CHAPTER 3

Riemann-Weyl quantization$^{[S.Riemann-Weyl.quantization]}$

Pseudodifferential operators on a compact manifold X are generally characterized by two basic properties. The first is pseudo-locality, meaning that the Schwartz kernel of such an operator is smooth ($\mathcal{C}^\infty$ in the present context) away from the diagonal of X^2. The second property is some form of 'conormality' of the kernel at the diagonal, Diag. This is readily described in terms of the Fourier transform of the kernel, localized near the diagonal, in transverse directions. More precisely, a neighborhood, U of Diag in X^2 can be identified with a neighborhood, V of the zero section in the normal bundle, $N\operatorname{Diag}$,

$$\psi : U \to V.$$

If k is a distribution on X^2 with support contained in U and singular support contained in Diag then $\psi^*(k)$ is a compactly supported distribution on $N\operatorname{Diag}$ with singular support contained in the zero section. Let (x,η) denote a point in $N_x\operatorname{Diag}$ and (x,ξ) a point in $N^*_x\operatorname{Diag}$ then there is a canonically defined fiber Fourier transform,

$$\sigma_\psi(k)(x,\xi) = < \psi^*(k)(x,\eta), e^{-<\eta,\xi>} > .$$

The correspondance $k \to \sigma_\psi(k)$ is the symbol map defined by the normal fibration, ψ. This we call the *amplitude* of the oscillatory integral. It is also called the full symbol of the operator, but since (even with the normal fibration of the boundary fixed, through choice of a metric as we shall do) this is not uniquely defined so we reserve the term for the residue class of ampltidue, modulo rapidly decreacsing amplitudes, which correspond to smooth Schwartz kernels. That is, the behavior of the amplitude at fiber infinity completely determines singularity of k along the diagonal.

One of the special features of the diagonal as a submanifold of X^2 is that there is a canonical identification of its normal bundle with a subbundle of TX^2 which is, in turn canonically isomorphic to TX itself, $T_xX \ni W \longrightarrow (W,-W) \in T_xX \oplus T_xX \simeq T_{(x,x)}X^2$. The image of this map is a complement to $T\operatorname{Diag} \hookrightarrow TX^2 \restriction_{\operatorname{Diag}}$ and hence isomorphic to $N\operatorname{Diag}$. With these identifications $\sigma_\psi(k)$ is identified with a function on T^*X. The choice of transversal, i.e. of a collar neighborhood or normal fibration around the diagonal must be taken into account in analyzing the regularity and covariance properties of the full symbol obtained by such Fourier transformation. In this section we analyze the properties of the symbol map defined by a special class of normal fibrations, the *Riemann-Weyl normal fibrations.*

1. Riemann-Weyl normal fibrations

Let X be any compact manifold (without boundary) and consider the Riemann-Weyl normal fibration of the diagonal in X^2 fixed by a choice of metric on X :

$$W : TX \supset U \ni (x,v) \longmapsto (\exp_x^g(\frac{1}{2}v), \exp_x^g(-\frac{1}{2}v)) \in X^2. \tag{75}$$ [HHH2.91]

Here $\exp_x^g : T_xX \supset U_x \longrightarrow X$ is the exponential map for the metric g from the point x, i.e. $\exp_x^g(w)$ is the point at parameter value 1 on the affinely-parametrized geodesic with intial point x and intial tangent vector w; U is an appropriately small neighbourhood of the zero section; it is useful to take this neighborhood to be geodesically convex.

[HHH2.96]

LEMMA 17. *If g_s is, for $s \in [0,1]$, a smooth 1-parameter family of metrics and W_s is the corresponding family of fibrations (75) then the family of local diffeomorphisms of X^2 near the diagonal defined by $F_s = W_s W_0^{-1}$ is generated by a vector field V_s which is of the form*

$$V_s = \sum_{\textit{finite}} f_j f_k V_{jk},\ \ f_i \in \mathcal{C}^\infty(X^2),\ \ f_j|_{\mathrm{Diag}} = 0,\ \ V_{jk}\ \textit{tangent to}\ \mathrm{Diag}$$

$$\frac{d}{ds} F_s^* k = F_s^*(V_s k), \quad \textit{near}\ \mathrm{Diag}\ \ \forall\ k \in \mathcal{C}^\infty(X^2).$$

PROOF. For a fixed metric, g consider W in any local coordinates, x in X. With (x,w) the corresponding local coordinates on TX, the exponential map takes the form,

$$\exp_x^g(w) = x + w + \sum h_{ij}(x,w) w_i w_j = x + w + h.w^2$$

so the Riemann-Weyl normal fibration is of the form

$$(x,v) \longrightarrow (x + \frac{1}{2}v + h'(x,v).v^2, x - \frac{1}{2}v + h'(x,-v).v^2)$$

where $h'(x,v) = h(x,\frac{1}{2}v)$ is a smooth matrix valued function. In terms of the change to flat 'Weyl coordinates' in X^2 from (x',x'') to $y = \frac{1}{2}(x'+x'')$ and $z = x' - x''$ this becomes

$$W : (x,v) \longrightarrow (x + (\frac{1}{2}(h'(x,v) + h'(x,-v))).v^2, v + (h'(x,v) - h'(x,-v).v^2) = (y,z).$$

This is true in *any* coordinates and shows that W is of the form $y = x + O(v^2)$ and $z = v + O(v^3)$. Thus the family F_s is necessarily of the form

$$F_s(y,z) = (y,z) + sO(z^2,z^3)$$

in any Weyl coordinates, with the diagonal given by $\{z = 0\}$. The generating vector field is therefore of the expected from,

$$V_s = O(z^2)\partial_y + O(z^3)\partial_z.$$

□

The inverse of the map (75) may be written

$$X^2 \supset O' \ni (x,y) \longmapsto (m(x,y), \tau(x,y)) \in O \subset TX \tag{76}$$ [HHH2.92]

where $m(x,y)$ is the mid-point of the unique shortest geodesic from y to x (which are close to each other) and $\tau(x,y) \in T_{m(x,y)}X$ is the tangent vector to this geodesic at the mid-point with respect the affine parameter in $[0,1]$.

2. Classical operators[SS.Classical.operators]

If V is a vector bundle over a compact manifold Y then the Fourier transform is a well-defined map from tempered distributional fibre densities to tempered distributions on the dual vector bundle,

$$
\begin{gathered}
\mathcal{C}^{-\infty}(\overline{V};\Omega_{\text{fib}}) \ni v \longleftrightarrow \hat{v} \in \mathcal{C}^{-\infty}(\overline{V^*}),\\
\hat{v}(y,\zeta) = \int e^{-i<z,\zeta>} v(y,z)dz, \qquad \text{[HHH2.93]}\\
v(y,z)dz = (2\pi)^{-\operatorname{rank} V} \int e^{i<z,\zeta>} \hat{v}(y,\zeta) d\zeta \cdot dz.
\end{gathered} \tag{77}
$$

Here $\mathcal{C}^{-\infty}(\overline{V})$ is the space of tempered distributions on V, it is naturally the topological dual of the schwartz space space, $\dot{\mathcal{C}}^{\infty}(\overline{V};\Omega)$ consisting of those smooth densities on the radial compactification which vanish, in the sense of Taylor series at the boundary. The fibre density bundles of V and V^* are canonically dual so the inverse formula also makes invariant sense as integration of a function, giving a fibre density on V. Assuming that V has a fibre metric, and hence fibre density, we may interpret (77) in terms of this and the induced fibre density on V^*.

The classical (i.e. 1-step polyhmogeneous) conormal distributions (with rapid decay at infinity) of order $m - \frac{1}{4}\operatorname{rank} V + \frac{1}{4}\dim Y$ with respect to the zero section $O \subset V$ consist precisely of the range of the Fourier tranform applied to $\rho^{-m}\mathcal{C}^{\infty}(\overline{V*};\Omega_{\text{fib}})$.

[HHH2.506]

Definition 1. The space of 'classical' pseudodifferential operators, $\Psi^m(X)$, on a compact manifold X is fixed in terms of their kernels

$$
\Psi^m(X) = \left\{K \in \mathcal{C}^{-\infty}(X^2;\pi_R^*\Omega); (78) \text{ and } 80 \text{ hold}\right\}^{[HHH2.507]} \tag{78}
$$

by the two conditions

$$
K \in \mathcal{C}^{\infty}(X^2 \setminus \operatorname{Diag};\pi_R^*\Omega)^{[HHH2.510]} \tag{79}
$$

and the requirement, for every $\phi \in \mathcal{C}^{\infty}(X^2)$ with support sufficiently near the diagonal, that under (75) for some metric,

$$
W^*(\phi K) \text{ is conormal of order } m \text{ with respect to } O \subset V^*.^{[HHH2.508]} \tag{80}
$$

The second of these conditions may be made more explicit by combining the Riemann-Weyl quantization with the Fourier transform and so write

$$
K(x,y) = (2\pi)^{-n} \int_{T^*_{m(x,y)}X} e^{i\tau(x,y)\cdot\xi} a(m(x,y),\xi)d\xi \text{ near Diag} \\
\text{with } a \in \rho^{-m}\mathcal{C}^{\infty}(\overline{T}^*X).^{[HHH2.94]} \tag{81}
$$

Here, a is the *amplitude* and the (Riemann-Weyl) *full symbol* is its residue class $a \in \rho^{-m}\mathcal{C}^{\infty}(\overline{T}^*X)/\dot{\mathcal{C}}^{\infty}(\overline{T}^*X)$. The full symbol is uniquely determed by K and the choice of metric; conversely it determines K up to a term with smooth kernel, i.e. in $\Psi^{-\infty}(X)$. Thus the full symbol map is an isomorphism

$$
\mathcal{A}^m(X) = \Psi^m(X)/\Psi^{-\infty}(X) \longleftrightarrow \rho^{-m}\mathcal{C}^{\infty}(\overline{T}^*X)/\dot{\mathcal{C}}^{\infty}(\overline{T}^*X).^{[HHH2.95]} \tag{82}
$$

Application of a smooth vector field, on X^2, to the kernel of $A \in \Psi^m(X)$ gives an element of $\Psi^{m+1}(X)$. If the vector field is tangent to the diagonal the order is preserved; multiplication of the kernel by a smooth function which vanishes on the diagonal lowers the order by 1. It follows from this and Lemma 17 that the Riemann-Weyl symbol is invariantly defined in $\rho^{-m}\mathcal{C}^\infty(\overline{T}^*X)/\rho^{-m+2}\mathcal{C}^\infty(\overline{T}^*X)$ independent of the choice of metric, i.e. the first two terms in the asymptotic expansion are well-defined. The second-order invariance is needed below to define the principal symbol of a Heisenberg pseudodifferential operator.

3. Contact manifolds and 1-forms[S.Contact.manifolds]

A short section to bring together the basic properties of contact manifolds.

4. Heisenberg pseudodifferential operators[SS.Heisenberg.pseudos]

If X is a contact manifold we may use the same definition, (81), to define both Heisenberg and extended Heisenberg pseudodifferential operators, simply changing the symbol class by changing the compactification of T^*X. The contact line bundle of the contact manifold is denoted by $L \subset T^*X$. The discussion of the compactifications in §2 then applies to $S = L$.

[HHH2.97]

Definition 2. The space $\Psi^k_{\mathrm{H}}(X)$ of Heisenberg pseudodifferential operators of order $z \in \mathbb{C}$ on a contact manifold X is defined as the space of linear operators with Schwartz kernels $K \in \mathcal{C}^{-\infty}(X^2; \pi_R^*\Omega)$ satisfying

$$K \in \mathcal{C}^\infty(X^2 \setminus \mathrm{Diag}; \pi_R^*\Omega)^{[HHH2.98]} \tag{83}$$

$$K \text{ satisfies (81) with } a \in \rho^{-z}\mathcal{C}^\infty({}^L\overline{T}^*X).^{[HHH2.100]} \tag{84}$$

We denote the larger space corresponding to $a \in \rho^{-m}\mathcal{A}({}^L\overline{T}^*X)$, where $m \in \mathbb{R}$ (since ρ^{it}, $t \in \mathbb{R}$, is a multiplier on $\mathcal{A}({}^L\overline{T}^*X)$) by $\Psi^m_{\mathrm{H},\infty}(X)$. Similarly the space of extended Heisenberg operators of double order z, w, $\Psi^{z,w}_{\mathrm{eH}}(X)$, corresponds to those kernels $K \in \mathcal{C}^{-\infty}(X^2; \pi_R^*\Omega)$ which satisfy (83) and in which (84) is replaced by

$$K \text{ satisfies (81) with } a \in \rho_R^{-z}\rho_H^{-w}\mathcal{C}^\infty({}^{\mathrm{eL}}\overline{T}^*X),\ z, w \in \mathbb{C}.^{[HHH2.101]} \tag{85}$$

The space $\Psi^{m,k}_{\mathrm{eH},\infty}(X)$, $m, k \in \mathbb{R}$, is similarly defined by admitting amplitudes $a \in \rho_R^{-m}\rho_H^{-k}\mathcal{A}({}^{\mathrm{eL}}\overline{T}^*X)$.

Recall that $\rho_L \in \mathcal{C}^\infty({}^{\mathrm{eL}}\overline{T}^*X)$ is a defining function for the 'Heisenberg' boundary face of ${}^{\mathrm{eL}}\overline{T}^*X$, whilst $\rho_R \in \mathcal{C}^\infty({}^{\mathrm{eL}}\overline{T}^*X)$ is a defining function for the 'radial' face. We shall generally consider the case where L is oriented, so in particular X admits a global contact form. In this case ${}^{\mathrm{eL}}\overline{T}^*X$ has three boundary hypersurfaces, as noted in §3. Where necessary we can let $\rho_{L,\pm}$ be defining functions for the two components of the parabolic (or Heisenberg) part of the boundary and $k_\pm$ be the corresponding two orders for an extended Heisenberg pseudodifferential operator. The operators with radial order $-\infty$ and lower (respectively upper) Heisenberg order also $-\infty$ form an important subspace which we denote by $\mathcal{I}^k_{\mathrm{H},+}(X)$ (resp. $\mathcal{I}^k_{\mathrm{H},-}(X)$) :

$$\mathcal{I}^k_{\mathrm{H},\pm}(X) = \left\{K \in \Psi^{-\infty,k}_{\mathrm{eH}}(X); a \in \rho^k_{L,\pm}\rho_R^M\rho^M_{L,\mp}\mathcal{C}^\infty({}^{\mathrm{eL}}\overline{T}^*X)\ \forall\ M\right\}.^{[HHH2.210]} \tag{86}$$

We call then the upper and lower Hermite spaces (ideals as we shall see) because of their relationship to the operators studied by Boutet de Monvel and Guillemin in [**7**].

Using the map defined fiberwise by (51) to identify the symbols, these Hermite subspaces can be identified with the subspaces of the Heisenberg algebra (for which we use the same notation),

$$\mathcal{I}^k_{\mathrm{H},\pm}(X) = \left\{ K \in \Psi^k_{\mathrm{H}}(X); a \in \rho^k_{L,\pm}\rho^M_{L,\mp}\mathcal{C}^\infty({}^L\overline{T}^*X)\ \forall\ M \right\}. \tag{87}$$ [HHH2.210.1]

Their importance is related to the fact that these subspaces are filtered ideals, in both the extended Heisenberg algebra and the Heisenberg algebra itself.

Let B be the inverse of the conormal bundle to the boundary of ${}^L\overline{T}^*X$ (i.e. the normal bundle) and similarlty let B_R and B_L be the normal bundles to the radial and Heisenberg boundary hypersurfaces of ${}^{\mathrm{eL}}\overline{T}^*X$.

[HHH2.102]

PROPOSITION 2. *The spaces $\Psi^k_H(X)$ and $\Psi^{m,k}_{eH}(X)$ as defined by* (84) *and* (85) *are independent of the metric used in* (81) *and the principal symbol maps*

$$\Psi^k_H(X) \ni A \longrightarrow [a] \in \rho^{-k}\mathcal{C}^\infty({}^L\overline{T}^*X)/\rho^{-k+1}\mathcal{C}^\infty({}^L\overline{T}^*X) \equiv \mathcal{C}^\infty({}^L\mathbb{S}^*X; B^k) \tag{88}$$ [HHH2.106]

$$\begin{aligned}\Psi^{m,k}_{eH}(X) \ni A \longrightarrow [a] \in \rho_H^{-k}\rho_R^{-m}\mathcal{C}^\infty({}^{eL}\overline{T}^*X)/\rho_L^{-k+1}\rho_R^{-m+1}\mathcal{C}^\infty({}^{eL}\overline{T}^*X)\\ \equiv \mathcal{C}^\infty({}^{eL}\mathbb{S}^*X; B_L^kB_R^m)\end{aligned} \tag{89}$$ [HHH2.104]

are also independent of choice and give short exact sequences

$$0 \longrightarrow \Psi^{k-1}_H(X) \longrightarrow \Psi^k_H(X) \longrightarrow \mathcal{C}^\infty({}^L\mathbb{S}^*X; B^k) \longrightarrow 0 \tag{90}$$ [HHH2.107]

$$0 \longrightarrow \Psi^{k-1,m-1}_{eH}(X) \longrightarrow \Psi^{k,m}_{eH}(X) \longrightarrow \mathcal{C}^\infty({}^{eL}\mathbb{S}^*X; B_L^kB_R^m) \longrightarrow 0 \tag{91}$$ [HHH2.105]

PROOF. All the spaces $\Psi^k_{\mathrm{H}}(X)$ and $\Psi^{m,k}_{\mathrm{eH}}(X)$ are $\mathcal{C}^\infty(X^2)$-modules. Since all classes contain the smoothing operators, it is enough to consider only kernels supported in an arbitrarily small neighbourhood, O, of the diagonal. If Kdg_R is such a kernel, let $K' = W^*K$ be the pull-back to TX under the Riemann-Weyl fibration. Since the space of metrics is connected it suffices to show the independence of the classes of pseudodifferential operators along a curve of metrics, g_s, as in Lemma 17. We may take the open set O to be contained in the range of W_s in (75) for all $s \in [0,1]$. Thus the class of distributions on TX is fixed, only the identifications with X^2 are varying. If $K_s dg_s \in \Psi^k_{\mathrm{H}}(X)$ is determined by the condition that $K'_s = W^*_s K_s$ is the Fourier transform of a curve in $\mathcal{C}^\infty({}^L\overline{T}^*X)$ then

$$\frac{d}{ds}K'_s = V_s K'_s \tag{92}$$ [HHH2.108]

where V_s is the vector field in Lemma 17.

For functions $f \in \mathcal{C}^\infty(X^2)$ with pull back, f', which is polynomial on the fibres of TX, the Fourier transform of $f'K'$ is given by a differential operator applied to the symbol, $P_f\hat{K}$, where P_f is tangent to, and constant coefficient on the fibres. The constant coefficient vector fields on the fibres lie in $\rho\mathcal{V}_{\mathrm{b}}(\overline{T}^*X)$ and $\rho\mathcal{V}_{\mathrm{b}}({}^L\overline{T}^*X)$ in the radial and L-parabolic compactifications and hence also lift to $\rho_R\rho_H\mathcal{V}_{\mathrm{b}}({}^{\mathrm{eL}}\overline{T}^*X)$. The Taylor series at the zero section decomposes any smooth function into such a fibre polynomial plus a smooth function vanishing to arbitrarily high order; the weak conormality of the kernels show that these terms are arbitrarily smooth. We may therefore use a partition of unity to localize the kernel and then use the local form of V_s.

Locally $TX \simeq U \times \mathbb{R}^n$ and the vector fields can be decomposed into base and tangent parts. The base parts commute with the Fourier transform. A calculation in local coordinates shows that differentiation in the base directions can raise the parabolic order by at most one. Combined with the fact that the coefficient of this vector field vanishes quadratically on the zero section shows that, overall the order is lowered by at least one – each order in the case of $\Psi^{m,k}_{\mathrm{eH}}(X)$. Similarly the fibre part of the V_{ij} can be written in terms of vector fields homogeneous of degrees 0 with respect to the radial homotheity and 0 or ± 1 in terms of the parabolic homotheity. One extra linear factor, from f_j, makes these homogeneous of non-negative degree. After Fourier transform, these operators preserve all the symbol spaces and the extra factor of f_i vanishing at the zero section lowers all orders by at least one.

Thus it follows that the Fourier transform of K'_s is constant, modulo lower order terms, as a density on the compactified space ${}^L\overline{T}^*X$. However, the density bundle is to be trivialized using the variable fibre density of the Riemann metric. This variation cancels, to leading order, the term dg_s dropped in the definition of K'. Thus the principal symbol itself, as defined above, is independent of all choices. $\square$

Several other invariance results can also be extracted from this proof. The full symbols, defined by a choice of Riemann metric,

$$\sigma_{\mathrm{RW}} : \Psi^k_{\mathrm{H}}(X) \longrightarrow \rho^{-k}\mathcal{C}^\infty({}^H\overline{T}^*X)/\dot{\mathcal{C}}^\infty({}^H\overline{T}^*X)^{[HHH2.109]} \tag{93}$$

is not invariant. However its covariance properties can be seen from the discussion above. Namely if a and a' are the symbols with respect to metric g and g' then

$$a' \sim \sum_{j=0}^{\infty} P_j a,\ \ P_j \in \rho^j \operatorname{Diff}^{2j}_{\mathrm{b}}({}^H\overline{T}^*X).^{[HHH2.110]} \tag{94}$$

This converges as a formal Taylor series; note that P_j is independent of k. A similar conclusion holds for the extended algebra where the full symbol map is

$$\begin{gathered}\sigma_{\mathrm{RW}} : \Psi^{m,k}_{\mathrm{eH}}(X) \longrightarrow \rho_R^{-m}\rho_H^{-k}\mathcal{C}^\infty({}^{\mathrm{eH}}\overline{T}^*X)/\dot{\mathcal{C}}^\infty({}^{\mathrm{eH}}\overline{T}^*X) \text{ and }_{[HHH2.111]} \\ a' \equiv Q_j a, \quad \bmod \rho_R^{2j-m}\rho_H^{j-k} \text{ with } Q_j \in \operatorname{Diff}^{2j}_{\mathrm{b}}({}^{\mathrm{eH}}\overline{T}^*X).\end{gathered} \tag{95}$$

Thus, although the full symbols are not invariantly defined the transformation law is local at the boundary of the corresponding compactification of T^*X; i.e. they form sheaves over this boundary. As we shall see below, the composition formula is not as local as this, so these spaces are not sheaves of algebras in this sense.

The two ideals in the Heisenberg algebra may be characterized by

$$\begin{aligned}\mathcal{I}^k_{\mathrm{H},+}(X) &= \{A \in \Psi^k_{\mathrm{H}}(X); \sigma_{\mathrm{RW}}(A) \equiv 0 \text{ at } {}^H\mathbb{S}^*_-X\},_{[HHH2.112]} \\ \mathcal{I}^k_{\mathrm{H},-}(X) &= \{A \in \Psi^k_{\mathrm{H}}(X); \sigma_{\mathrm{RW}}(A) \equiv 0 \text{ at } {}^H\mathbb{S}^*_+X\}.\end{aligned} \tag{96}$$

These are the subspaces with full symbols (with respect to any Riemann-Weyl quantization) vanishing to infinite order at the boundary in either the lower or upper half of the parabolic compactification of T^*X.