CHAPTER 4

Isotropic algebras$^{[S.symbol.star]}$

In Chapter 5 we discuss the composition properties of Heisenberg and extended Heisenberg operators. The 'leading part' of the diagonal singularity of such an operator is determined by its principal symbol. The standard algebra is commutative at this top level, however (in general) the Heisenberg algebras are not. In this chapter we discuss the isotropic algebras in which the Heisenberg principal symbol takes its values.

The isotropic algebra in its basic form is very well known, under a variety of names ([**26**], [**27**], [**19**], [**20**]). We first discuss the usual realization of this algebra as operators on functions on $\mathbb{R}^n$. Somewhat more abstractly the algebra can simply be regarded as arising from a non-commutative product on functions on $\mathbb{R}^{2n}$. This in turn can be further abstracted to a non-commutative product on functions on a symplectic vector space, in which case the action as operators is no longer canonical. For our purposes it is useful to consider even more generally the product which arises from the choice of an antisymmetric bilinear form on a vector space, whether degenerate or not. The 'isotropic' product in this case depends smoothly on the 2-form, varying between the commutative product in case the 2-form vanishes and the standard 'isotropic' algebra in case the form is non-degenerate. We still call this an isotropic algebra, or more specifically ω-isotropic for an antisymmetric form ω.

1. Isotropic algebra on $\mathbb{R}^n$ $^{[SS.Isotropic.algebra]}$

We start with a short discussion of the composition properties of the 'isotropic' algebra of operators on $\mathbb{R}^n$ and its variants. The fundamental example of an elliptic operator in this algebra, and one in which we are particularly interested, is the harmonic oscillator, $H = \Delta + |x|^2$.

As discussed in the Introduction, pseudodifferential operators on $\mathbb{R}^n$ are here identified directly in terms of their Schwartz kernels and the Fourier transform. Weyl quantization (which is Riemann-Weyl quantization for the Euclidean metric on $\mathbb{R}^n$) gives an automorphism of $\mathcal{S}'(\mathbb{R}^{2n})$ which is used to identify the kernel, A, of an operator and the corresponding amplitude, a, by

$$A(x,x') = q_W(a) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} a(\frac{x+x'}{2},\xi)d\xi^{[14.7.1998.4]} \tag{97}$$

$$a(x,\xi) = \sigma_W(A) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi}A(x+\frac{y}{2},x-\frac{y}{2})dy.^{[14.7.1998.3]} \tag{98}$$

Different spaces of operators correspond to different choices for the space of amplitudes (often called 'symbols' but we reserve that term for functions satisfying 'symbolic estimates'). If as amplitude we take an element of the space of classical symbols of order m on $\mathbb{R}^{2n}$ jointly in x and ξ then the corresponding operator A is an element of the *classical isotropic space*, written $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$. Here 'iso' stands for 'isotropic operator'. The space of classical symbols of order m is identified, under radial compactification, as the space $\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$ in terms of the $\mathcal{C}^\infty$-structure on $\overline{\mathbb{R}^{2n}}$ as the closure of image of $\mathbb{R}^{2n}$ under the radial compactification map in (3) (so in all variables, x, ξ). As we shall see below, $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is an algebra of operators on $\mathcal{S}(\mathbb{R}^n)$. Later we use a slightly different notation and write

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) = \Psi^m_{\mathrm{is}}(\mathbb{R}^{2n}) \tag{99}$$ [HHH2.511]

when we think of the isotropic algebra as being a non-commutative product on functions on $\mathbb{R}^{2n}$ so on the left we consider the operator A and on the right its Weyl amplitude a.

The *symbolic estimates*, which are satisfied by any classical isotropic symbol $a \in \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$, are the bounds on derivatives

$$|\partial_x^\alpha \partial_\xi a(x,\xi)| \le C_{\alpha\beta}\langle (x,\xi)\rangle^{m-|\alpha|-\beta|} \tag{100}$$ [c10.23.1998.1]

for some constants $C_{\alpha\beta}$. The space of functions satisfying these estimates is usually denoted $S^m(\mathbb{R}^{2n})$; we identify it with its image under radial compactification as a space of functions on the interior of the upper half sphere and denote it $\mathcal{A}^{-m}(\mathbb{S}^{2n,1})$; this is the space of ($L^\infty$*-based) conormal functions* on $\mathbb{S}^{2n,1}$. Thus

$$\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}) \subset \mathcal{A}^{-m}(\mathbb{S}^{2n,1}) \tag{101}$$ [HHH2.512]

is a strict inclusion. If $S^m(\mathbb{R}^{2n}) = \mathcal{A}^{-m}(\mathbb{S}^{2n,1})$ is topologized with the seminorms given by the best constants in (100) then

(102)
$$\dot{\mathcal{C}}^\infty(\mathbb{S}^{2n,1}) \subset \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}) \text{ is dense in the topology of } \mathcal{A}^{-m'}(\mathbb{S}^{2n,1}) \text{ for } m' > m.$$ [HHH2.513]

In terms of this notation we have isomorphisms

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi^m_{\mathrm{is}}(\mathbb{R}^{2n}) = \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}) . \tag{103}$$ [14.7.1998.5]

If the space of classical symbols is replaced by the space of conormal functions $\mathcal{A}^{-m}(\mathbb{S}^{2n,1}) = \rho^{-m}\mathcal{A}^0(\mathbb{S}^{2n,1})$ where

$$\mathcal{A}^0(\overline{\mathbb{R}^{2n}}) = \left\{ a \in L^\infty(\overline{\mathbb{R}^{2n}}); \mathrm{Diff}^k_{\mathrm{b}}(\overline{\mathbb{R}^{2n}}) \cdot a \subset L^\infty(\overline{\mathbb{R}^{2n}}) \ \forall\ k \in \mathbb{N} \right\} \tag{104}$$ [14.7.1998.6]

then quantization gives the *isotropic calculus with bounds*,

$$\Psi^m_{\mathrm{iso},\infty}(\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \mathcal{A}^{-m}(\overline{\mathbb{R}^{2n}}) . \tag{105}$$ [14.7.1998.7]

In the other direction, of refinement, we may replace $\overline{\mathbb{R}^{2n}}$ by the quadratic compactification ${}^q\overline{\mathbb{R}^{2n}}$. We denote the resulting space of operators by

$$\Psi^m_{\mathrm{Is}}(\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \rho_q^{-m/2}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}}) . \tag{106}$$ [HHH2.211]

Clearly $\Psi^m_{\mathrm{Is}}(\mathbb{R}^n) \subset \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$. The change from the radial to the quadratic compactification does not change the isotropic calculus with bounds except for its effect on the apparent order.

If $A = q_W(a)$ is of order m and $B = q_W(b)$ is of order m' with $m, m' < -n$ then the kernels are continuous and the composite $C = A \circ B$ of these two operators is given by a convergent integral with $C = q_W(c)$ where

$$C(x,x') = (2\pi)^{-2n} \int e^{i(x-z)\xi + i(z-x')\eta} a(\frac{x+z}{2},\xi) b(\frac{z+x'}{2},\eta) dz d\xi d\eta$$

$$c(x,\zeta) = (2\pi)^{-2n} \int \exp(-iy\cdot\zeta + i(x+\frac{y}{2}-z)\cdot\xi + i(z-x+\frac{y}{2})\cdot\eta) \quad [14.7.1998.8]$$
$$a(\frac{x}{2}+\frac{y}{4}+\frac{z}{2},\xi) b(\frac{z}{2}+\frac{x}{2}-\frac{y}{4},\eta) dy dz d\xi d\eta.$$

Making the change of variables from z, y to s, t where $t = \frac{x}{2}+\frac{y}{4}+\frac{z}{2}$ and $s = \frac{z}{2}+\frac{x}{2}-\frac{y}{4}$ gives $dsdt = 2^{-2n} dzdy$ and

(107)
$$c(x,\zeta) = \pi^{-2n} \int \exp(-2is\cdot\xi + 2it\cdot\eta) a(t+x,\xi+\zeta) b(s+x,\eta+\zeta) ds dt d\xi d\eta. \quad [14.7.1998.9]$$

Note that $\omega((t,\xi),(s,\eta)) = t\cdot\eta - s\cdot\xi$ is the symplectic form on $\mathbb{R}^{2n}$, identified as $T^*\mathbb{R}^n$. The composition properties and symbolic composition formula follow from (107) and the principle of stationary phase.

[HHH2.514]

PROPOSITION 3. *The spaces $\Psi^m_{\text{iso},\infty}(\mathbb{R}^n)$ form a filtered $*$-algebra of operators on $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$*

(108)
$$\Psi^m_{\text{iso},\infty}(\mathbb{R}^n) \circ \Psi^{m'}_{\text{iso},\infty} \subset \Psi^{m+m'}_{\text{iso},\infty}(\mathbb{R}^n), \quad [14.7.1998.13]$$

where $\sigma_W(A^*) = \overline{\sigma_W(A)}$ and the product has the asymptotic description

(109)
$$\sigma_W(A\circ B)(x,\xi) = \sigma_W(A)\#\sigma_W(B)$$
$$\sim \sum_{k\ge 0} \frac{i^k}{2^k k!} (D_x\cdot D_\eta - D_y\cdot D_\xi)^k [\sigma_W(A)(x,\xi)\sigma_W(B)(y,\eta)]\big|_{y=x,\eta=\xi}$$
$$= \sum_{k\ge 0} \rho^{2k} B_k(\sigma_W(A),\sigma_W(B)) \quad [14.7.1998.11]$$

with the B_k being smooth bilinear differential operators tangent to the boundary of $\overline{\mathbb{R}^{2n}}$ and where ρ is a defining function for the boundary.

Note that we have defined $a\#b = \sigma_W(q_W(a)\circ q_W(b))$.

PROOF. This is a special case of the standard theory of pseudodifferential operators on $\mathbb{R}^n$. Since $\dot{\mathcal{C}}^\infty(\mathbb{S}^{2n,1})$ is dense in $\mathcal{A}^{-m}(\mathbb{S}^{2n,1})$ in the topology of $\mathcal{A}^{-m'}(\mathbb{S}^{2n,1})$ for any $m' > m$ it is convenient to first show that (107) defines a bilinear map on $\dot{\mathcal{C}}^\infty(\mathbb{S}^{2n,1}) = \mathcal{S}(\mathbb{R}^{2n})$ which is continuous in the topologies of symbols of order m on a, m' on b and with range in symbols of some order $M(m,m')$:

(110)
$$\mathcal{A}^{-m}(\mathbb{S}^{2n,1}) \times \mathcal{A}^{-m'}(\mathbb{S}^{2n,1}) \longrightarrow \mathcal{A}^{-M(m,m')}(\mathbb{S}^{2n,1}). \quad [HHH2.516]$$

Then, once proved, (109) gives the precise composition properties.

If $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^{4n})$ and $\phi = 1$ near 0 then inserting the cut off function

(111)
$$1 - \widetilde{\phi} = 1 - \phi\left(\frac{(t,\xi,s,\eta)}{(1+|x|^2+|\zeta|^2)^{\frac{1}{2}}}\right) \quad [HHH2.515]$$

into the integrand in (107) limits the support to a region where $|t|+|\xi|+|s|+|\eta| \geq C(1+|x|+|\zeta|)$, $C>0$. Integration by parts in t, ξ, s and η using the identity

$$\begin{aligned} (112) \quad & 2(|s|^2+|\xi|^2+|t|^2+|\eta|^2)\exp(-2is\cdot\xi+2it\cdot\eta) \\ & = i(s\cdot\partial_\xi + i\xi\cdot\partial_s - it\cdot\partial_\eta - i\eta\cdot\partial_t)\exp(-2is\cdot\xi+2it\cdot\eta)^{[HHH2.529]} \end{aligned}$$

shows that the resulting bilinear form

$$\begin{aligned} (113) \quad & \mathcal{A}^{-m}\times\mathcal{A}^{-m'} \ni (a,b) \longmapsto \pi^{-2n}\int \exp(-2is\cdot\xi+2it\cdot\eta) \\ & (1-\widetilde{\phi})a(t+x,\xi+\zeta)b(s+x,\eta+\zeta)dsdtd\xi d\eta \in \mathcal{A}^{-M} \ \forall\ M^{[HHH2.536]} \end{aligned}$$

is continuous. This shows that inserting $1-\tilde{\phi}$ gives a bilinear from

$$\mathcal{A}^{-m}(\mathbb{S}^{2n,1})\times\mathcal{A}^{-m'}(\mathbb{S}^{2n,1}) \longrightarrow \dot{\mathcal{C}}^\infty(\mathbb{S}^{2n,1}) = \mathcal{S}(\mathbb{R}^{2n}).$$

Thus we may insert the factor $\widetilde{\phi}$ in (107) and consider the bilinear form

$$\begin{aligned} (114) \quad & \mathcal{A}^{-m}\times\mathcal{A}^{-m'} \ni (a,b) \longmapsto \pi^{-2n}\int \exp(-2is\cdot\xi+2it\cdot\eta) \\ & \widetilde{\phi}a(t+x,\xi+\zeta)b(s+x,\eta+\zeta)dsdtd\xi d\eta^{[HHH2.517]} \end{aligned}$$

in which the integral is absolutely convergent, since the integrand has compact support locally uniformly in (x,ζ). Choosing ϕ appropriately we may arrange that on the support of the integrand

$$|(s,\xi,t,\eta)| \leq \frac{1}{2}|(x,\zeta)|+1.$$

It follows that the amplitude in the integral satisfies the estimates

$$(115) \qquad |D^\gamma[\widetilde{\phi}a(t+x,\xi+\zeta)b(s+x,\eta+\zeta)]| \leq C_\gamma(1+|x|+|\zeta|)^{m+m'-|\gamma|^{[HHH2.537]}}$$

for derivatives in all variables. These estimates and the support property of the integrand then show that (114) defines a continuous bilinear map (110) with $M = m+m'+4n$.

Now, if we replace the factors involving a and b in (114) by their Taylor series around $s=0$, $\xi=0$, $t=0$, $\eta=0$ respectively we obtain an expansion of the bilinear form

$$\begin{aligned} (116) \quad & \sum_{|\alpha|+|\beta|+|\gamma|+|\delta|\leq N} c_{\alpha,\beta,\gamma,\delta}\frac{i^{|\alpha|+|\beta|+|\gamma|+|\delta|}}{\alpha!\beta!\gamma!\delta!}D_x^\alpha D_\zeta^\beta a(x,\zeta)D_x^\gamma D_\zeta^\delta b(x,\zeta) \\ & c_{\alpha,\beta,\gamma,\delta} = \pi^{-2n}\int \exp(-2is\cdot\xi+2it\cdot\eta)\widetilde{\phi}t^\alpha\xi^\beta s^\gamma\eta^\delta dsdtd\xi d\eta. \end{aligned}^{[HHH2.518]}$$

Differentiation of the coefficient functions $c_{\alpha,\beta,\gamma,\delta}$ with respect to any x or ζ variable falls on the factor $\tilde{\phi}$ in the integrand and, since this has support contained in that of $1-\tilde{\phi}$, this gives a Schwartz function by the argument above. Thus these coefficients are of the form of a constant plus a term in $\mathcal{S}(\mathbb{R}^{2n})$. The constants are determined by the composition properties of differential operators, and may in any case be readily computed to give the expansion (109). That this expansion is truly asymptotic follows from essentially the same estimates as above applied to the remainders in the Taylor series:

$$(117)\quad \int_0^1 \frac{(1-r)^N}{N!}\frac{d^{N+1}}{dr^{N+1}}a(rs+x,r\xi+\zeta)b(rt+x,r\eta+\zeta)dr = \sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=N+1}\frac{s^\alpha\xi^\beta t^\gamma\eta^\delta}{\alpha!\beta!\gamma!\delta!}\int_0^1\frac{(1-r)^N}{N!}a_{\alpha,\beta}(rs+x,r\xi+\zeta)b_{\gamma,\delta}(rt+x,r\eta+\zeta)dr.^{[HHH2.530]}$$

where the suffixes denote derivatives in the respective variables. Since $a_{\alpha,\beta}$ is a symbol of order $m-|\alpha|-|\beta|$ and similarly for $b_{\gamma,\delta}$, these terms give continuous bilinear forms as in (110) with M replaced by $M-|\alpha|-|\beta|-|\gamma|-|\delta| = M-N-1$.

The basic composition property, (108), then follows from (110) and the fact that the terms in (109) are of order $m+m'-2k$. The properties of the bilinear differential operators in the second form of the expansion in (109) are a consequence of the fact, noted at the end of SS1 above, that a constant coefficient vector field on the radial compactification is smooth and of the form ρV where V is smooth and tangent to the boundary of which ρ is a defining function. □

[HHH2.519]

COROLLARY 1. *The subspaces* $\Psi^m_{\mathrm{Iso}}(\mathbb{R}^n) \subset \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^m_{\mathrm{iso},\infty}(\mathbb{R}^n)$ *also determine filtered algebras.*

PROOF. From (108) and (109) it also follows that if $a \in \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$ and $b \in \rho^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$ then $c = a\#b \in \rho^{-m-m'}(\overline{\mathbb{R}^{2n}})$. To see the same for the quadratic compactification it is enough to note that a constant coefficient vector field on $\mathbb{R}^{2n}$ lifts to be of the form $\rho_q^{\frac12}V$ where V is smooth on ${}^q\overline{\mathbb{R}^{2n}}$ and is tangent to the boundary. Thus the bilinear differential operators in (109) are also of the form

$$\rho^{2k}B_k = \rho_q^k\tilde{B}_k$$

where $\tilde{B}_k$ is a smooth bilinear differential operator on ${}^q\overline{\mathbb{R}^{2n}}$ tangent to the boundary. □

If $A \in \Psi^m_{\mathrm{iso},\infty}(\mathbb{R}^n)$ for an $m < -2n$ then A is a trace class operator and its trace is given by

$$(118)\qquad \mathrm{Tr}(A) = \int\limits_{\mathbb{R}^{2n}} a(x,\xi)d\xi dx.^{[HHH.11.18.1998.1]}$$

This functional satisfies

$$(119)\quad \mathrm{Tr}(a\#b) = \mathrm{Tr}(b\#a)\ \forall\ a \in \rho^{-m}\mathcal{A}(\overline{\mathbb{R}^{2n}}),\ b \in \rho^{-m'}\mathcal{A}(\overline{\mathbb{R}^{2n}}) \quad \text{provided } m+m' < -2n.^{[HHH.11.18.1998.2]}$$

2. The ω'-isotropic algebras [SS.omega.isotropic]

In the form (107) the product makes invariant sense on any symplectic vector space W, with symplectic form $\omega \in \Lambda^2 W'$, namely

$$(120)\qquad a\#b(u) = \pi^{-\dim W}\int_{W^2} e^{2i\omega(v,w)}a(v+u)b(w+u)|\omega_v^{\frac12\dim W}\omega_w^{\frac12\dim W}|^{[14.7.1998.10]}$$

Here ω is the symplectic form on W so $|\omega|^{\frac12\dim W}$ is a smooth density on W. By the linear equivalence of symplectic forms an isomorphism $W \simeq \mathbb{R}^{\dim W}$ can be found reducing ω to the Darboux form. Thus the product # in (120) makes the space $\rho^{-\mathbb{Z}}\mathcal{C}^\infty(\overline{W})$ (where the minus sign is just a reminder that the order of $a \in \rho^m\mathcal{C}^\infty(\overline{W})$ is $-m$) into an associative algebra isomorphic to $\Psi^{\mathbb{Z}}_{\text{iso}}(\mathbb{R}^{\dim W})$; we denote it $\Psi^{\mathbb{Z}}_{\text{is}}(W)$.

The product # is also well-defined on $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty({}^q\overline{W})$, the space of 'Laurent' functions on the quadratic compactification of W; as an algebra we denote this $\Psi^{\mathbb{Z}}_{\text{Is}}(W)$. The asymptotic expansion in (109) may be written invariantly in terms of the dual antisymmetric form $\omega' \in \Lambda^2 W$

$$a\#b(u) \sim \sum_{k\geq 0} \frac{i^k}{k!}\omega'(D_v, D_w)^k a(v)b(w)\big|_{v=w=u};^{[HHH2.262]} \tag{121}$$

if either factor is a polynomial this formula is exact. In any case, it makes the quotients, the spaces of Laurent series at the boundary, $\mathcal{L}(\overline{W}) = \rho^{-\mathbb{Z}}\mathcal{C}^\infty(\overline{W})/\dot{\mathcal{C}}^\infty(\overline{W})$ and $\mathcal{L}({}^q\overline{W}) = \rho_q^{-\mathbb{Z}}\mathcal{C}^\infty({}^q\overline{W})/\dot{\mathcal{C}}^\infty({}^q\overline{W})$ into associative algebras. The resulting product is often called the *star product.* These are spaces of 'formal' isotropic pseudodifferential operators of integral order (and in the quadratic case even-integral order).

The dual form $\omega' \in \Lambda^2 W$ is determined by the original symplectic form $\omega \in \Lambda^2 W'$ in terms of the isomorphism $\tilde{\omega} : W \longrightarrow W'$ arising from the non-degeneracy of ω through

$$\begin{gathered}\omega'(v', w') = \omega(v, w) \text{ if } \tilde{\omega}(v) = v',\ \tilde{\omega}(w) = w'_{[HHH2.521]}\\ \text{where } \tilde{\omega}(v)\cdot w = \omega(v, w)\ \forall\ v, w \in W.\end{gathered} \tag{122}$$

The form and its dual are related through the Fourier transform of their oscillatory exponents. In local coordinates this relationship may be written

$$\exp\left(2i(t\cdot\eta - s\cdot\xi)\right) = 2^{-4n}\pi^{-2n}\int_{\mathbb{R}^{4n}} e^{-it\cdot T - i\eta\cdot H - is\cdot S - i\xi\cdot\Xi} \exp\left(-\frac{i}{2}(T\cdot H - S\cdot\Xi)\right) dT\,dH\,dS\,d\Xi.^{[HHH2.522]} \tag{123}$$

Inserting this into the local coordinate form of (120) gives

(124)
$$a\#_{\omega'}b(u) = (2\pi)^{-2\dim W}\int_{W^2\times(W')^2} e^{-iv\cdot v' - iw\cdot w'} e^{-\frac{i}{2}\omega'(v',w')} a(v+u)b(w+u)\,\nu^{2^{[HHH2.523]}}$$

where ν is the canonical volume form on $W \times W'$ (corresponding to the canonical symplectic structure on this vector space but unrelated to the choice of the 2-form ω).

Now, this expression makes direct sense as an iterated integral for a, b Schwartz functions for an arbitrary element $\omega' \in \Lambda^2 W$. We do not even need to assume that the dimension of W is even. The product depends smoothly on ω' as a bilinear form with values in Schwartz functions on W; it is always an associative product. For even-dimensional W this follows from the density of the symplectic forms in $\Lambda^2 W$; it is shown in the odd-dimensional case below. In fact, we shall show that the resulting product extends to $\rho^{-\mathbb{Z}}\mathcal{C}^\infty(\overline{W})$ as in the symplectic case.

[HHH2.525]

LEMMA 18. *For any $\omega' \in \Lambda^2 W$, the associate product $\#_{\omega'}$ in* (124) *extends by continuity to have the composition properties* (110) *and* (121) *and defines continous* *filtered $*$-products on $\rho^{-\mathbb{R}}\mathcal{A}(\overline{W})$, on $\rho^{-\mathbb{Z}}\mathcal{C}^\infty(\overline{W})$ and on $\rho_q^{-\mathbb{Z}}({}^q\overline{W})$ which all depend smoothly on $\omega' \in \Lambda^2 W$.*

PROOF. Note first the extreme case $\omega' = 0$. Then (124) is simply the integral, over $(W')^2$, of the Fourier transform in the variables v and w in W. Thus, by the Fourier inversion formula,

$$a\#_0 b = ab^{[HHH2.524]} \tag{125}$$

is simply the commutative product.

The general case can be analysed similarly. Indeed, for any $\omega' \in \Lambda^2 W$ the space W may be decomposed into $W = W_1 \times W_2$ where W_2 is the annihilator of null space of ω'. In terms of this and the dual decomposition $W' = W_1' \times W_2'$ ω' is determined uniquely by its restriction to W_2' on which it is symplectic, i.e. non-degenerate. Corresponding to this decomposition the product $\#_{\omega'}$ takes the form

$$a\#_{\omega'}b(\zeta_1,\zeta_2) = (a(\zeta_1,\bullet)\#b(\zeta_1,\bullet))(\zeta_2)^{[HHH2.526]} \tag{126}$$

where $\#$ is the isotropic product corresponding to the symplectic form on W_2, $\omega' \in \Lambda^2 W_2$. The composition properties above follow readily from arguments similar to those in the proof of Proposition 3.

Thus, in (124) we may insert a cutoff function as before, but in all the integrated variables:

$$\begin{gathered} 1 - \widetilde{\phi} = 1 - \phi\left(\frac{(X,X',Y,Y')}{(1+|x|^2+|\zeta|^2)^{\frac12}}\right), \quad {}^{[HHH2.534]} \\ X = s,\xi,\ X' = t,\eta,\ Y = S,\Xi, Y' = T,H. \end{gathered} \tag{127}$$

Replacing (112) by the similar formula which arises from the non-degeneracy of the quadratic form

(128)

$$\begin{aligned} \Phi =& t\cdot T + \eta\cdot H + s\cdot S + \xi\cdot\Xi + \frac12\omega'(S,\Xi;T,H) \\ =& X\cdot Y + X'\cdot Y' + \frac12\omega'(Y,Y') \qquad {}^{[HHH2.535]} \\ (|X|^2 + &|X'|^2 + |Y|^2 + |Y'|^2)\exp(-i\Phi) = -V\exp(-i\Phi), \\ V =& Y\cdot D_X + Y'\cdot D_{X'} + X\cdot D_Y - \frac12\omega'(X,D_{X'}) + X'\cdot D_{Y'} + \frac12\omega'(X',D_X). \end{aligned}$$

and integrating by parts repeatedly shows that the modified quadratic form again satisfies (113).

This it suffices to consider (124) with $\tilde{\phi}$ inserted. Using the estimates corresponding to (115) shows that this part of the quadratic form satisfies (110) with $M(m,m') = m + m' + 8n$. Replacing the symbols by their Taylor series gives the asymptotic expansion (121) for the product with remainder, estimated as before, mapping into symbols of low order. The composition properties therefore follow and each of the terms in the expansion is clearly smooth in ω'. □

We denote the algebras defined by the product $\#_{\omega'}$ corresponding to a choice of $\omega' \in \Lambda^2 W$

$$\begin{gathered} \Psi^{\mathbb{Z}}_{\mathrm{is}(\omega')}(W) = \rho^{-\mathbb{Z}}\mathcal{C}^\infty(\overline{W}) \text{ with } \#_{\omega'} \quad {}^{[HHH2.528]} \\ \Psi^{\mathbb{Z}}_{\mathrm{is}(\omega')}(W) = \rho_q^{-\mathbb{Z}}\mathcal{C}^\infty({}^q\overline{W}) \text{ with } \#_{\omega'}. \end{gathered} \tag{129}$$

There are related 'formal' algebras on the spaces of Laurent series.

More generally we may consider variants of all these algebras, for instance $\Psi^m_{\mathrm{is}}(W;F) = \Psi^m_{\mathrm{is}}(W) \otimes F$ for a vector space F, Formally, these are the isotropic operators 'with values in F.' If $F = \hom(G)$ then this is an algebra and more generally there is a well-defined, non-commutative composition,

$$\Psi^m_{\mathrm{is}(\omega')}(W;F) \cdot \Psi^{m'}_{\mathrm{is}(\omega')}(W;G) \subset \Psi^{m+m'}_{\mathrm{is}(\omega')}(W;F \otimes G) \tag{130}$$ [HHH2.232]

satisfying the corresponding associativity condition.

3. The harmonic oscillator[SS.Harmonic.oscillator]

If W is a symplectic vector space then $J \in \hom(W)$ is a compatible complex structure if

$$\begin{gathered} J^2 = -\operatorname{Id},\ \text{i.e. } J \text{ is a complex structure,} \\ \omega(JX,Y) + \omega(X,JY) = 0 \text{ or equivalently } \omega(JX,JY) = \omega(X,Y), \\ \text{and the quadratic form } h(X,Y) = \omega(JX,Y) \text{ is positive definite.} \end{gathered} \tag{131}$$ [HHH2.243]

The metric necessarily satisfies $h(JX,JY) = h(X,Y)$ so J is an orthogonal transformation. Taking linear coordinates in which h is the Euclidean metric, it follows that W decomposes as a sum of 2-dimensional spaces on which J acts. Thus there are orthogonal coordinates (x,ξ) in terms of which$Jx = \xi$ and $J\xi = -x$. The form ω is then the Darboux form.

Thus the structure of a symplectic vector space with compatible complex structure can be reduced to the model case of $\mathbb{R}^{2n}$ with the Darboux form and J the standard complex structure corresponding to the realization of $\mathbb{C}^n$ as $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Conversely, by the linear equivalence of symplectic forms, there is always such a compatible complex structure on a symplectic vector space.

In terms of the quantization, (97), the quadratic function

$$h = |x|^2 + |\xi|^2 = h(\cdot,\cdot) \in \rho_q^{-1}\mathcal{C}^\infty({}^q\overline{W}) \subset \rho^{-2}\mathcal{C}^\infty(\overline{W})$$

is the symbol of the harmonic oscillator $\Delta + |x|^2$. In this section we discuss the well-known basic properities of this operator, including its spectral decomposition, from the point of view of the isotropic algebra $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$. Once they are stated intrisically in terms of the algebra $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ all results have an immediate interpretation in terms of the algebra $\Psi^{\mathbb{Z}}_{\mathrm{is}}(W)$ for a symplectic vector space with compatible complex structure. In fact we prefer to carry through the discussion directly in the context of a symplectic vector space with compatible complex structure and only resort to an explicit identification with $\mathbb{R}^{2n}$ when it simplifies proofs (as it often does). For example, we define idempotent elements, $s_k \in \Psi^{-\infty}_{\mathrm{is}}(W)$ which correspond to the projectors onto the eigenspaces of the harmonic oscillator. This more algebraic approach is useful when we transfer these constructions to a symplectic vector bundle in the context of a contact manifold. In the proofs we sometimes suppress the isomorphism σ_W and its inverse q_W, as we have done above, and in so doing identify $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ and $\Psi^{\mathbb{Z}}_{\mathrm{is}}(\mathbb{R}^{2n})$ between which they are algebra isomorphisms.

The complex structure, J defines a dual complex structure on W' and hence a complex vector space, denoted, $W^{1,0}$ as the i-eigenspace of J acting on the complexification of W'. If we let $W^{0,1}$ denote the complex conjugate of $W^{1,0}$ then

$$W' \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1} \text{ over } \mathbb{C}. \tag{132}$$ [HHH2.214]

Thinking of $W \otimes \mathbb{C}$ as a complex symplectic vector space, (132) is a complex Lagrangian decomposition, or polarization, of $W \otimes \mathbb{C}$. Indeed, ω vanishes on $W^{1,0}$. It follows from this that the holomorphic polynomials, i.e. elements of the symmetric powers of $W^{1,0}$, form a commutative subalgebra of $\Psi^{\mathbb{Z}}_{\text{is}}(W)$. The operators they define are certainly differential operators in the isotropic calculus and hence so are any composites. Since $\omega(X,Y) = h(X, JY)$, the dual form ω' in (121) is of type $1,1$. Indeed representing J as the standard complex structure we can express the differential operators appearing in (109) in terms of complex derivatives,

$$i[D_x \cdot D_\eta - D_y \cdot D_\xi] = \frac{1}{2}[(D_x - iD_\xi)\cdot(D_y + iD_\eta) - (D_x + iD_\xi)\cdot(D_y - iD_\eta)]$$

Hence, from (109), the holomorphic (and similarly the anti-holomorphic) polynomials do indeed form a subalgebra of $\Psi^{\mathbb{Z}}_{\text{is}}(W)$ with the composition just given by the term with $k=0$, i.e. on these subalgebras the isotropic product reduces to the standard pointwise product

The elements of $W^{1,0}$, are the symbols of the 'creation' operators on $\Psi^{*}_{\text{is}}(W)$ and the elements of $W^{0,1}$ are the annihilation operators. Indeed, if $z_i \in W^{1,0}$ is an orthonormal basis with respect to the Hermitian metric defined by h then

$$h = \sum_{i=1}^{n} z_i \# \bar{z}_i + n,\ [z_i, \bar{z}_j] = -2\delta_{ij},\ [h, z_i] = 2z_i. \tag{133}$$ [HHH2.219]

Perhaps the simplest way to verify these relations is by using the quantization, (97). It leads to these correspondances

$$\begin{aligned} &z_i \leftrightarrow C_i = -\partial_{x_i} + x_i, \quad \bar{z}_i \leftrightarrow C_i^* = \partial_{x_i} + x_i, \quad [C_i, C_j^*] = -2\delta_{ij}, \quad H \leftrightarrow h \\ &\text{where } H = \Delta + |x|^2 = \sum_i (-\partial_{x_i}^2 + x_i^2) = \sum_i C_i C_i^* + n, \quad [H, C_i] = 2C_i. \end{aligned} \tag{134}$$ [HHHc.10.14.98.1]

As $\sigma_W(A \circ B) = \sigma_W(A) \# \sigma_W(B)$ the relations in (133) follow immediately from these easily verified identities.

That H is an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$ with discrete spectrum follows from the existence of a parametrix in the calculus and the compactness of operators of negative order acting on L^2. All the L^2 eigenfunctions are necessarily in $\mathcal{S}(\mathbb{R}^n)$. In fact, as is well known, the commutation relations in (134) allow the spectrum of H to be described explicitly.

The *ground state* is the lowest eigenspace. From the decomposition of H in (134) it follows that $H \geq n$. The joint null space of the C_i^* is the span of $\exp(-\frac{1}{2}|x|^2) \in \mathcal{S}(\mathbb{R}^n)$. Thus n is a simple eigenvalue. Let $s \in \Psi^{-\infty}_{\text{is}}(W)$ be the self-adjoint projection onto this ground state eigenspace. Thus, s is determined by the conditions

$$h\#s = s\#h = ns,\ s\#s = s \neq 0, \ \text{ then } h^* = h \Longrightarrow s^* = s \tag{135}$$ [HHH2.215]

which do not involve the action on $\mathcal{S}(\mathbb{R}^n)$ and so are directly interpretable as equations in $\Psi^{\mathbb{Z}}_{\text{is}}(W)$. Since

$$\int_{\mathbb{R}^n} \exp(-\frac{1}{2}|x|^2)dx = (2\pi)^{\frac{n}{2}},$$

the kernel on $\mathbb{R}^{2n}$ for the model structure is

$$q_W(s) = (2\pi)^{-n} \exp(-\frac{1}{2}(|x|^2 + |x'|^2)) = (2\pi)^{-n} \exp(-|\frac{x+x'}{2}|^2 - \frac{1}{4}|x-x'|^2).$$

The symbol s is therefore the pointwise positive function

$$s = 2^n \exp(-|z|^2) = 2^n e^{-h} \tag{136}$$ [HHH2.539]

which therefore holds in $\Psi^{-\infty}_{\mathrm{is}}(W)$.

All the eigenvalues of H are of the form $n+2k$, $k \in \mathbb{N}$. This follows from (134) which shows that each of the annihilation operators C_i^* maps any eigenspace with eigenvalue λ into an eigenspace with eigenvalue $\lambda - 2$; moreover the joint null space is empty if $\lambda \neq n$. For each positive integer, k we let s_k be the symbol of the self-adjoint projection onto the span of the eigenvectors with eigenvalue $n+2k$. It is uniquely determined by the conditions in $\Psi^{\mathbb{Z}}_{\mathrm{is}}(W)$

$$\begin{gathered} s_k \in \Psi^{-\infty}_{\mathrm{is}}(W),\ s_k\#s_k = s_k^* = s_k,\ h\#s_k = s_k\#h = (n+2k)s_k, \text{ and} \\ g \in \Psi^{-\infty}_{\mathrm{is}}(W),\ h\#g = (n+2k)g \Longrightarrow s_k g = g. \end{gathered} \tag{137}$$ [HHH2.216]

It is useful to have several representations of these symbols.

[HHH.50.1]

LEMMA 19. *For each $k \geq 0$ there is a real polynomial p_k of degree k such that*

$$s_k(z,\bar{z}) = p_k(h(z))e^{-h(z)}. \tag{138}$$ [HHH.50.2]

PROOF. There are many ways to prove this statement. One approach is to use the classical formula for the Weyl symbol of the heat kernel:

$$\begin{aligned} \sigma_W(e^{-tH}) &= \cosh^{-n}(t)\exp(-h(z)\tanh t) \\ &= \exp(-h(z))\cosh^{-n}(t)\exp(-h(z)[\tanh t - 1])_{[HHH.50.3]} \\ &= \exp(-h(z))2^n e^{-nt}\sum_{k=0}^{\infty}\frac{[-2h(z)e^{-2t}]^k}{k!(1+e^{-2t})^{n+k}}. \end{aligned} \tag{139}$$

Expanding the terms $(1+e^{-2t})^{n+k}$ as Taylor series in e^{-2t} and reversing the order of summation we obtain the formula

$$\sigma_W(e^{-tH}) = \exp(-h)\sum_{k=0}^{\infty} p_k(h(z))e^{-(n+2k)t}.$$

Here p_k is a polynomial of degree k. The proof follows from the functional calculus by recalling that the spectrum of H consists precisely of the set $\{n+2k : k \in \mathbb{N}_0\}$. □

It is an immediate consequence of this lemma that the symbols $s_k(z,\bar{z})$ are invariant under the action of the unitary group.

We can also give a representation of these symbols using an eigenbasis expansion. If v_0 is the orthonormalized ground state then the eigenstate corresponding to the eigenvalue, $n+2k$ is spanned by the functions, $\{C^\alpha v_0 : \alpha \in \mathcal{I}_k\}$. Here $\mathcal{I}_k$ denotes the set of multi-indices, $\{\alpha = (\alpha_1, \ldots, \alpha_n) \ :\ |\alpha| = \alpha_1 + \cdots + \alpha_n = k\}$. Using (134) one can show that

$$< C^\alpha v_0, C^\beta v_0 >= c_\alpha \delta_{\alpha\beta},$$

where the positive constants satisfy, $\{c_\alpha = 2^{|\alpha|}\alpha!\}$. From this we easily conclude that

$$s_k = \sum_{\alpha\in\mathcal{I}_k} \frac{z^\alpha \# s \# \bar z^\alpha}{c_\alpha}. \tag{140}$$ [HHHc.10.12.1998.1]

Note that as $s_k\#s_l = 0$ if $k \neq l$ if we set

$$s_{(N)} = \sum_{k\le N} s_k$$

then $s_{(N)}\#s_{(N)} = s_{(N)}$ is the orthogonal projection onto the span of the eigenspaces with eigenvalues at most $n + 2N$.

As projections the $s_{(N)}$ define subalgebras, which we call the higher Toeplitz symbol algebras. Let us set

$$\begin{aligned} t_k &= \{a \in \Psi^{-\infty}_{\mathrm{is}}(W); a = s_k\#a'\#s_k,\ a' \in \Psi^{-\infty}_{\mathrm{is}}(W)\} \\ &= \{a \in \Psi^{-\infty}_{\mathrm{is}}(W); a = s_k\#a = a\#s_k\} \text{ and} \\ t_{(N)} &= \{a \in \Psi^{-\infty}_{\mathrm{is}}(W); a = s_{(N)}\#a = a\#s_{(N)}\}. \end{aligned} \tag{141}$$ [HHH2.222]

Cleary $t_0 = \mathbb{C}$ is the one-dimensional subalgebra spanned by $s = s_0$. More generally we can consider the modules

$$t_{k,l} = \{a \in \Psi^{-\infty}_{\mathrm{is}}(W); a = s_k\#a'\#s_l,\ a' \in \Psi^{-\infty}_{\mathrm{is}}(W)\}. \tag{142}$$

From the relations above we have

$$\begin{gathered} t_{k,l}\#t_{l,p} \subset t_{k,p},\ t_{k,l}\#t_{l',p} = \{0\},\ l' \neq l, \\ t_{(N)} = \sum_{k,l\le N} t_{k,l}. \end{gathered} \tag{143}$$ [HHH2.223]

[HHH2.540]

LEMMA 20. *For* $a \in \Psi^{-\infty}_{\mathrm{is}}(W)$

$$s\#a\#s = \mathrm{Tr}(s\#a\#s)s$$

where $\mathrm{Tr}(b)$ *is defined in* (118).

For each l let F_l denote the space of homogeneous holomorphic polynomials of degree l and let $\bar F_l$ denote their complex conjugates. As abstract vector spaces these are simply the lth symmetric powers

$$F_l = \bigodot^l W^{1,0},\ \bar F_l = \bigodot^l W^{0,1}, \tag{144}$$ [HHH2.217]

The maps

$$F_l \ni p \longmapsto p\#s \in \Psi^{-\infty}_{\mathrm{is}}(W),\ \bar F_l \ni p \longmapsto s\#p \in \Psi^{-\infty}_{\mathrm{is}}(W) \tag{145}$$ [HHH2.229]

are respectively isomorphisms onto $t_{l,0}$ and $t_{0,l}$. If we give F_l the Hermitian inner product arising this way,

$$s\#\bar p\#p\#s = |p|^2_{\mathrm{BF}}s \tag{146}$$ [HHH2.224]

then $F_l \otimes \bar F_k \equiv \hom(F_k, F_l)$ is the usual sesquilinear identification.

Using these identifications we can further identify $t_{k,l}$ with $\hom(F_l, F_k)$. Applying the second map in (145) in the second variable we obtain

$$
\begin{array}{ccc}
\hom(F_l) \equiv F_l \otimes \bar{F}_l & \longrightarrow & \Psi^{-\infty}_{\mathrm{is}}(W; F_l), \\
\uparrow & & \uparrow \\
\mathrm{Id} & \longrightarrow & q_l
\end{array}
\tag{147}
$$
[HHH2.230]

More concretely, written as a vector parameterized by $\mathcal{I}_l$,

$$q_l(z, \bar{z}) = (s\#\bar{z}^\alpha c_\alpha^{-\frac{1}{2}}; \alpha \in \mathcal{I}_l).$$

It is useful to think of the range of q_l as a column vector. The adjoint is then given by $q_l^*(z) = (z^\alpha \# s c_\alpha^{-\frac{1}{2}}; \alpha \in \mathcal{I}_l)$ which we think of as a row vector.

[HHH.40.041]

Lemma 21. *For each k, l the map*

$$t_{k,l} \ni a \longmapsto q_k \# a \# q_l^* \in \Psi^{-\infty}_{\mathrm{is}}(W; \hom(F_l, F_k)) \tag{148}$$
[HHH2.231]

is an isomorphism onto $s \otimes \hom(F_l, F_k)$.

Proof. The isomorphism to $s \otimes \hom(F_k, F_l)$ is obtained by observing that for all j, $F_j \simeq F_j \# s$. This vector space has an orthonormal basis $\{c_\alpha^{-\frac{1}{2}} z^\alpha \# s; |\alpha| = j\}$. An element $A \in t_{k,l}$ has the form

$$
\begin{aligned}
A =& s_k \# a \# s_l \\
=& \sum_{\alpha \in \mathcal{I}_k,\, \beta \in \mathcal{I}_l} \frac{1}{c_\alpha c_\beta} z^\alpha \# s \# \bar{z}^\alpha \# a \# z^\beta \# s \# \bar{z}^\beta \\
=& \sum_{\alpha \in \mathcal{I}_k,\, \beta \in \mathcal{I}_l} a_{\alpha\beta} \frac{z^\alpha \# s \# \bar{z}^\beta}{\sqrt{c_\alpha c_\beta}},
\end{aligned}
$$

where

$$a_{\alpha\beta} = \mathrm{Tr}(s \# \bar{z}^\alpha \# a \# z^\beta \# s)[c_\alpha c_\beta]^{-\frac{1}{2}}. \tag{149}$$
[8.5.99.12]

The action on an element $f = \sum c_\gamma^{-\frac{1}{2}} b_\gamma z^\gamma \# s \in F_k \# s$ is

$$f \to A \# f = c_k^{-\frac{1}{2}} \sum_{\alpha \in \mathcal{I}_k, \beta \in \mathcal{I}_l} z^\alpha \# s a_{\alpha\beta} b_\beta, \tag{150}$$
[HHHc.10.14.1998.3]

from which it is evident that $a_{\alpha\beta} \in \hom(F_l, F_k)$. The composition, $q_k \# a \# q_l^*$ is then the $|\mathcal{I}_k| \times |\mathcal{I}_l|$-matrix, $(a_{\alpha\beta} s)$. $\square$

Summing these over $0 \le l \le N$ we obtain the vector spaces,

$$F_{(N)} = \bigoplus_{0 \le l \le N} F_l = \bigoplus_{0 \le l \le N} \bigodot^{l} W^{1,0} \tag{151}$$
[HHH2.330]

Thus we have proved

[HHH2.541]

Lemma 22. *The subalgebra $t_{(N)} \subset \Psi^{-\infty}_{\mathrm{is}}(W)$ in (143) is naturally isomorphic to $s \otimes \hom(F_{(N)})$ with the isomorphism realized by*

$$t_{(N)} \ni a \longmapsto q_{(N)} \# a \# q_{(N)}^* \in s \otimes \hom(F_{(N)}) \subset \Psi^{-\infty}_{\mathrm{is}}(W; \hom(F_{(N)})). \tag{152}$$
[HHH2.228]

Alternately, if we use the isomorphism $F_{(N)} \simeq F_{(N)}\#s$ then every element $A \in \hom(F_{(N)})$ has unique representative $a \in t_{(N)}$ where the action is $A : f\#s \mapsto a\#f\#s$. To demonstrate this we first prove a lemma which is a crucial step in the proof of the index theorem for the Heisenberg algebra.
[HHH2.384]

LEMMA 23. *Let $a \in \Psi_{\mathrm{is}}^{-\infty}(W)$ then as $N \to \infty$ $s_{(N)}\#a\#s_{(N)}$, $s_{(N)}\#a$ and $a\#s_{(N)}$ all converge to a in $\mathcal{S}(W) = \dot{\mathcal{C}}^\infty(\overline{W})$.*

PROOF. We work in the model case, taking $W = \mathbb{R}^{2n}$ and only give the proof for the first sequence, the other cases are very similar. Let $A = q_W(a)$ and $A_N = q_W(s_{(N)}\#a\#s_{(N)})$. We also let $A(x,y)$ and $A_N(x,y)$ respectively, denote their Schwartz kernels, which belong to $\mathcal{S}(\mathbb{R}^{2n})$. Finally let $\pi_N = q_W(s_{(N)})$ and H denote the harmonic oscillator on $\mathbb{R}^n$. By the Sobolev embedding theorem, the seminorms

$$\|f\|_j = \|H^j f\|_{L^2}$$

induce the standard topology on Schwartz space, as for example [**24**]. We obtain the analogous operators on $\mathbb{R}^{2n}$ letting $P_N = \pi_N \otimes \pi_N$ and $\mathcal{H} = H_x \oplus H_y$ denote the harmonic oscillator acting on $\mathbb{R}^{2n}$. It is immediate that $\mathcal{H}P_N = P_N\mathcal{H}$ for every N and that

$$A_N = P_N A.$$

Since $A \in L^2(\mathbb{R}^{2n})$ and P_N is a sequence of orthogonal projections converging to the identity we conclude that

$$\lim_{N\to\infty} \|A - A_N\|_{L^2} = 0.$$

As $\mathcal{H}^k A_N = P_N \mathcal{H}^k A$ and $\mathcal{H}^k A \in L^2(\mathbb{R}^{2n})$ for every k it again follows that

$$\lim_{N\to\infty} \|A - A_N\|_k = \lim_{N\to\infty} \|\mathcal{H}^k(A - A_N)\|_{L^2} = 0.$$

As these semi-norms define the standard topology on $\mathcal{S}(\mathbb{R}^{2n})$ we conclude that $A_N(x,y)$ converges to $A(x,y)$ in the Schwartz space topology. As $s_{(N)}\#a\#s_{(N)}$ is obtained from $A_N(x,y)$ by formula (98), the conclusion of the lemma follows from the convergence of the A_N. $\square$

Note that we may interpret the maps $q_{(N)}$ and $q^*_{(N)}$ obtained by summing the map in (145) as elements

$$\tilde{q}_{(N)} \in \Psi_{\mathrm{is}}^{-\infty}(W; \hom(F_{(N)}, \mathbb{C})),\ \tilde{q}^*_{(N)} \in \Psi_{\mathrm{is}}^{-\infty}(W; \hom(\mathbb{C}, F_{(N)})). \tag{153}$$ [HHH2.254]

Using our description of these maps as row and column vectors it follow from (140) that

$$\tilde{q}_{(N)}\#\tilde{q}^*_{(N)} = \mathrm{Id}_{F_{(N)}} \otimes s,\ \tilde{q}^*_{(N)}\tilde{q}_{(N)} = s_{(N)}. \tag{154}$$ [HHH2.255]

In applications it is useful to have similar projections acting on vector valued functions. If E is a complex vector space then for each k

$$s_{k,E} = s_k \otimes \mathrm{Id}_E$$

defines a symbol with values in $\hom(E)$. If a is a symbol taking values in $\hom(E)$ then using Lemma 21 we can identify $s_{k,E}\#a\#s_{l,E}$ as an element of

$$\hom(E \otimes F_l, E \otimes F_k).$$

Analogously, if we let $s_{(N,E)} = \mathrm{Id}_E \otimes s_{(N)}$ then $s_{(N,E)}\#a\#s_{(N,E)}$ is identified with an element of $\hom(E \otimes F_{(N)})$.

4. Representations of the isotropic algebra [S.Bargmann.Fock]

The isotropic algebra is defined on a symplectic vector space and more generally on a symplectic vector bundle over a manifold. In the case of a vector space one can choose a Lagrangian splitting and this allows one to define a quantization as in (97). In the bundle case this is generally not possible, a symplectic vector bundle does not usually have a global splitting as a sum of complementary Lagrangian subspaces. It is therefore useful to have representations for the isotropic algebra which are do not require such a splitting. As is already explained in [**13**], the choice of a field of harmonic oscillators leads to such representations. In this subsection we consider two such representations.

Let $a \in \Psi^*_{\mathrm{is}}(W)$ if $f \in \Psi^{-\infty}_{\mathrm{is}}(W)$ the $a\#f \in \Psi^{-\infty}_{\mathrm{is}}(W)$ as well. The symbolic estimates show that for any s the map from $\Psi^s_{\mathrm{is}}(W) \times \Psi^{-\infty}_{\mathrm{is}}(W)$ to $\Psi^{-\infty}_{\mathrm{is}}(W)$ defined by $(a, f) \to a\#f$ is continuous. Recall that $a \in \Psi^s_{\mathrm{is}}(W)$ is elliptic element if there is a $b \in \Psi^{-s}_{\mathrm{is}}(W)$ such that

$$a\#b - 1, b\#a - 1 \in \Psi^{-\infty}_{\mathrm{is}}(W).$$

For classical symbols this is equivalent to the non-vanishing of the principal symbol. If we fix a Lagrangian splitting and thereby a quantization map q_W, then an elliptic symbol defines Fredholm operator on $\mathcal{S}(\mathbb{R}^n)$. In this case the ellipticity condition above implies that the $\operatorname{Nul} q_W(a)$ and $\operatorname{Nul} q_W(a)^*$ are finite dimensional. Without fixing a quantization we define the abstract nullspace of a to be

$$\operatorname{Nul}_a(a) = \{f \in \Psi^{-\infty}_{\mathrm{is}}(W) \mid a\#f = 0\}.$$

[1.30.00.1]

LEMMA 24. *If a is an elliptic element of $\Psi^s_{\mathrm{is}}(W)$ then $\operatorname{Nul}_a(a)$ is a finite dimensional right $\Psi^{-\infty}_{\mathrm{is}}(W)$-module.*

PROOF. Let $f \in \operatorname{Nul}_a(a)$ and $g \in \Psi^{-\infty}_{\mathrm{is}}(W)$ then as the isotropic algebra is associative $a\#(f\#g) = (a\#f)\#g = 0$ showing that $\operatorname{Nul}_a(a)$ is a right $\Psi^{-\infty}_{\mathrm{is}}(W)$-module. To show that it is finite dimensional we fix a quantization q_W for the isotropic algebra. Let $\{v_1, \dots, v_l\}$ be a basis for the null space of $q_W(a)$. An element $f \in \operatorname{Nul}_a(a)$ if and only if $\operatorname{ran} q_W(f) \subset \operatorname{Nul} q_W(a)$. This implies that the Schwartz-kernel of $q_W(f)$ is of the form

$$k_f = \sum_{i=1}^{l} v_i \otimes z_i$$

for some collection of functions $z_i \in \mathcal{S}(\mathbb{R}^n)$. For each i let f_i denote the rank one element of $\Psi^{-\infty}_{\mathrm{is}}(W)$ whose Schwartz kernel is $v_i \otimes z_0$, where z_0 is the normalized vacuum state for a choice of harmonic oscillator on $\mathbb{R}^n$. If we let g_i denote the rank one elements with Schwartz kernel $z_0 \times z_i$ then it follows from the composition formula that

$$f = \sum_{i=1}^{l} f_i\#g_i.$$

This completes the proof of the lemma. □

For our applications to K-theory it is preferrable to have an abstract notion of the null space of a symbol which is vector space rather than an $\Psi^{-\infty}_{\mathrm{is}}(W)$-module. Here it is very convenient to fix a choice of compatible almost complex structure and thereby a harmonic oscillator. Let s be the symbol of the projector onto the vacuum state. For an elliptic symbol we define the reduced abstract null space by

$$\operatorname{Nul}_r(a) = \{f \in \Psi_{\mathrm{is}}^{-\infty}(W) \mid a\#f = 0 \text{ and } f\#s = f\}. \tag{155}$$
[1.30.00.5]

[1.30.00.2]

LEMMA 25. *If $a \in \Psi_{\mathrm{is}}^s(W)$ is an elliptic symbol then $\operatorname{Nul}_r(a)$ is a finite dimensional complex vector space with dimension equal to the null space of $q_W(a)$ for any choice of quantization. If $a(x)$ is a continuous family of elliptic symbols for which $\dim \operatorname{Nul}_r(a(x))$ is locally constant then the reduced kernels form a continuous family of subspaces of $\Psi_{\mathrm{is}}^{-\infty}(W)$.*

PROOF. Since $\Psi_{\mathrm{is}}^{-\infty}(W)$ is a $\mathbb{C}$-algebra it is clear that $\operatorname{Nul}_r(a)$ is a complex vector space. Fix a quantization map compatible with our choice of almost complex structure. Let $\{v_1, \dots, v_l\}$ be an orthonormal basis for the null space of $q_W(a)$. As before, if $f \in \operatorname{Nul}_r(a)$ then the Schwartz kernel of $q_W(f)$ is of the form

$$k_f = \sum_{i=1}^{l} v_i \otimes z_i.$$

The fact that $f\#s = f$ implies that each $z_i = c_i z_0$ where $c_i \in \mathbb{C}$ and z_0 is the normalized vacuum state for the harmonic oscillator. With f_i the symbol whose quantization has Schwartz kernel $v_i \otimes z_0$ we have shown that

$$f = \sum_{i=1}^{l} c_i f_i.$$

It is obvious that the $\{f_i\}$ are linearly independent.

The continuity of the reduced kernels for a continuous family follows from similar considerations. Let $A(x)$ denote the quantizations. If we fix a parameter x_0 then for x sufficiently close to x_0 there is an $\epsilon > 0$ so that the spectrum of $A(x)$ intersects $B_\epsilon(0)$ only at 0. A standard argument in perturbation theory shows that we can find bases $\{v_1(x), \dots, v_l(x)\}$ for $\operatorname{Nul} A(x)$ depending continuously on x. The symbols $\{f_i(x) \mid i = 1, \dots, l\}$ whose quantizations have Schwartz kernels $\{v_i(x) \otimes z_0 \mid i = 1, \dots, l\}$ define a continuous family of bases for $\operatorname{Nul}_r(a(x))$. $\square$

There is a different though related representation which can be defined on the Bargmann-Fock space. The classical Bargmann-Fock space is defined as the closure of polynomials on $\mathbb{C}^n$ with respect to the inner product defined by:

$$|p|_{\mathrm{BF}}^2 = \int_{\mathbb{C}^n} |p(z)|^2 e^{-\frac{1}{2}h(z)} dVol. \tag{156}$$
[HHH.50.4]

Let $\mathcal{H}_{\mathrm{BF}}^n$ be the Bargmann-Fock space, it is the Hilbert space of entire functions on $\mathbb{C}^n$ with $|f|_{\mathrm{BF}} < \infty$. A holomorphic polynomial p of degree d defines a symbol in $\Psi_{\mathrm{is}}^d(W)$ and therefore $p\#s \in \Psi_{\mathrm{is}}^{-\infty}(W)$. The Bargmann-Fock inner product is represented by

$$|p|_{\mathrm{BF}}^2 = \operatorname{Tr} s\#\bar{p}\#p\#s.$$

Note however that the closure is not a subset of $\Psi_{\mathrm{is}}^{\mathbb{Z}}(W)$, i.e. under this closure operation we do not get symbols. This is easy to see because differentiation is not a bounded operator on the Bargmann-Fock space.

If $a \in \Psi_{\mathrm{is}}^0(W)$ and p is a holomorphic polynomial of degree d then it follows from the symbolic composition formula for the isotropic algebra (in terms of holomorphic and anti-holomorphic derivatives) that $a\#p$ is again a holomorphic polynomial of degree at most d. We have the following estimate:

[HHH.50.7]

LEMMA 26. *If $a \in \Psi^0_{\text{is}}(W)$ and f is a holomorphic polynomial then*

$$|a\#f|_{\text{BF}} \le \|A\||f|_{\text{BF}}. \quad [HHH.50.7.1] \tag{157}$$

Here $A = q_W(a)$ and $\|A\|$ is the L^2-operator norm.

PROOF. If we let $F = q_W(f)$ and $S = q_W(s)$ then it is a simple consequence of the fact that FS is trace class and (118) that

$$|a\#f\|_{\text{BF}} = \text{Tr}(SF^*A^*AFS) = \text{Tr}(A^*AFSF^*).$$

Let $\{v_i\}$ be an eigenbasis for the compact, non-negative self adjoint operator FSF^* with $\{\lambda_i \ge 0\}$ the corresponding eigenvalues. From the following identities

$$\begin{aligned} \|f\|^2_{\text{BF}} &= \text{Tr}(FSF^*) = \sum_{j=1}^{\infty} \lambda_j, \\ \|a\#f\|_{\text{BF}} &= \sum_{j=1}^{\infty} \lambda_j < Av_i, Av_i >_{L^2} \end{aligned} \tag{158}$$

the conclusion of the lemma is immediate. □

This lemma leads to a second representation:

[HHH.50.8]

PROPOSITION 4. *For an element $a \in \Psi^o_{\text{is}}(W)$ the action on holomorphic polynomials $f \to a\#f$ extends by continuity to define a bounded linear operator on the Bargmann-Fock space. For a continuous family of symbols the associated operators are continuous in the operator norm topology.*

PROOF. Since the holomorphic polynomials are dense in Bargmann-Fock space, the lemma show that the action of an element $a \in \Psi^0_{\text{is}}(W)$ on holomorphic polynomials extends by continuity to define a bounded linear operator on the Bargmann-Fock space. If $a(x)$ denotes a continuous family of symbols in $\Psi^0_{\text{is}}(W)$ then for any pair of parameters x_1, x_2 and element f of the Bargmann-Fock space, we have the estimate

$$|a(x_1)\#f - a(x_2)\#f|_{\text{BF}} \le \|A(x_1) - A(x_2)\||f|_{\text{BF}}.$$

Since the operator norm of the quantization is controlled by a finite number of symbolic semi-norms, the continuity of the family of symbols implies that

$$\lim_{x_1 \to x_2} \|A(x_1) - A(x_2)\| = 0.$$

□

5. Reversal of orientation [SS.Reverse.orientation]

Let (W, ω) be a symplectic vector space. The $\#$-product of a pair of isotropic symbols defined on W is given in formula (120). This product is an essential part of the composition formula for a the Heisenberg calculus and in the symbolic composition formula (263) we use the $\#$-product with respect to the symplectic structure defined by both ω and $-\omega$. In this section we consider the relationship between these two products. For a, b a pair of isotropic symbols on W we let

$$a\#_{\pm}b(w) = \iint\limits_{W\times W} a(u+w)b(v+w)e^{\pm 2i\omega(u,v)}dudv.$$

So that no ambiguity should arise we establish the convention that $\# = \#_+$.

The following identities are immediate consequences of the definition

$$a\#_+b = b\#_-a, \tag{159}$$ [8.5.99.1]

$$a\#_+b = \overline{\bar{a}\#_-\bar{b}}. \tag{160}$$ [8.5.99.2]

If $\{a_1, \ldots, a_m\}$ is a collection of isotropic symbols then using these identities recursively we easily see that

$$a_1\#_+\ldots\#_+a_m = a_m\#_-\ldots\#_-a_1, \tag{161}$$ [8.5.99.6]

$$a_1\#_+\ldots\#_+a_m = \overline{\bar{a}_1\#_-\ldots\#_-\bar{a}_m}. \tag{162}$$ [8.5.99.7]

If J is an almost complex structure compatible to ω then $-J$ is an almost complex structure compatible to $-\omega$; this amounts to replacing the almost complex structure by its conjugate. If $\{z_1, \ldots, z_n\}$ are linear holomorphic coordinates with respect to J then $\{\bar{z}_1, \ldots, \bar{z}_n\}$ are holomorphic coordinates with respect to $-J$. The symbol of the harmonic oscillator then becomes:

$$h = \sum_{i=1}^{\infty} \bar{z}_i\#_-z_i + n = \sum_{i=1}^{\infty} z_i\#_+\bar{z}_i + n, \tag{163}$$ [8.5.99.3]

where the second equality follows from (159). The basic commutation relation becomes:

$$[h, \bar{z}_i]_- = 2\bar{z}_i. \tag{164}$$ [8.5.99.4]

The symbol of the ground state projection remains $2^n e^{-h}$. Using (164) we obtain formulæ analogous to (140) for the symbols of the higher eigenprojections with respect to the conjugate complex structure:

$$s_{k-} = \sum_{\alpha\in\mathcal{I}_k} \frac{\bar{z}^\alpha\#_-s\#_-z^\alpha}{c_\alpha}. \tag{165}$$ [8.5.99.5]

It is a simple consequence of (161) that $s_{k-} = s_k$ and therefore it is not necessary to distinguish which product we are using when referring to these projections.

For later application we consider the relationship between the two isomorphisms given by Lemma 21 for $\pm\omega$. To avoid confusion we use the notation $W^{1,0}$ and $W^{0,1}$ to refer to the $(1,0)$ and $(0,1)$-subspaces of $W\otimes\mathbb{C}$ with respect to the almost complex structure defined by J. We then use the definitions of $F_l, \bar{F}_l$ given in (144). Generalizing slightly we let

$$t^{\pm}_{k,l} = \{a \in \Psi^{-\infty}_{\text{is}}(W); a = s_k\#_{\pm}a'\#_{\pm}s_l,\ a' \in \Psi^{-\infty}_{\text{is}}(W)\}. \tag{166}$$

A simple application of (161) shows that

$$t^+_{k,l} = t^-_{l,k}. \tag{167}$$

We define the vector valued symbols

$$q^-_l = (s\#_-z^\alpha c_\alpha^{-\frac{1}{2}} : \alpha\in\mathcal{I}_l) \text{ and } q^{-*}_l = (\bar{z}^\alpha\#_-s : \alpha\in\mathcal{I}_l).$$

[8.5.99.8]

LEMMA 27. *For each k, l the map*

$$t^-_{k,l} \ni a \longmapsto q^-_k \#_- a \#_- q^{-*}_l \in \Psi^{-\infty}_{\text{is}}(W; \hom(\bar{F}_l, \bar{F}_k))^{[8.5.99.9]} \tag{168}$$

is an isomorphism onto $s \otimes \hom(\bar{F}_l, \bar{F}_k)$.

The action of $a \in t^-_{k,l}$ on an element f of $\bar{F}_k \# -s \simeq \bar{F}_k$ is given by $A : f \mapsto a\#_f$. Using the identities above we obtain the matrix representation for A with respect to the orhtonormal bases of $\bar{F}_k$ ($\bar{F}_l$ respectively $\{c_\alpha^{-\frac{1}{2}} \bar{z}^\alpha \#_- s : |\alpha| = k(l)\}$

$$A_- = s_k \#_- a \#_- s_l \tag{169}$$

$$= \sum_{\alpha \in \mathcal{I}_k,\, \beta \in \mathcal{I}_l} \frac{1}{c_\alpha c_\beta} \bar{z}^\alpha \#_- s \#_- z^\alpha \#_- a \#_- \bar{z}^\beta \#_- s \#_- z^\beta \tag{170}$$

$$= \sum_{\alpha \in \mathcal{I}_k,\, \beta \in \mathcal{I}_l} a^-_{\alpha\beta} \frac{\bar{z}^\alpha \#_- s \#_- z^\beta}{\sqrt{c_\alpha c_\beta}}, \tag{171}$$

where

$$a^-_{\alpha\beta} = \operatorname{Tr}(s \#_- z^\alpha \#_- a \#_- \bar{z}^\beta \# s)[c_\alpha c_\beta]^{-\frac{1}{2}}.^{[8.5.99.11]}$$

As an immediate consequence of these definitions and the identities (161) and (162) we have

[8.5.99.10]

LEMMA 28. *Let E be a complex vector space and a_{ij} be an isotropic symbol taking values in $\hom(E)$. If $a^+_{ij;\alpha\beta}$ is given by (149) then we have the identities*

$$a^-_{ij;\alpha\beta} = \overline{(\bar{a})^+_{ij;\alpha\beta}}.^{[8.5.99.13]} \tag{172}$$

If a is a scalar symbol then

$$a^-_{\alpha\beta} = \overline{a^+_{\beta\alpha}}.^{[8.5.99.15]} \tag{173}$$

REMARK 2. If a_{ij} is a symbol with values in $\hom(E)$ then $\bar{a}_{ij}$ is naturally a symbol with values in $\hom(\bar{E})$. If E has a Hermitian inner product, $< e, f >$ then the map

$$j : f \mapsto < \cdot, \bar{v} >$$

is a complex linear isomorphism of $\bar{E}$ to E', the dual of E. If A is an element of $\hom(E)$ then there is a canonically element $A^t \in \hom(E')$ defined by

$$(A^t l)(e) = l(Ae).$$

A simple calculation shows that

$$j(\bar{A}v)(e) = j(v)(A^* e), \quad j(A^t v)(e) = j(v)(Ae).$$

In other words: under the dual inner product induced on E' by the inner given product on E the linear transformations $\bar{A}$ and A^t are adjoints.

We let C denote the operation of complex conjugation. For each $l \in \mathbb{N}$ we have

$$C : \bar{E} \otimes F_l \#_+ s \longleftrightarrow E \otimes \bar{F}_l \#_- s.$$

Let a_{-N} be the element of $\hom(E \otimes \bar{F}_{(N)} \#_- s)$ defined by $s_{(N,E)} \#_- a \#_- s_{(N,E)}$ and $\bar{a}_{+N}$ the element of $\hom(\bar{E} \otimes F_{(N)} \#_+ s)$ defined by $s_{(N,\bar{E})} \#_+ \bar{a} \#_+ s_{(N,\bar{E})}$. From (172) we conclude that for a symbol a with values in $\hom(E)$ we have, for each N the identity:

$$a_{-N} = C\bar{a}_{+N}C. \tag{174}$$ [8.5.99.14]

If a is a scalar symbol then we can use (173) to obtain that

$$a_{-N} = Ca^*_{+N}C. \tag{175}$$ [8.5.99.16]

For a matrix valued symbol a_{ij} we can also use (173) to conclude that

$$a^-_{ij;\alpha\beta} = \overline{a^+_{ij,\beta\alpha}}.$$

However, if $\dim(E) > 1$ then the linear tranformation of $\bar{E} \otimes F_{(N)}$ defined by the right hand side of this equation is not related to that defined by $a^+_{ij;\alpha\beta}$ in a simple way.

6. Symplectic bundles[SS.Almost.complex.structures]

A symplectic bundle W is an even-dimensional real vector bundle, with base X, with a symplectic structure on the fibers W_x which varies smoothly with $x \in X$. The construction above gives a bundle of isotropic algebras, for example making $\mathcal{C}^\infty(\overline{W})$ into an algebra where the product is fiber-by-fiber the isotropic product on $\mathcal{C}^\infty(\overline{W_x})$ given by the symplectic structure on that fiber.

As is well-known,the fibers of any symplectic bundle, W may be given a complex structure compatible with the symplectic structure on the fibers in the sense of (131). Indeed, choosing any fibre metric h' on W the bundle isomorphism defined by the third identity in (131) J', is real and automatically satisfies the second identity and hence is skew-symmetric for the metric h'. Its eigenvalues are therefore pure imaginary, non-zero and in conjugate pairs. The span of the eigenfuntions with eigenvalues having positive imaginary parts gives a smooth complex subbundle of the complexification, $W_{\mathbb{C}} = W \otimes \mathbb{C}$, which defines a smooth family of complex structures, and hence an isomorphism J. Since the space of fibre metrics is contractible this shows that the space of complex structures is also contractible.

A symplectic bundle with compatible complex structure admits a connection for which both the symplectic form and the tensor J are parallel. To construct such a connection, let $P \to X$ denote the principal bundle of symplectonormal frames of W:

$$\begin{aligned}(X_1, \dots, X_n, Y_1, \dots, Y_n) \text{ such that } \omega(X_i, X_j) = \omega(Y_i, Y_j) = 0 \text{ for all } i, j,\\ \omega(X_i, Y_j) = \delta_{ij},\ JX_i = Y_i, \quad JY_i = -X_i, \quad i = 1, \dots, n.\end{aligned}$$

The structure group of this bundle is $U(n)$ embedded as a subgroup of the symplectic group $\mathrm{Sp}(n)$ via

$$(A + iB) \longrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \tag{176}$$ [HHH.40.01]

From this formulation it is apparent that there is an ad-equivariant matrix of forms, ψ defined on P taking values in the Lie algebra of the structure group and reducing to the Maurer-Cartan form on the fibers of P. Indeed, locally one can take the product connection and then sum over a partition of unity in the base.

Let ∇ denote the induced affine connection on sections of W. Let $(X_1, \dots, X_{2n})$ denote a local section of P and Ψ the connection matrix relative to this local framing. The functions $\omega(X_i, X_j)$ and $h(X_i, X_j)$ are locally constant. Let Z be a vector field defined in this neighborhood then

$$\nabla_Z X_i = \Psi(Z)_i^j X_j.$$

As $\Psi(Z)_i^j$ is in the intersection of the Lie algebras of the "orthogonal groups" for ω and h it follows that

$$\omega(\nabla_Z X_i, X_j)) + \omega(X_i, \nabla_Z(X_j)) = 0 \text{ and } h(\nabla_Z X_i, X_j) + h(X_i, \nabla_Z(X_j)) = 0.$$

Thus $\nabla\omega_0 = \nabla h = 0$.

Using the definition of the covariant derivative of a endomorphism, we obtain

$$(\nabla J)(X) = \nabla(JX) - J(\nabla X).$$

Since $Zh(JX_i, X_j) = 0$ it follows that

$$0 = h(\nabla_Z(JX_i), X_j) + h(JX_i, \nabla_Z X_j).$$

Using the formula for ∇J we conclude that

$$\begin{aligned} -h((\nabla_Z J)X_i, X_j) &= h(J\nabla_Z X_i, X_j) + h(JX_i, \nabla_Z X_j) \\ &= -[\omega(\nabla_Z X_i, X_j) + \omega(X_i, \nabla_Z X_j)] = 0. \end{aligned}$$

Thus $\nabla J = 0$ as well.

7. Associated Bundles and Connections[S.Assoc.bundles]

This discussion here of connections on the bundle of isotropic algebras to some extent follows Fedosov in [**13**], although he does not use the approach through the principal bundle; for this see [**3**].

In the previous section we constructed a $U(n)$–principal bundle, P as the bundle of symplectonormal frames for the symplectic bundle, W. This arose from the choice of a compatible almost complex structure on the fibers of W. We now use the fiber product to construct several associated bundles. This facilitates the construction of connections on these bundles. For the convenience of the reader we recall this construction: If $\pi : P \to X$ is a G-principal bundle and $\rho : G \to Gl(V)$ is a linear representation of G then the fiber product

$$\begin{aligned} &P \times_\rho V = P \times V/\sim \text{ where} \\ &(p, v) \sim (g \cdot p, \rho(g^{-1})v) \text{ for all } g \in G \end{aligned}$$

is a vector bundle over X with fiber, V. Let $p_\rho : P \times V \to P \times_\rho V$ denote the canonical projection. If V has an algebra structure which is respected by the representation, ρ then $P \times_\rho V$ is a bundle of algebras. More concretely, suppose that there is a bilinear operation, $V \times V \to V$, denoted by $(v, w) \to v \circ w$ which gives V the structure of a real or complex algebra. If we have the identity

$$(\rho(g)v) \circ (\rho(g)w) = \rho(g)(v \circ w) \text{ for all } g \in G,\ u, v \in V$$

then the associated bundle is a bundle of algebras.

A connection on the principal bundle, P is a choice of subbundle $H_P \subset TP$ which satisfies:

1. H_P is G–invariant
2. $\pi_* H_{Pq} \to T_{\pi(q)}X$ is an isomorphism for each $q \in P$.

This induces a connection on every associated vector bundle. Recall that a connection on a vector bundle is defined by a choice of complement for the vertical (or fiber) tangent subbundle. We include H_P into $T(P \times V) \simeq TP \oplus TV$ as $H_P \oplus \{0\}$. The horizontal subspace in $T(P \times_\rho V)$, H_ρ. is simply $p_{\rho *}(H_P \oplus \{0\})$. The G–invariance of H_P implies that H_ρ is well defined and is a complement to the vertical subbundle on $T(P \times_\rho V)$. The connection one form takes values in the Lie algebra of the structure group of the fiber. If ϕ is the connection one form on P then $\rho_*(\phi)$ is the connection one form on $P \times_\rho V$. Here ρ_* denotes the homomorphism of Lie algebras defined by $d\rho_e$.

First of all let $\rho_1 : U(n) \to Gl(n, \mathbb{C})$ be the canonical representation. The associated bundle is simply $W^{1,0}$. The symmetric tensor products, $\{\rho_k\}$ of ρ_1 define the bundles $\odot_k W^{1,0}$. We next consider the action of $U(n)$ on $\mathcal{S}(\mathbb{R}^{2n})$. Let $i : U(n) \to Sp(n)$ be the map defined in (176) then we define a representation, ρ of $U(n)$ on $\mathcal{S}(\mathbb{R}^{2n})$ by

$$\rho(g) \cdot f(x) = f(i(g)x).$$

In (120) we defined an associative product on $\mathcal{S}(\mathbb{R}^{2n})$

$$(a, b) \to a\#b(w) = \frac{1}{(2\pi)^n (n!)^2} \int\limits_{\mathbb{R}^{2n}} \int\limits_{\mathbb{R}^{2n}} e^{2i\omega(u,v)} a(u + w) b(v + w) \omega_u^n \omega_v^n.$$

As $i(g) \in Sp(n) \cap O(2n)$ it is a simple matter to show that

$$(\rho(g) \cdot a)\#(\rho(g) \cdot b) = \rho(g) \cdot (a\#b). \quad [HHH.40.02] \tag{177}$$

By the remarks above it follows that $\Psi^{-\infty}_{\mathrm{is}}(X; W) \simeq P \times_\rho \mathcal{S}(\mathbb{R}^{2n})$ as a bundle of algebras over X with fiber isomorphic to $\Psi^{-\infty}_{\mathrm{is}}(\mathbb{R}^{2n})$. Indeed this product extends to a product on $\rho^{-\mathbb{Z}} \mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}})$, the space of quadratic classical isotropic symbols on $\mathbb{R}^{2n}$. As $\Psi^{-\infty}_{\mathrm{is}}(\mathbb{R}^{2n})$ is dense the equivariance, (177) extends to the full symbol space. We denote this corresponding bundle by $\Psi^{\mathbb{Z}}_{\mathrm{is}}(X; W) = \rho_q^{\mathbb{Z}} \mathcal{C}^\infty(X; {}^q\overline{W})$. Note that $\Psi^{-\infty}_{\mathrm{is}}(X; W)$ is a bundle of two sided ideals in $\Psi^{\mathbb{Z}}_{\mathrm{is}}(X; W)$.

A section a of $\Psi^s_{\mathrm{is}}(X; W)$ is elliptic if there exists a section b of $IPs{-}sX; W$ such that $a\#b - 1, b\#a - 1 \in \Psi^{-\infty}_{\mathrm{is}}(X; W)$. As usual it suffices that there exists a symbol $b_0 \in \Psi^{-s}_{\mathrm{is}}(X; W)$ so that $a\#b_0 - 1, b_0\#a - 1 \in \Psi^{-1}_{\mathrm{is}}(X; W)$. This condition is equivalent to a having a nowhere vanishing principal symbol. For an elliptic symbol we define an abstract null space. Let s denote the symbol of the projection onto the vacuum state for a field of harmonic oscillators. For an elliptic symbol $a \in \Psi^s_{\mathrm{is}}(X; W)$ we define the reduced null space to be

$$\mathrm{Nul}_r(a) = \{f \in \Psi^{-infty}_{\mathrm{is}}(\,)|\; a\#f = 0 \text{ and } f\#s = f\}. \quad [1.30.00.4] \tag{178}$$

The following is an immediate consequence of Proposition 25.

[1.30.00.3]

Proposition 5. *Let* $a \in \Psi^s_{\mathrm{is}}(X; W)$ *be an elliptic symbol such that for* $x \in X$ *the* $\dim \mathrm{Nul}_r(a(x))$ *is locally constant then* $\mathrm{Nul}_r(a)$ *is the space of smooth sections of a vector bundle over* X *which we also denote by* $\mathrm{Nul}_r(a)$. *The isomorphism class of this vector does not depend on the choice of almost complex structure.*

Proof. Only the last statement requires further justification. It is a consequence of the fact that the space of compatible almost complex structures is path connected. If s_0 and s_1 are two choices of vacuum projector then there is a smooth one parameter family $\{s_t\}$ joining them. This allows the smooth deformation of the null bundle defined by s_0 to that defined by s_1. □

This concept is needed in our treatment of K-theory and to prove the index theorem for the Heisenberg calculus.

In §3 we identified several families of $U(n)$-invariant subspaces of $\Psi^{\mathbb{Z}}_{\text{is}}(W)$, e.g. the vector spaces, $\{F_k : k = 1, 2, \dots\}$ defined by (144). Restricting the representation, ρ to each of these subspaces thereby defines a subbundle of $\rho_q^{-\mathbb{Z}}\mathcal{C}^\infty(X; {}^q\overline{W})$. We have the isomorphisms,

$$P \times_\rho F_k \simeq \bigodot_k W^{1,0}.$$

The construction of associated bundles gives very useful representations for sections of the associated bundles. Let $\mathcal{C}^\infty(P; V)^G$ denote the space of V−valued functions on P which satisfy:

$$f(g \cdot p) = \rho(g^{-1})f(p).$$

One easily sees that $x \to p_\rho(q, f(q))$ where $q \in \pi^{-1}(x)$ gives a well defined section, $F(x)$ of $P \times_\rho V$. These functions are in one-to-one correspondence with the sections of $P \times_\rho V$. If ρ is the representation of $U(n)$ on $\Psi^{-\infty}_{\text{is}}(\mathbb{R}^{2n})$ defined above then any function in $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty({}^q\overline{\mathbb{R}}^{2n})$ which is invariant under the action of $U(n)$ defines a global section of $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X; {}^q\overline{W})$. Using Lemma 19 we see that the symbol of the harmonic oscillator and the functional calculus it generates define canonical sections of $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X; {}^q\overline{W})$. If h_x is the quadratic form on $\mathbb{R}^{2n}$ which is invariant under the action of $i(U(n))$ then h_x defines a section $h \in \rho_q^{-1}\mathcal{C}^\infty(X; {}^q\overline{W})$ which is, on each fiber, the symbol of the harmonic oscillator. This symbol is of course identical to that defined above directly in terms of the symplectic form and choice of compatible almost complex structure.

If $f \in \rho^{\mathbb{Z}}\mathcal{C}^\infty(\overline{\mathbb{R}})$ then $f(h)$ defines a global section of $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X; {}^q\overline{W})$. For example the symbol of the projection onto the ground state, e^{-h} defines a global section $s_0 \in \Psi^{-\infty}_{\text{is}}(X; W)$. The relations, (135) are transfered verbatim to the bundle context if one simply interprets, # as the fiberwise product on the algebra fibers. In Lemma 19 we show that the symbols of the projections, $\{s_k\}$ onto the eigenstates of the harmonic oscillator are functions of h and are therefore also $U(n)$-invariant. We denote the global sections of $\Psi^{-\infty}_{\text{is}}(X; W)$ which they define by s_k as well. The formula, (140) again transplants verbatim to the bundle context as well as the relations, (137) between these sections and h. Thus, once a compatible almost complex structure is fixed on W the field of harmonic oscillators and its functional calculus is well defined.

The subbundle defined by s_0 is a trivial oriented line bundle and therefore the bundles, $\mathcal{F}_k = P \times_\rho F_k \# s$ are canonically isomorphic to $P \times_\rho F_k$. The modules, $\{t_{k,l} : k, l = 0, 1, \dots\}$ define subbundles of $\mathcal{T}_{k,l} \subset \Psi^{-\infty}_{\text{is}}(X; W)$ which are isomorphic to $\hom(\mathcal{F}_l, \mathcal{F}_k)$. The action is by fiberwise #-product. We let $\bar{\mathcal{F}}_k$ be the complex congujate bundles and

$$\mathcal{F}_{(N)} = \bigoplus_{k=0}^{N} \mathcal{F}_k,$$

be the Whitney sum.

There is also a nice description of the action of the covariant derivative defined by a connection. If ∇ is the covariant derivative induced on sections of V by a connection on P then we have the following formula:

$$\nabla_Y F(x) = p_\rho(q, (Y_P f)(q)) \text{ where } q \in \pi^{-1}(X) \tag{179}$$ [HHH.40.03]

and Y_P is the horizontal lift of Y to $T_q P$. Using these formulæ we see that the space $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X; {}^q\overline{W})$ has a family of canonical flat sections. Let $h(x)$ denote the quadratic form on $\mathbb{R}^{2n}$ which is invariant under the action $i(U(n))$.

[HHH.40.04]

PROPOSITION 6. *If $f \in \rho^{\mathbb{Z}}\mathcal{C}^\infty(\bar{\mathbb{R}})$ then, with respect to any connection on P the function $f(h(x))$ defines a flat section of $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X; {}^q\overline{W})$. If $f \in \mathcal{S}(\mathbb{R})$ then $f(h(x))$ defines a flat section of $\Psi_{\text{is}}^{-\infty}(X; W)$.*

PROOF. That $f(h(x))$ defines a section is immediate from the fact that

$$\rho(g^{-1})f(h(x)) = f(h(i(g^{-1})x)) = f(h(x)).$$

Flatness follows from (179) as this section is simply a constant vector. □

The covariant derivative on the bundle of algebras is a derivation of the algebra,

$$\nabla(a\#b) = \nabla a\#b + a\#\nabla b.$$

As the subbundles, $\mathcal{F}_k, \mathcal{T}_{k,l}$ are defined by the conditions

$$s_k\#a = a, \quad a = s_k\#a\#s_l,$$

respectively it is clear that the applying the covariant derivative to a section of one of these bundles produces a section of $\mathcal{F}_k \otimes \Lambda^1, \mathcal{T}_{k,l} \otimes \Lambda^1$ respectively. In other words the restriction of ∇ to these subbundles defines a connection on each which we continue to denote by ∇. We represent sections of the vector bundle, $\mathcal{F}_k$ in the form, $a\#s_0$ where a is a section of $P \times_\rho F_k$. As the covariant derivative is a derivation and $\nabla s_0 = 0$ we see that

$$\nabla(a\#s_0) = (\nabla a)\#s_0.$$

Observe also that, as ∇ defines connections on $\mathcal{F}_k$ and $\mathcal{F}_l$ it induces a second connection, ∇' on $\mathcal{T}_{l,k} \simeq \hom(\mathcal{F}_k, \mathcal{F}_l)$ via the rule

$$(\nabla' b) \cdot (f) = \nabla(b \cdot f) - b \cdot \nabla f.$$

As $b \cdot f = b\#f$ and ∇ is a derivation of the algebra bundle it follows that these two definitions coincide, i.e. $\nabla' = \nabla$.

[HHH.40.05]

PROPOSITION 7. *For each k, l the connection induced on the bundle, $\mathcal{T}_{l,k}$ by restricting the representation, ρ agrees with the connection defined on it under the identification, $\mathcal{T}_{l,k} \simeq \hom(\mathcal{F}_k, \mathcal{F}_l)$. Here $\mathcal{F}_k, \mathcal{F}_l$ are given the connections induced by restricting the representation, ρ to $\mathcal{F}_k$ and $\mathcal{F}_l$ respectively.*

In [**13**], Fedosov shows that the covariant derivative on $\Psi_{\text{is}}^{-\infty}(X; W)$ has a representation in terms of the algebra structure on the fibers. We let $\{e_1, \ldots, e_{2n}\}$ be a local section of the principal bundle, P and set $y^i = \omega(e_i, y)$. Here y denotes a point in the fiber of H. If Γ_{ij} is the pulled down connection form relative to this choice of local framing then we set

$$\Gamma_w = \frac{1}{2}\Gamma_{ij} y^i y^j.$$

This is evidently a one-form with values in $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X;{}^q\overline{W})$. If a is a smooth section of $\rho_q^{\mathbb{Z}}\mathcal{C}^\infty(X;{}^q\overline{W})$ then

$$\nabla a = da + [\Gamma_w, a]^{[HHH.40.05.01]} \tag{180}$$

The commutator is computed in the fiber #–product structure. A formula for the action of ∇ on sections of $\mathcal{F}_{(N)}$ also follows from this expression

$$\nabla f = df + (\Gamma_w - s_0 \operatorname{Tr} s_0 \# \Gamma_w \# s_0) \# f.$$

These formulæ give expressions for the action of the curvature tensor on a section, a of $\Psi_{\mathrm{is}}^{-\infty}(X;W)$ and a section f of a vector bundle, $\mathcal{F}_{(N)}$,

$$\nabla^2 a = [\Omega_w, a] \text{ where } \Omega_w = \frac{i}{2}\Omega_{ij}y^i y^j, \tag{181}$$

$$\nabla^2 f = (\Omega_w - s_0 \operatorname{Tr}[s_0 \# \Omega_w \# s_0]) \# f.^{[HHH.40.05.02]} \tag{182}$$