the fact that

$$[P_i, P_{i+n}] = -P_0$$

along with (274) and (281) in (275) we obtain that

$$\sigma(S(ASB - BSA)S) = \left[\frac{i}{\sigma_0}\right] \sum_{j=1}^{n} [P_j b P_{j+n} a - P_j a P_{j+n} b] s + \text{l.o.t.} \tag{282}$$ [HHH.5.27.99.2]

The ray subbundle L of $T^*\mathbb{H}^n$, defined by the positive contact direction,

$$\theta = dt + \frac{1}{2}[x \cdot dy - y \cdot dx]$$

is a symplectic submanifold. The symplectic structure defines a canonical Poisson bracket on $\mathcal{C}^\infty(L)$. Let $\{f, g\}_L$ denote this Poisson bracket, it is given by

$$\{f, g\}_L = \frac{1}{\sigma_0} \sum_{j=1}^{n} [P_j f P_{j+n} g - P_j g P_{j+n} f] + [\partial_{\sigma_0} f P_0 g - P_0 f \partial_{\sigma_0} g].$$

Combining (282) with (269) we see that

$$\sigma(S(AT^m SBT^{m'} - BT^{m'} SAT^m)S) = \frac{1}{i}\{a\sigma_0^m, b\sigma_0^{m'}\}_L s. \tag{283}$$