CHAPTER 2

Parabolic compactification and symbols [S.Parabolic.compactification]

In this chapter we discuss variants of the compactification of a vector space to a compact manifold with boundary or with corners. We use these compactifications to introduce and characterize spaces of symbols. Working fiber by fiber we then compactify the total space of a vector bundle over a compact manifold. In case the vector bundle is the cotangent space the symbols can be quantized, as is done in the next chapter, to give corresponding calculi of pseudodifferential operators.

1. Radial compactification [SS.radial.compactification]

Since it is the model for the discussion below of parabolic compactification, we first briefly describe the radial compactification of a vector space and then of vector bundle. Let V be a finite-dimensional real vector space. The radial $\mathbb{R}^+$-action on V, $V \ni v \longmapsto \tau v$, $0 < \tau \in \mathbb{R}$, allows the sphere (at infinity) of V to be defined by

$$\mathbb{S}V = (V \setminus \{0\})/\mathbb{R}^+,\ \pi_{\mathbb{R}^+} : V \setminus \{0\} \longrightarrow \mathbb{S}V.$$

Pull-back from $\mathbb{S}V$ to $V \setminus \{0\}$ under $\pi_{\mathbb{R}^+}$ identifies functions on $\mathbb{S}V$ with functions which are homogeneous of degree zero on $V \setminus \{0\}$. The subspace of smooth functions on $V \setminus \{0\}$ gives $\mathbb{S}V$ a natural $\mathcal{C}^\infty$ structure and the linear vector fields on V, i.e. the elements of the Lie algebra of $\mathrm{GL}(V)$, project to smooth vector fields on $\mathbb{S}V$; with the radial vector field projecting to zero.

As a set, the radial compactification of V is the disjoint union

$$\overline{V} = V \sqcup \mathbb{S}V. \tag{29}$$ [HHH2.490]

To define the $\mathcal{C}^\infty$ structure on $\overline{V}$ we choose a Euclidean metric $|\cdot|$ on V and use it to embed V as a subspace of $\mathbb{R} \times V$ and at the same time embed $\mathbb{S}V$:

$$\begin{gathered}\mathrm{RC} : V \ni v \mapsto \Big(\frac{1}{(1+|v|^2)^{\frac12}}, \frac{v}{(1+|v|^2)^{\frac12}}\Big) \in \mathbb{R} \times V, \\ \mathbb{S}V \ni [w] \mapsto \Big(0, \frac{w}{|w|}\Big) \in \mathbb{R} \times V.\end{gathered} \tag{30}$$ [HHH2.128]

The combined map $\mathrm{RC} : \overline{V} \longrightarrow \mathbb{R} \times V$ is an injection with range precisely the upper half of the unit sphere

$$\{(t, v) \in \mathbb{R} \times V; t \geq 0, t^2 + |v|^2 = 1\}. \tag{31}$$ [HHH2.491]

This is the $\mathcal{C}^\infty$ structure we give $\overline{V}$, making it into a compact manifold with boundary which is, by definition, diffeomorphic to a closed ball the interior of which is canonically identified with V. The 'radial inversion' map

$$V \setminus \{0\} \ni v \longrightarrow \Big(\frac{1}{r}, \frac{v}{|v|}\Big) \in (0, \infty) \times \mathbb{S}V,\ r = |v| \tag{32}$$ [HHH2.492]

identifies a neighbourhood of infinity in V with $(0,1)\times\mathbb{S}V$ in such a way that a function f on $\overline{V}$ is $\mathcal{C}^\infty$ precisely if it is $\mathcal{C}^\infty$ on V and is such that $F(s,w)=f(s^{-1}w)$ extends to be smooth down to $s=0$ on $[0,1)\times\mathbb{S}V$. This follows from the fact that $t\longmapsto(1+t^{-2})^{-\frac12}$ extends by continuity to a diffeomorphism of $[0,1]$ onto $[0,2^{-\frac12}]$.

To see that the $\mathcal{C}^\infty$ structure is independent of the choice of the metric it suffices to show that linear transformations of V lift to global diffeomorphism of $\overline{V}$ since any two metrics are linearly equivalent. Linear transformations are finite products of exponentials of linear vector fields, so it suffices to show that the linear vector fields themselves lift to be smooth. The linear vector fields are homogeneous of degree zero, so in terms of the identification near infinity, given by (32), they are each of the form

$$L=as\partial_s+L'$$

where L' is the smooth vector field on $\mathbb{S}V$ which is the projection of L and $a\in\mathcal{C}^\infty(\mathbb{S}V)$, since it is homogeneous of degree zero. Thus L extends to be smooth on $\overline{V}$ (and to be tangent to the sphere at infinity). Thus the $\mathcal{C}^\infty$ structure on $\overline{V}$ is independent of the choice of metric.

This $\mathcal{C}^\infty$ structure may also be defined by reference to a coordinate covering of $\overline{V}$. We need to define coordinate charts which cover the boundary. The unit sphere for some choice of metric, $S_1=\{v\,:\,|v|=1\}$ may be coverd by coordinate charts, $\{(U_i;(x_1^i,\ldots,x_{d-1}^i)\}$. Extend the functions $\{x_j^i\}$ to the cone $U_i'=\{rx\,:\,r\geq1,\,x\in U_i\}$ as homogeneous functions of degree 0 and denote the extended functions by $\{X_j^i\}$. Let $R=|v|^{-1}$. The sets,

$$U_i''=U_i'\sqcup U_i,$$

where we identify U_i with a subset of $\mathbb{S}V$ in an obvious way, give open sets which cover $\partial\overline{V}$. The functions $\{X_j^i,R\}$ extend to U_i'' and $\{(U_i'',(X_1^i,\ldots,X_{d-1}^i,R)\}$ is a cover of a neighborhood of $\partial\overline{V}$ by coordinate charts. A function is smooth on U_i'' if it can be expressed as $f(X_1^i,\ldots,X_{d-1}^i,R)$ for a function $f\in\mathcal{C}^\infty(V_i\times[0,1])$. Here V_i is a open subset of $\mathbb{R}^{d-1}$. Thus we say that the $\mathcal{C}^\infty$ structure on $\overline{V}$ is generated by the smooth functions of compact support on V together with the smooth functions on V which are homogeneous of degree 0 or -1 outside a compact set, extended to $\mathbb{S}V$ in the obvious way.

As already noted, $\mathrm{GL}(V)$ lifts to act (smoothly) as a group of diffeomorphism of $\overline{V}$ and its Lie algebra, the space of linear vector fields, lifts into the smooth vector fields on $\overline{V}$ which are tangent to the boundary, $\mathbb{S}V$. In fact their lifts span, over $\mathcal{C}^\infty(\overline{V})$, the space $\mathcal{V}_{\mathrm{b}}(\overline{V})$ of all such smooth vector fields tangent to the boundary. A similar homogeneity argument to that above shows that the constant vector fields, and hence the translations themselves, lift to be smooth on $\overline{V}$. Thus $\overline{V}$ with its $\mathcal{C}^\infty$ structure is really an invariant of the affine, rather than the full linear, structure of V.

2. Standard symbol spaces [SS.Standard.Symbols]

The smooth action of $\mathrm{GL}(V)$ on $\overline{V}$ means that for a vector bundle, V, over a manifold X, the fibre-by-fibre compactification

$$\overline{V}=\bigcup_{x\in X}\overline{V_x}$$

is naturally a manifold with boundary and is compact if X is. The main reason that this manifold, and its parabolic variant, is of interest here is that it gives a simple way of describing the symbols on V.

Namely, the 'standard' symbol spaces on V are identified with spaces of conormal distributions on $\overline{V}$. In the case of a vector space the symbol spaces $S^m(V) \subset \mathcal{C}^\infty(V)$ consists of those functions, a which satisfy

$$|\partial_v^\alpha a| \le C_\alpha \langle v\rangle^{m-|\alpha|},\ \langle v\rangle^2 = 1 + |v|^2 \quad (33)$$

[HHH2.129]

in terms of any linear coordinates and Euclidean norm. Note that $\rho = \langle v\rangle^{-1}$ is a defining function for $\partial\overline{V}$. On $\overline{V}$ we may define the bounded conormal functions

$$\mathcal{A}(\overline{V}) = \{a \in L^\infty(\overline{V}); \mathrm{Diff}_{\mathrm{b}}^*(\overline{V})a \subset L^\infty(V)\}. \quad (34)$$

[HHH2.130]

Since $\mathrm{Diff}_{\mathrm{b}}^*(\overline{V})$ is the enveloping algebra of $\mathcal{V}_{\mathrm{b}}(\overline{V})$ these are just the functions that remain bounded after the repeated action of vector fields tangent to the boundary. Then

$$\mathrm{RC}^* : \rho^{-m}\mathcal{A}(\overline{V}) \longrightarrow S^m(V) \text{ is an isomorphism } \forall\ m \in \mathbb{R}. \quad (35)$$

[HHH2.131]

We generally regard this as an identification, so symbols in the conventional sense on V are just conormal functions on $\overline{V}$. To prove (35) it suffices to note that (33) can be rewritten in the form

$$\|U^\alpha(\langle v\rangle^{-m}a)\|_{L^\infty} < \infty\ \forall\ \alpha \text{ with } U_i = \langle v\rangle\partial_{v_i}$$

and then to observe that the vector fields U_i lift to be smooth on $\overline{V}$ and to span $\mathcal{V}_{\mathrm{b}}(\overline{V})$ over $\mathcal{C}^\infty(\overline{V})$. The same identification (35) holds in general for a vector bundle, since the base variables are in essence parameters.

For any $m \in \mathbb{R}$, the space of '1-step polyhomogeneous symbols' on V of order m is identified by (35) with $\rho^{-m}\mathcal{C}^\infty(\overline{V})$. That is, the 1-step polyhomogeneous symbols of order 0 are identified with $\mathcal{C}^\infty(\overline{V})$ and ρ, as a function on V, is a non-vanishing elliptic symbol of order -1. For a symbol of order 0 the usual notion of asymptotic expansion reduces to the Taylor series at the sphere at infinity in powers of ρ. For a symbol of order m we get such a Taylor series multiplied by ρ^{-m}, that is:

$$a \in \rho^{-m}\mathcal{C}^\infty(\overline{V}) \Longleftrightarrow a \sim \rho^{-m}\sum_{j\ge 0} a_j\rho^j,\quad a_j \in \mathcal{C}^\infty(\mathbb{S}V).$$

It is useful to have a direct characterization of $\rho^{-m}\mathcal{C}^\infty(\overline{V}) \subset \rho^{-m}\mathcal{A}(\overline{V})$. This is easily given in terms of the action of the radial vector field for V, e.g. $R = v\cdot\partial_v$. A radial vector field is any smooth vector field, R on V which lifts to an element of $\mathcal{V}_{\mathrm{b}}(\overline{V})$ with the property that it *vanishes* at the boundary (as a smooth vector field in the usual sense) and satisfies $R\rho = \rho + O(\rho^2)$ for any (one) boundary defining function $\rho \in \mathcal{C}^\infty(\overline{V})$. Another vector field with these two properties differs from R by an element of $\rho\mathcal{V}_{\mathrm{b}}(\overline{V})$.

[HHH2.505]

LEMMA 1. *For* any *such radial vector field*

(36)

$$a \in \rho^{-m}\mathcal{C}^\infty(\overline{V}) \Longleftrightarrow a \in \rho^{-m}\mathcal{A}(\overline{V}),\ \left[\prod_{j=0}^{k}(R - j + m)\right] a \in \rho^{-m+k+1}\mathcal{A}(\overline{V})\ \forall\ k.$$

[HHH2.132]

PROOF. A commutation argument shows that (36) is independent of the choice of radial vector field, i.e. if it holds for one it holds for all. For the standard radial vector field in local coordinates its verification is elementary. □

Radial compactification is natural with respect to linear (even affine) subspaces. Thus if $S \subset V$ is a subspace then the closure of S in $\overline{V}$ is canonically diffeomorphic to $\overline{S}$. In particular the boundary of $\overline{S}$, $\mathbb{S}S$, is an associated spherical subspace of $\mathbb{S}V = \partial\overline{V}$.

As well as the radial compactification we can alternatively give the same set, $V \sqcup \mathbb{S}V$ a 'cruder' $\mathcal{C}^\infty$ structure (i.e. with fewer $\mathcal{C}^\infty$ functions) generated, as above by $\mathcal{C}^\infty_{\mathrm{c}}(V)$ and the elements of $\mathcal{C}^\infty(V)$ which are homogeneous of degrees 0 or -2 outside a compact set. We denote the resulting manifold ${}^q\overline{V}$. This $\mathcal{C}^\infty$ structure also only depends on the linear structure, but is not an invariant of the affine structure, i.e. translations of V do not lift to be smooth on ${}^q\overline{V}$. The analogue of (30) is the compactifying map

$$
{}^{\mathrm{q}}\mathrm{RC} : V \ni v \longmapsto \frac{v}{(1+|v|^2)^{\frac{1}{2}}} \in \mathbb{B}V = \{v \in V; |v| \leq 1\}
$$

which extends to a diffeomorphism of ${}^q\overline{V}$ and $\mathbb{B}V$ for any choice of metric. For this compactification $\langle v\rangle^{-2} \in \mathcal{C}^\infty({}^q\overline{V})$ is a boundary defining function and the linear vector fields still lift into $\mathcal{V}_{\mathrm{b}}({}^q\overline{V})$ and span it over $\mathcal{C}^\infty({}^q\overline{V})$. The linear vector fields *do not* lift to be smooth on ${}^q\overline{V}$.

3. Parabolic compactification of a vector space [SS.parabolic.compactification.v.s]

Now suppose that the finite-dimensional vector space, V, has a specified linear subspace, $S \subset V$. The direct product, $S \times (V/S)$ has an 'S-parabolic' $\mathbb{R}^+$-action,

$$
(s, w) \longmapsto (\tau^2 s, \tau w),\ t \in (0, \infty). \tag{37}
$$
[HHH2.146]

We identify the quotient

$$
{}^S\mathbb{S}V = ((S \times V/S) \setminus \{0, 0\}) / \mathbb{R}^+ \tag{38}
$$
[HHH2.184]

as the 'S-parabolic sphere at infinity' to V. As a set, the S-parabolic compactification of V is

$$
{}^S\overline{V} = {}^S\mathbb{S}V \sqcup V. \tag{39}
$$
[HHH.28]

As in the radial case, we shall give this set a natural $\mathcal{C}^\infty$ structure so that it is diffeomorphic to a closed ball.

Choose a complement to S, $V = S \oplus T$ and fix Euclidean metrics on S and T. Then consider the map

$$
\begin{gathered}
V = S \oplus T \ni (s,t) \longmapsto \left(\frac{1}{\langle s,t\rangle_S}, \frac{s}{\langle s,t\rangle_S^2}, \frac{t}{\langle s,t\rangle_S}\right) \in \mathbb{R} \times V, \\
\langle s,t\rangle_S^4 = 1 + |s|^2 + |t|^4.
\end{gathered} \tag{40}
$$
[HHH.29]

This is a $\mathcal{C}^\infty$ diffeomorphism of V onto the set

$$
\{(\rho, s', t') \in \mathbb{R} \times V; |\rho|^4 + |s'|^2 + |t'|^4 = 1,\ \rho > 0\}.
$$

Since the non-round sphere in $\mathbb{R} \times V$, $\{|\rho|^4 + |s'|^2 + |t'|^4 = 1\}$ is a compact smooth embedded submanifold, this gives ${}^S\overline{V}$ a $\mathcal{C}^\infty$ structure as a manifold with boundary (diffeomorphic to a ball) by identifying V/S with T and hence

$$^S\mathbb{S}V \simeq \{(s',t') \in S \oplus T; |s'|^2 + |t'|^4 = 1\}.$$

The function,

$$\rho = \langle s,t\rangle_S^{-1}$$

is a smooth defining function for $\partial^S\overline{V}$. A more intrinsic description of the $\mathcal{C}^\infty$ structure on $^S\overline{V}$ is given by:

[HHH.31]

LEMMA 2. *The $\mathcal{C}^\infty$ structure on $^S\overline{V}$ is generated by the $\mathcal{C}^\infty$ functions of compact support on V and those $\mathcal{C}^\infty$ functions on $V \setminus \{0\}$ which are homogeneous of degree 0 or -1 under the $\mathbb{R}^+$-action, $V = S \oplus T \ni (s,t) \longmapsto (\tau^2 s, \tau t)$ for $\langle s,t\rangle_S \geq 0$.*

PROOF OF LEMMA 2. By definition, the $\mathcal{C}^\infty$ functions near the boundary of $^S\overline{V}$ are generated by $\frac{1}{\zeta}$, $\frac{s}{\zeta^2}$ and $\frac{t}{\zeta}$ with $\zeta = \langle s,t\rangle_S$. Since $\frac{1}{\zeta}$ can be replaced by $(|s|^2 + |t|^4)^{-\frac{1}{4}}$ away from 0, and this is homogeneous of degree -1, the result follows. $\square$

[HHH.32]

LEMMA 3. *Any linear vector field on $S \oplus T$ which is tangent to S extends to a $\mathcal{C}^\infty$ vector field on $^S\overline{V}$ which is tangent to the boundary.*

PROOF. Let s_i be linear coordinates in S and t_j linear coordinates in T, so $S = \{t = 0\}$. The linear vector fields tangent to S are then spanned by

$$s_{i'}\frac{\partial}{\partial s_i},\ t_j\frac{\partial}{\partial s_i} \text{ and } t_{j'}\frac{\partial}{\partial t_j}.$$

With respect to the parabolic dilation, these are all homogeneous of degree 0 or -1. Lifted to $^S\overline{V}$ they are therefore of the form

$$R^r(a\rho\partial_\rho + T),\ R = (|s|^2 + |t|^4)^{\frac{1}{4}} \tag{41}$$ [HHH.35]

where $a \in \mathcal{C}^\infty(^S\mathbb{S}V)$, T is a smooth vector field on $^S\mathbb{S}V$ and $r = 0$ or -1 with the homogeneity. It follows that the lifted vector fields are smooth. $\square$

[HHH.33]

REMARK 1. The vector fields $t_j t_{j'}\partial_{s_i}$ and the second order differential operators with linear coefficients of the form $s_l\partial/\partial t_k\partial/\partial t_j$ are homogeneous of degree 0. It follows from the argument above that they all lift to be smooth differential operators on $^S\overline{V}$. In fact, from their homogeneity, it follows that they are of orders 1 and 2 in the filtered algebra $\mathrm{Diff}^*_{\mathrm{b}}(^S\overline{V})$ generated by $\mathcal{V}_{\mathrm{b}}(^S\overline{V})$.

[HHH.30]

LEMMA 4. *Any $A \in \mathrm{GL}(V)$ which satisfies $A(S) = S$ extends from V to a diffemorphism of $^S\overline{V}$ for the $\mathcal{C}^\infty$ structure described above.*

PROOF OF LEMMA 4. For a decomposition $V = S \oplus T$, if $A \in \mathrm{GL}(V)$ preserves both S and T it commutes with the $\mathbb{R}^+$ action and hence the spaces of functions homogeneous near ∞. Thus, by Lemma 2, it lifts to be smooth on $^S\overline{V}$. We are therefore reduced to consideration of elements of the form,

$$A = \begin{pmatrix} \mathrm{Id} & B \\ 0 & \mathrm{Id} \end{pmatrix}.$$

If A_τ is the corresponding element with B replaced by τB then, for $\tau \in [0,1]$, A_τ is a family connecting A to Id. This family is defined by integration of a linear vector field tangent to S :

$$\frac{d}{d\tau} A_\tau^* f = A_\tau^*(Wf);$$

indeed, W is homogeneous of degree -1. By Lemma 3, W lifts to be smooth on ${}^S\overline{V}$ and tangent to the boundary, so A_τ, $\tau \in [0,1]$ lifts to a diffeomorphism of ${}^S\overline{V}$ determined by the lifted differential equation. □

As a direct corollary of Lemma 4, the $\mathcal{C}^\infty$ structure introduced above on ${}^S\overline{V}$ is independent of the decomposition $V = S \oplus T$ used to define it. Thus ${}^S\overline{V}$ has a natural structure as a compact $\mathcal{C}^\infty$ manifold with boundary, with $\partial\, {}^S\overline{V} = {}^S\mathbb{S}V$, and on which the Lie group $\mathrm{GL}(V;S) \subset \mathrm{GL}(V)$, the subgroup leaving S invariant, has a smooth action.

If $\rho \in \mathcal{C}^\infty({}^S\overline{V})$ is a defining function for the boundary, for instance $\rho = 1/\langle s,t\rangle_S$ in (40), then any polynomial of degree p on V lifts to be an element of $\rho^{-2p}\mathcal{C}^\infty({}^S\overline{V})$. The polynomials of degree p which lift into the smaller space $\rho^{-p}\mathcal{C}^\infty({}^S\overline{V})$ are precisely those which are generated by the linear functions vanishing on S. Reversing the discussion of symbols in the previous section we may interpret the parabolic compactification as introducing a corresponding class of 'S-parabolic' symbols on V. Namely the conormal functions on ${}^S\overline{V}$ are, by definition, the S-parabolic symbols on V and the space $\cup_{m\in\mathbb{R}}\rho^{-m}\mathcal{C}^\infty({}^S\overline{V})$ is the 'classical' subspace.

4. Parabolic symbols [SS.Parabolic.symbols]

Since ${}^S\overline{V}$ is diffeomorphic to the radial compactification, the space $\mathcal{A}({}^S\overline{V})$ is defined exactly as in (34):

$$\mathcal{A}({}^S\overline{V}) = \{a \in L^\infty({}^S\overline{V}); \mathrm{Diff}^*_{\mathrm{b}}({}^S\overline{V})a \subset L^\infty(V)\}. \quad (42)$$
[HHH2.133]

Of course such a diffeomorphism is far from the identity on V so the resulting classes of parabolic symbols on V are quite different from the usual symbols.

[HHH2.134]

LEMMA 5. *A function $a \in \mathcal{C}^\infty(V)$ lifts to an element of $\rho^{-m}\mathcal{A}({}^S\overline{V})$ if and only if it satisfies the estimates*

$$|\partial_t^\alpha \partial_s^\beta a(s,t)| \le C_{\alpha,\beta}\langle s,t\rangle_S^{m-|\alpha,\beta|_S},\ |\alpha,\beta|_S = |\alpha| + 2|\beta| \quad (43)$$
[HHH2.135]

for any (one) set of linear coordinates (s,t) and Euclidean metric. The subspace $\dot{\mathcal{C}}^\infty({}^S\overline{V})$ is dense in $\rho^{-m}\mathcal{A}({}^S\overline{V})$ in the topology of $\rho^{-m'}\mathcal{A}({}^S\overline{V})$ for any $m' > m$.

PROOF. By a straightforward commutation argument, the estimates (43) are equivalent to the iterative estimates

$$\begin{gathered} \|V^\alpha W^\beta(\langle s,t\rangle_S^{-m}a)\|_{L^\infty} < \infty\ \forall\ \alpha,\beta, \\ V_i = \langle s,t\rangle_S\partial_{t_i},\ W_k = \langle s,t\rangle_S^2\partial_{s_k}. \end{gathered} \quad (44)$$
[HHH2.136]

To complete the proof we note that these vector fields, $\{V_i, W_k\}$ lift to be smooth on ${}^S\overline{V}$ and to span $\mathcal{V}_{\mathrm{b}}({}^S\overline{V})$ over $\mathcal{C}^\infty({}^S\overline{V})$. Thus (44) is just a reformulation of the conormality condition. □

As before the smooth ('classical') subspace of $\mathcal{A}({}^S\overline{V})$ may be easily characterized in terms of the effect of a radial vector field. Indeed this is the same result as (36); we only restate it to emphasize that the class of radial vector fields has changed.

[HHH2.137]

LEMMA 6. *The vector field $R_S = t\cdot\partial_t + 2s\cdot\partial_s$ lifts to a radial vector field on ${}^S\overline{V}$ and if $a \in \rho^{-m}\mathcal{A}({}^S\overline{V})$ then $a \in \rho^{-m}\mathcal{C}^\infty({}^S\overline{V})$ if and only if it satisfies*

$$\left[\prod_{j=0}^{k}(R_S - j + m)\right] a \in \rho^{-m+k+1}\mathcal{A}({}^S\overline{V}) \ \forall\ k. \tag{45}$$ [HHH2.138]

This in turn can be restated as a more familiar asymptotic statement characterizing the 'classical' S-parabolic symbols.

[HHH2.85]

LEMMA 7. *An element of $a \in \mathcal{C}^\infty(V)$ lifts into $\mathcal{C}^\infty({}^S\overline{V})$ if and only if it has an expansion near infinity with terms which are homogeneous of degree $-k$ under the parabolic homogeneity*

$$a(s,t) \sim \sum_{k=0,-1,\dots} a'_k,\ a'_k = a_k(|s|^2+|t|^4)^{-k/4},\ a_k \in \mathcal{C}^\infty({}^S\mathbb{S}V).$$

Asymptotic equivalence here means, as usual, that for any polynomial differential operator P on V there exists an N such that $P(a - \sum_{k>-N'} a'_k)$ is bounded near infinity for $N' \ge N$.

PROOF. This is essentially a restatement of the definition of the $\mathcal{C}^\infty$ structure. □

The closure in ${}^S\overline{V}$ of the parabolic subspace S is clearly naturally diffeomorphic to the 'totally' parabolic compactification ${}^S\overline{S}$ of S, a defining function for $\partial{}^S\overline{S}$ is $|s|^{-\frac{1}{2}}$. The boundary of ${}^S\overline{S}$ is still the radial sphere, so $\mathbb{S}S \subset \partial{}^S\overline{V}$ is a natural spherical submanifold of the boundary of ${}^S\overline{V}$. Similarly if T is a transversal subspace to S, $V = S\oplus T$ then the closure of T in ${}^S\overline{V}$ is canonically isomorphic to the radial compactification $\overline{T}$.

[HHH2.69]

LEMMA 8. *The boundary $\partial\overline{T} \subset \partial{}^S\overline{V}$ of the closure of any transversal subspace to S in V is independent of choice and hence a well-defined 'equatorial subsphere' $\mathbb{S}T \subset \partial{}^S\overline{V} = {}^S\mathbb{S}V$.*

The description of this submanifold of the S-parabolic sphere at infinity as the 'equatorial' subsphere is perhaps only justified in the case (of primary interest here) where S is one dimensional.

PROOF. In linear coordinates s, t, given by one splitting as in (40), T is the subspace, $\{s = 0\}$. Any other transversal subspace is then of the form $T' = \{s = At\}$, for some matrix, A and corresponding coordinates are $s' = s - At$ and $t' = t$. The sphere at infinity of T' in the image in (40) therefore consists of the limiting points of $(1/\zeta, At/\zeta^2, t/\zeta)$, $\zeta^4 = |s|^2 + |t|^4 + 1 = |At|^2 + |t|^4 + 1$, as $|t| \to \infty$. The first and second terms vanish in the limit which is therefore independent of the choice of A. □

As a slight refinement of Lemma 4 we have

[HHH2.70]

LEMMA 9. *The action of the Lie group* $\mathrm{GL}(V,S)$ *of linear transformations of* V *preserving* S *lifts to be smooth on* ${}^S\overline{V}$, *with the action preserving the equatorial subsphere* $\mathbb{S}T$ *of* ${}^S\overline{V}$. *The lift of the Lie algebra,* $gl(V;S)$ *spans, over* $\mathcal{C}^\infty({}^S\overline{V})$ *the algebra* $\mathcal{V}_b({}^S\overline{V},\mathbb{S}T)$ *of all vector fields tangent to the boundary and to the equatorial subsphere within the boundary.*

It is convenient below to consider the covering of the boundary of ${}^S\overline{V}$ given by two regions which are neighbourhoods of $\mathbb{S}S$ and $\mathbb{S}T$. We may take the first, $F_1 \subset {}^S\overline{V}$, to be, for any $\epsilon > 0$, the union of the region $F_1' = \{|s| > \epsilon|t|^2; |s|^2 + |t|^4 > 1/\epsilon^4\}$ in V and the corresponding region $F_1'' \subset {}^S\mathbb{S}V$, forming part of the boundary of the image of this set under (40). Since $|s|^{\frac12}/\zeta$, $\zeta = \langle s,t\rangle_S$ as in (40), is smooth and bounded below on this set it follows that the more 'projective' map

$$V \supset F_1' \ni (s,t) \longmapsto (\frac{1}{|s|^{\frac12}}, \frac{s}{|s|}, \frac{t}{|s|^{\frac12}}) \in [0,\epsilon) \times \mathbb{S}S \times \{t' \in T; |t'| < 1/\epsilon\} \tag{46}$$ [HHH2.86]

extends by continuity to a diffeomorphism of F_1 onto the range space.

Similarly we may consider $F_2' = \{|s| < \epsilon'|t|^2; |s|^2 + |t|^4 > 1/\epsilon^4$ and the corresponding sets $F_2'' \subset {}^S\mathbb{S}V$ and $F_2 = F_2' \cup F_2'' \subset {}^S\overline{V}$. Then the map

$$V \supset F_2' \ni (s,t) \longmapsto (\frac{1}{|t|}, \frac{s}{|t|^2|}, \frac{t}{|t|}) \in [0,\epsilon') \times \{s' \in S; |s'| < \epsilon'\} \times \mathbb{S}T \tag{47}$$ [HHH2.87]

extends to a diffeomorphism of F_2 onto the range space. Since ${}^S\overline{V} = V \cup F_1 \cup F_2$, provided $\epsilon < \epsilon'$, these maps determine the $\mathcal{C}^\infty$ structure on ${}^S\overline{V}$.

The radial compactification $\overline{V}$ is the special case of the parabolic compactification where the parabolic subspace is $\{0\}$, thus ${}^0\overline{V} = \overline{V}$. Although the S-parabolic compactification is always a closed ball, in general it and the radial compactification are quite different. The only other case in which the natural identification of the interiors, as V, extends to be smooth in either direction is the other 'trivial' case $S = V$. Even then it is not a diffeomorphism. Recall from [**12**] the general notion of the parabolic blow-up of a submanifold, $Y \subset Z$ with respect to a subbundle of its conormal bundle, $H \subset N^*Y$. This is denoted by $[Z; Y, H]$. The V-parabolic compactification of V can be naturally identified with the parabolic blow up of the boundary of $\overline{V}$ in the sense of [**12**] and then the smooth extension of the identification of the interiors is the blow-down map

$$\beta : {}^V\overline{V} \equiv [\overline{V}; \partial\overline{V}, N^*\partial\overline{V}] \longrightarrow \overline{V}. \tag{48}$$ [HHH2.71]

Indeed, $\mathcal{C}^\infty({}^V\overline{V})$ is just obtained from $\mathcal{C}^\infty(\overline{V})$ by adjoining the square-root of a defining function for the boundary. In the next section, (48) will be generalized to give a relationship, through blow up, between $\overline{V}$ and ${}^S\overline{V}$ in the general case.

There is a similar relationship between the quadratic and the radial compactifications of a vector space. Namely

$$\overline{V} = [{}^q\overline{V}; \partial{}^q\overline{V}, N^*\partial{}^q\overline{V}]. \tag{49}$$ [HHH2.206]

This is essentially the definition of the two spaces, since it means that the $\mathcal{C}^\infty$ structure on $\overline{V}$, on the same set, is obtained from the that of ${}^q\overline{V}$ by admitting the square-root, $\langle v\rangle^{-1}$ of the boundary defining function, $\langle v\rangle^{-2} \in \mathcal{C}^\infty({}^q\overline{V})$ in $\mathcal{C}^\infty(\overline{V})$ with tangential regularity unchanged.

We are particularly interested in ${}^S\overline{V}$ in the case that $S \subset V$ is one-dimensional. Then the boundary of the S-parabolic compactification of V is divided into two half-spheres by the equatorial sphere corresponding to any transversal. The labelling of the two halves of ${}^S\mathbb{S}V_\pm$ corresponds to a choice of orientation for S. The compactification of V induces a $\mathcal{C}^\infty$ structure on each hemisphere. It turns out that these hemispheres are canonically identified with the quadratic compactification of a naturally occuring vector space. Let S_+ denote the positive half of S then the vector space we require is $(V/S)\otimes S_+^{-\frac{1}{2}}$. In our applications V is replaced by a vector bundle and S by a one dimensional subbundle. The factor $S_+^{-\frac{1}{2}}$ appears so that we get a global object with the correct invariance properties. The following two lemmas are crucial in understanding our approach to the symbolic and invariance properties of the composition formula for the Heisenberg algebra.
[HHH2.183]

LEMMA 10. *In the case that $S \subset V$ is one-dimensional, and S is oriented, the natural map*

$$W = (V/S)\otimes S_+^{-\frac{1}{2}} \ni v'\otimes s^{-\frac{1}{2}} \longrightarrow [(s,v')] \in {}^S\mathbb{S}V,\ s>0, \tag{50}$$ [HHH2.185]

extends to a $\mathcal{C}^\infty$ map ${}^q\overline{W} \longrightarrow {}^S\mathbb{S}V$ which is a diffeomorphism onto the upper half-sphere.

PROOF. Recall that ${}^S\mathbb{S}V$ is defined by (38). In view of the parabolic homogeneity, (37) the map (50) is well-defined. The radial regularity is also clear. Taking metrics on V/S and S, the $\mathcal{C}^\infty$ structure on the quotient (38) is the same as given by the identification with the non-round sphere $s^2+|v'|^4=1$. The obvious boundary defining function is then s, but on the non-round sphere $\rho = s(s+|v'|^2)^{-1}$ is also a boundary defining function. This lifts to W as the function

$$\langle v'\otimes s^{-\frac{1}{2}}\rangle^{-2} = (1+|v'\otimes s^{-\frac{1}{2}}|^2)^{-1} = s(s+|v'|^2)^{-1}$$

which is a boundary defining function on ${}^q\overline{W}$. □

For the invariant treatment of the symbol calculus below we also need to analyse the normal bundle to the boundary.
[HHH2.233]

LEMMA 11. *If $S \subset V$ is a one dimensional subspace then over the complement in ${}^S\mathbb{S}V$ of the equatorial subsphere there is a natural isomorphism of B, the normal bundle to ${}^S\mathbb{S}V$ as the boundary of ${}^S\overline{V}$ and the trivial line bundle $S^{\frac{1}{2}}$. This isomorphism extends up to $\mathbb{S}T$, the equatorial subsphere, so that*

$$\mathcal{C}^\infty({}^S\mathbb{S}_+V;B) = \rho^{-\frac{1}{2}}\mathcal{C}^\infty({}^q\overline{W};S^{\frac{1}{2}}),\ W=V/LS.$$

PROOF. An element of $0 < s \in S^{\frac{1}{2}}$ defines a linear function on $V/S\times S$ by $f(w,ts^2)=t$. This function is homogeneous of degree 2 under the parabolic homogeneity and is positive near the part of the boundary of ${}^S\overline{V}$ consisting of the interior of the upper half sphere ${}^S\mathbb{S}_+V$. Thus $f^{-\frac{1}{2}}$ is a defining function for the boundary of ${}^S\overline{V}$ over ${}^S\mathbb{S}_+V$, giving a linear identification of $S^{\frac{1}{2}}$ and the normal bundle to the boundary of ${}^S\overline{V/S\times S}$. A choice of transversal identifies $V/S\times S$ with V and this identifies f up to a term homogeneous of degree 1. This does not change the identification of $df^{\frac{1}{2}}$ as a section of the conormal bundle to the boundary, so the identification of $S^{\frac{1}{2}}$ and B is canonical over ${}^S\mathbb{S}_+V$.

In terms of a choice of transversal, T and metric, a true boundary defining function is $(f^2+|t|^4)^{\frac14} = f^{\frac12}(1+|t|^4/f^2)^{\frac14}$. The second factor here is a smooth function on $W \equiv T \otimes S^{-\frac12}$ and is a positive element of $\rho^{-\frac12}\mathcal{C}^\infty({}^q\overline{W})$. This proves the lemma. □

5. Extended parabolic compactification [SS.extended.parabolic]

The behaviour of the linear vector fields on ${}^S\overline{V}$ given in Lemma 9 suggests that it may be natural to blow-up the equatorial submanifold. We define the *extended* S-parabolic compactification of V to be the compact manifold with corners resulting from this (radial) blow up, together with its natural blow-down map

$$\beta_e : {}^{\mathrm{eS}}\overline{V} = [{}^S\overline{V}; \mathbb{S}T] \longrightarrow {}^S\overline{V}, \tag{51}$$ [HHH2.72]

where $\mathbb{S}T \subset \partial{}^S\overline{V}$ is the boundary of the closure of any transversal subspace to S.

From Lemma 9 and the properties of radial blow-up it follows that

$$\mathrm{GL}(V;S) \text{ lifts to } {}^{\mathrm{eS}}\overline{V} \text{ to span } \mathcal{V}_{\mathrm{b}}({}^{\mathrm{eS}}\overline{V}) \text{ over } \mathcal{C}^\infty({}^{\mathrm{eS}}\overline{V}). \tag{52}$$ [HHH2.73]

We recall that the $\mathcal{C}^\infty$ functions on ${}^{\mathrm{eS}}\overline{V}$ are generated by the $\mathcal{C}^\infty$ functions on ${}^S\overline{V}$ together with the smooth functions in polar coordinates around the submanifold $\mathbb{S}T$. In this case, since it is not only an embedded manifold but has trivial normal bundle, $\mathbb{S}T$ has an open neighbourhood in ${}^S\overline{V}$ which is a product of the form

$$\mathbb{S}T \times \mathbb{B}^{k,1}(\epsilon)_y,\ \mathbb{S}T \simeq \mathbb{S}T \times \{0\} \times \{0\},$$

with $\mathbb{B}^{k,1}(\epsilon) = \{y \in \mathbb{R}^k; y_1 \ge 0, |y|^2 < \epsilon\}$ a Euclidean half ball. The polar coordinates are then simply introduced at the origin in the last factor. That is a neighbourhood of the 'new' boundary hypersurface of ${}^{\mathrm{eS}}\overline{V}$, compared to ${}^S\overline{V}$, introduced by the blow up is

$$\mathbb{S}T \times \mathbb{S}^{k-1,1} \times [0,\epsilon^{1/2}) \ni (t,\omega,r) \longmapsto \\ (t,r\omega) \in \mathbb{S}T \times \mathbb{B}^{k,1}(\epsilon),\ \mathbb{S}^{k-1,1} = \mathbb{S}^{k-1} \cap \mathbb{R}^{k,1}. \tag{53}$$ [HHH2.77]

Thus ${}^{\mathrm{eS}}\overline{V}$ is covered by the two open sets ${}^S\overline{V} \setminus \mathbb{S}T$ and (53).

We shall let $\rho_S \in \mathcal{C}^\infty({}^{\mathrm{eS}}\overline{V})$ denote a defining function for the lift of the 'old' boundary hypersurface of ${}^S\overline{V}$ and $\rho_R \in \mathcal{C}^\infty({}^{\mathrm{eS}}\overline{V})$ be a defining function for the new boundary hypersurface produced by the blow up in (51). If $\rho \in \mathcal{C}^\infty({}^S\overline{V})$ is a boundary defining function then lifted to ${}^{\mathrm{eS}}\overline{V}$ it satisfies

$$\rho = a\rho_R\rho_S,\ 0 < a \in \mathcal{C}^\infty({}^{\mathrm{eS}}\overline{V}). \tag{54}$$ [HHH2.207]

The main geometric reason that this more complicated compactification of V is of interest here is that it simultaneously resolves ${}^S\overline{V}$ and $\overline{V}$. The resolution of the latter follows from (51); we proceed to consider the former and then compare the two resulting spaces.

Consider the space obtained by the parabolic blow up the boundary of the closure of S in $\overline{V}$ with parabolic direction the boundary conormal; it may written $[\overline{V}; \mathbb{S}S, N^*\mathbb{S}V]$. We shall show that it is the same space as ${}^{\mathrm{eS}}\overline{V}$. As above, $\mathbb{S}S \subset \overline{V}$ is a subsphere of the boundary, so has a trivial normal bundle. In $\overline{V}$ it has a product neighbourhood

$$\mathbb{S}S \times \{y = (y_0, y') \in \mathbb{R}^l; y_0 \ge 0, y_0^2 + |y'|^4 < \epsilon\}.$$

The neighbourhood is written in 'parabolic' form to simplify the introduction of parabolic polar coordinates with respect to the boundary. Namely the corresponding neighourhood of the new boundary hypersurface of $[\overline{V};\mathbb{S}S,N^*\mathbb{S}V]$ introduced by the blow up of $\mathbb{S}S$ is, with its blow-down map,

$$\begin{aligned}(55)\quad \mathbb{S}S\times\{z=(z_0,z')\in\mathbb{R}^{l,1};z_0^2+|z'|^4=1,\ z_0\geq 0\}\times[0,\epsilon^{1/4})\ni(s,z,R)\longmapsto\\ (s,R^2z_0,Rz')\in\mathbb{S}S\times\{y=(y_0,y')\in\mathbb{R}^l;y_0\geq 0,y_0^2+|y'|^4<\epsilon\}.^{[HHH2.76]}\end{aligned}$$

[HHH2.75]

Proposition 1. *The canonical identification with V of the interiors of ${}^{\mathrm{e}S}\overline{V}$ and $[\overline{V};\mathbb{S}S,N^*\mathbb{S}V]$ extends to a diffeomorphism as manifolds with corners.*

Proof. Let (s,t) be linear coordinates in $V=S\times T$ for some choice of transversal to S. Thus $S=\{t=0\}$ and a neighbourhood of the sphere at infinity in $\overline{V}$ is diffeomorphic to $\mathbb{S}V\times[0,\epsilon)$, with $\epsilon>0$ and small, under the extension of the map

$$(56)\qquad (s,t)\longmapsto\left(\frac{s}{|(s,t)|},\frac{t}{|(s,t)|},\frac{1}{|(s,t)|}\right)^{[HHH2.78]}$$

for a product metric $|(s,t)|^2=|s|^2+|t|^2$ on V. The sphere at infinity of S is then the submanifold $\mathbb{S}S=\{(\frac{s}{|s|},0,0)\}$. A neighbourhood of this submanifold is, as noted above, diffeomorphic to $\mathbb{S}S\times\mathbb{B}(\epsilon)\times[0,\epsilon)$ for a small ball around the origin in $\mathbb{B}(\epsilon)\subset T$, under the extension of the map

$$(57)\qquad (s,t)\longmapsto\left(\frac{s}{|s|},\frac{t}{|s|},\frac{1}{|s|}\right).^{[HHH2.79]}$$

Under the parabolic blow up of $\mathbb{S}S$, to construct $[\overline{V};\mathbb{S}S,N^*\mathbb{S}V]$, the first factor in (57) is left unchanged. In view of (56), we may take as radial variable for the parabolic blow up

$$(58)\qquad \rho=(\frac{|t|^4}{|s|^4}+\frac{1}{|s|^2})^{\frac14}=S|s|^{-1},\ S=(|s|^2+|t|^4)^{\frac14}.^{[HHH2.80]}$$

Thus a neighbourhood of the new boundary hypersurface is diffeomorphic to $\mathbb{S}S\times\mathbb{S}T\times[0,\delta)\times[0,\delta)$, for some $\delta>0$, under the extension of the map

$$(59)\qquad (s,t)\longmapsto\left(\frac{s}{|s|},\frac{t}{\rho|s|},\frac{1}{\rho^2|s|},\rho\right)=\left(\frac{s}{|s|},\frac{t}{S},\frac{s}{S^2},\frac{S}{|s|}\right).^{[HHH2.81]}$$

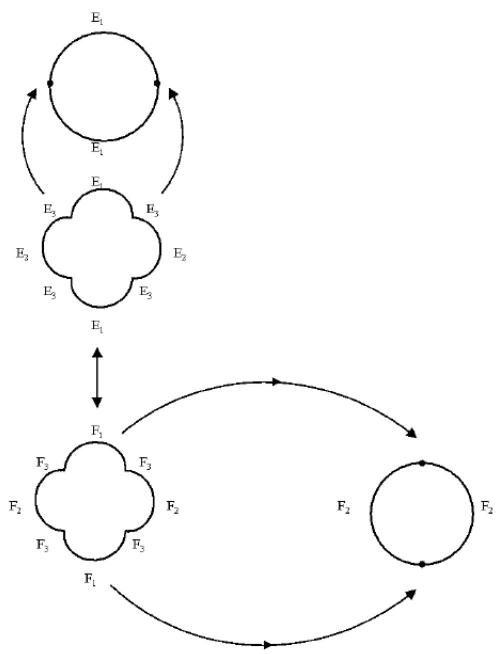
We divide a neighbourhood of the boundary of $[\overline{V};\mathbb{S}S,N^*\mathbb{S}V]$ into three open sets. The first is disjoint from the blow up of $\mathbb{S}S$ just described. Namely we may take the region of V where $|s|<\epsilon|t|$ for any $\epsilon>0$ together with the corresponding part of the boundary. In this region $|t|^2/|s|^2$ is smooth up to the boundary of $\overline{V}$. Thus (56) may be replaced by

$$(60)\qquad (s,t)\longmapsto\left(\frac{s}{|t|},\frac{t}{|t|},\frac{1}{|t|}\right)^{[HHH2.82]}$$

as a map which extends to a diffeomorphism of an open set $E_1\subset{}^{\mathrm{eS}}\overline{V}$ to some manifold with boundary

$$E_1'=\{(s',t')\in S\times\mathbb{S}T;|s'|<\epsilon\}\times[0,c),\ c>0.$$

Figure for paper with Epstein and Mendoza



The second region in $[\overline{V}; \mathbb{S}S, N^*\mathbb{S}V]$ will cover an aritrarily large part of the interior of the boundary face produced by the blow up of $\mathbb{S}S$. Namely if we consider a region $|s| > \delta|t|^2$, for any $\delta > 0$ then S may be replaced by $|s|^{\frac{1}{2}}$ and the map (59) similarly replaced by

$$(s,t) \longrightarrow \left(\frac{s}{|s|}, \frac{t}{|s|^{\frac{1}{2}}}, \frac{1}{|s|^{\frac{1}{2}}}\right) \in E_2' = \mathbb{S}S \times \{t' \in T; |t'| < 1/\delta\} \times [0, c')^{[HHH2.83]} \tag{61}$$

which extends to a diffeomorphism of $E_2 \subset [\overline{V}; \mathbb{S}S, N^*\mathbb{S}V]$ onto E_2' (for $c' > 0$ small).

The third region of $[\overline{V}; \mathbb{S}S, N^*\mathbb{S}V]$ will cover the corner, i.e. the remainder of the boundary. It arises from a region $\{\epsilon'|t| < |s| < \delta'|t|^2\}$. Then, in (59), S may be replaced by $|t|$ resulting in the map

$$(s,t) \longmapsto \left(\frac{s}{|s|}, \frac{t}{|t|}, \frac{|s|}{|t|^2}, \frac{|t|}{|s|}\right) \in E_3' = \mathbb{S}S \times \mathbb{S}T \times [0, c_1) \times [0, c_2)^{[HHH2.88]} \tag{62}$$

which extends to a diffeomorphism of E_3 (so defined) onto E_3', for $c_1, c_2 > 0$ small. From the discussion above it follows that E_1, E_2, and E_3 are open in $[\overline{V}; \mathbb{S}S, N^*\mathbb{S}V]$ and cover a neighbourhood of the boundary, for appropriate choices of the constants $\epsilon < \epsilon'$, and $\delta < \delta'$.

Now we proceed with a similar decomposition of ${}^{\mathrm{eS}}\overline{V} = [{}^{S}\overline{V}; \mathbb{S}T]$. Initially we consider a region of the boundary in ${}^{S}\overline{V}$ disjoint from the equatorial subsphere $\mathbb{S}T$, thus this is also well-defined in ${}^{\mathrm{eS}}\overline{V}$ and arises from a region $|s| > \delta|t|^2$. Then we may use $|s|^{\frac{1}{2}}$ as radial variable instead of $(|s|^2 + |t|^4)^{\frac{1}{4}}$. The map (40) may therefore be replaced by (61), which is a diffeomorphism of E_2 onto a large part of the complement of $\mathbb{S}T$ in the boundary.

The other two regions cover the boundary hypersurface produced by the blow up of $\mathbb{S}T$. Now $\mathbb{S}T$ itself is contained in any region $|s|/\rho^2 < \epsilon, 1/\rho < \epsilon$ with $\epsilon > 0$. Thus it arises from a region of V near infinity where $|s|^2 < \epsilon^2(|s|^2 + |t|^4)$, and hence $|s| < \epsilon''|t|^2$ for some $\epsilon'' > 0$. Thus we may replace $1/\rho$ in (40) by $1/|t|$ and we obtain a neighbourhood of $\mathbb{S}T$ by continuous extension of the map

$$(s,t) \longmapsto (\frac{1}{|t|}, \frac{s}{|t|^2}, \frac{t}{|t|}) \in [0,\epsilon) \times \{s' \in S; |s'| < \epsilon''\} \times \mathbb{S}T. \tag{63}$$ [HHH2.84]

A neighbourhood including an arbitrarily large part of the interior of this boundary face arises as the image of the region where $|s|/|t|^2 < \delta/|t|$, $1/|t| < \delta$, that is $|s| < \delta|t|$. Thus as radial function defining the interior of the new boundary face we may use $1/|t|$ and then the map (60) extends to a diffeomorphism of E_1 to such a neighbourhood. Finally a neighbourhood of the corner of ${}^{\mathrm{eS}}\overline{V}$ arises from some region $\delta'|t| < |s| < \epsilon'|t|^2$. As in the previous case this can be seen to be diffeomorphic to E_3 in (62).

This completes the proof that ${}^{\mathrm{eS}}\overline{V}$ and $[\overline{V}; \mathbb{S}S, N^*\mathbb{S}V]$ are canonically diffeomorphic. □

Thus we identify these two compact manifolds with corners and regard them both as the extended S-parabolic compactification of V. Note that if $\rho' \in \mathcal{C}^\infty(\overline{V})$ is a boundary defining function then lifted to ${}^{\mathrm{eS}}\overline{V}$ it satisfies

$$\rho' = b\rho_R\rho_S^2,\ 0 < b \in \mathcal{C}^\infty({}^{\mathrm{eS}}\overline{V}) \tag{64}$$ [HHH2.208]

in place of (54).

6. Extended parabolic symbols[SS.Extended.parabolic.symbols]

Just as for the parabolic compactification, the space ${}^{\mathrm{eS}}\overline{V}$ is associated with symbol spaces on V. Although ${}^{\mathrm{eS}}\overline{V}$ is a compact manifold with corners, rather than simply a compact manifold with boundary, the definition of conormality can be taken over directly. Thus $\mathrm{Diff}^*_{\mathrm{b}}({}^{\mathrm{eS}}\overline{V})$ is the enveloping algebra of the space $\mathcal{V}_{\mathrm{b}}({}^{\mathrm{eS}}\overline{V})$ of smooth vector fields tangent to all boundary faces (which is equivalent to tangency to the boundary hypersurfaces). Then the same definition, (42) may be used for $\mathcal{A}({}^{\mathrm{eS}}\overline{V})$ by replacing ${}^S\overline{V}$ by ${}^{\mathrm{eS}}\overline{V}$ throughout. The most important difference between these space is that ${}^{\mathrm{eS}}\overline{V}$ has two (types of) boundary hypsurfaces so there is a double filtration of the spaces of conormal functions

$$\rho_R^{-m}\rho_S^{-k}\mathcal{A}({}^S\overline{V}) = \{a \in \rho_R^{-m}\rho_S^{-k}L^\infty({}^S\overline{V}); \mathrm{Diff}^*_{\mathrm{b}}({}^S\overline{V})a \subset \rho_R^{-m}\rho_S^{-k}L^\infty(V)\}. \tag{65}$$ [HHH2.140]

When S has dimension 1, ${}^{\mathrm{eS}}\overline{V}$ has three boundary faces rather than two as $\{\rho_S = 0\}$ has two components, indexed by the orientation of S. We can allow independent orders at these two boundary hypersurfaces, simply by letting $\rho_S = \rho_{S,+}\rho_{S,-}$ be a product of defining functions for the two parts and inserting appropriate factors in (65).

Again we may easily relate these conormal estimates to 'symbol' estimates on V.

[HHH2.141]

LEMMA 12. *A function $a \in \mathcal{C}^\infty(V)$ lifts to an element of $\rho_R^{-m}\rho_S^{-k}\mathcal{A}({}^S\overline{V})$ if and only if it satisfies the estimates*

$$|\partial_t^\alpha \partial_s^\beta a| \le C_{\alpha,\beta}\langle s,t\rangle^{-m+k-|\beta|}\langle s,t\rangle_S^{2m-k-|\alpha|}. \tag{66}$$ [HHH2.142]

PROOF. As with the proof of Lemma 5 the estimates (66) may be rewritten as

$$\|H^\alpha G^\beta(\langle s,t\rangle^{m-k}\langle s,t\rangle_S^{-2m+k}a\|_{L^\infty} < \infty \ \forall\ \alpha,\beta \quad \text{where } H_i = \langle s,t\rangle_S\partial_{t_i},\ G_j = \langle s,t\rangle\partial_{s_j}. \tag{67}$$ [HHH2.143]

Then the proof of the lemma amounts to showing that the vector fields H_i and G_j lift to be smooth on ${}^{\mathrm{eS}}\overline{V}$ and span $\mathcal{V}_{\mathrm{b}}({}^{\mathrm{eS}}\overline{V})$, together with the fact, which follows from (54) and (64), that $\rho_R = \langle s,t\rangle\langle s,t\rangle_S^{-2}$ and $\rho_S = \langle s,t\rangle_S\langle s,t\rangle^{-1}$ are defining functions for the two boundary hypersurfaces. □

There is a corresponding characterization of the 'classical' subspaces, which we use below in discussing the composition of operators. Namely on a manifold with corners such as ${}^{\mathrm{eS}}\overline{V}$ there is a 'radial' vector field for each boundary hypersurface H. This is a smooth vector R_H, which is tangent to all boundary faces, which vanishes at the given boundary hypersurface and satisfies $R_H\rho_H = \rho_H + O(\rho_H^2)$ for any (one) defining function for that boundary hypersurface. If $\rho_{H'}$ are boundary defining functions for the other boundary faces it can always be chosen to satisfy $R_H\rho_{H'} = 0$ near H' if $H' \ne H$.

[HHH2.144]

LEMMA 13. *An element $a \in \rho_R^{-m}\rho_S^{-k}\mathcal{A}({}^S\overline{V})$ is in $\rho_R^{-m}\rho_S^{-k}\mathcal{C}^\infty({}^S\overline{V})$ if and only if it satisfies the conditions*

$$\left[\prod_{j=0}^{J}(R_R - j + m)\prod_{i=0}^{I}(R_S - i + m)\right] a \in \rho_R^{-m+p+1}\rho_S^{-k+q+1}\mathcal{A}({}^S\overline{V})\ \forall\ I, J. \tag{68}$$ [HHH2.145]

PROOF. See [**22**] and for a detailed proof [**21**]. □

[HHH2.89]

LEMMA 14. *An element of $a \in \mathcal{C}^\infty(V)$ lifts into $\mathcal{C}^\infty({}^{eS}\overline{V})$ if and only if it has complete asymptotic expansions at infinity of the following type, in terms of coordinates (s,t) in $V = S \oplus T$, for any cut-off functions $\phi,\psi,\phi',\psi' \in \mathcal{C}_c^\infty(\mathbb{R})$:*

$$a(s,t)\phi(t|s|^{-\frac12}) \sim \sum_{j\ge 0}|s|^{-j/2}a_j(t|s|^{-\frac12}, s|s|^{-1}),\ a_j \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}) \tag{69}$$

$$a(s,t)\psi(s|t|^{-1}) \sim \sum_{j\ge 0}|t|^{-j}b_j(t|t|^{-1}, s|t|^{-1}),\ b_j \in \mathcal{C}_c^\infty(\mathbb{S}\times\mathbb{R}) \tag{70}$$

$$a(s,t)\phi'(s|t|^{-2})\psi'(t|s|^{-1}) \sim \sum_{j\ge 0}|t|^{-j}c_j(t|s|^{-\frac12}, s|s|^{-1}),\ c_j \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}). \tag{71}$$ [HHH2.90]

Note that in linear coordinates s,t with S defined by $t=0$,

$$R_R = s\cdot\partial_s + t\cdot\partial_t \text{ and } R_S = 2s\cdot\partial_s + t\cdot\partial_t \tag{72}$$ [HHH2.209]

are respectively radial vector fields for the two boundary hypersurfaces.

7. Compactifications of a vector bundle[SS.compactifications.bundle]

Now, let $V \longrightarrow Y$ be a vector bundle. If $S \subset V$ is a subbundle we can give the S-parabolic compactification of V, defined as the union of the compactifications of the fibres

$$^{S}\overline{V} = \bigsqcup_{y \in Y} {}^{S_y}\overline{V_y},$$

the structure of a manifold with boundary, compact with Y.

In case both V and S are products this is a consequence of the discussion above. It then follows that any $\mathcal{C}^\infty$ linear diffeomorphism of V which preserves S lifts to a diffeomorphism of $^{S}\overline{V}$. The general case can now be deduced by taking a local trivialization of V which also locally trivializes S, since in the overlaps the transformations preserve S. Thus we conclude that $^{S}\overline{V}$ is a manifold with boundary, compact with Y, to which any fibre-linear diffeomorphsim of V preserving S lifts.

Given the invariance properties of the compactifications discussed above similar conclusions hold for the extended S-parabolic compactification $^{\mathrm{eS}}\overline{V}$ of a vector bundle with respect to a subbundle (orientable or not). The spaces of symbols are defined as in (65) and there is a similar characterization of the classical elements; indeed the extra 'base' variables are essentially smooth parameters. However this last point must be considered carefully as the differential geometry of the subbundle, S can have a dramatic effect on the classes of symbols.

Of particular interest is the case $V = T^*X$ where, X is a smooth manifold. The bundle, S is then a subbundle of T^*X. Its annihilator, $S^\perp$ is a subbundle of the tangent bundle. This subbundle is either integrable and defines a foliation or not. If $S^\perp$ is integrable then the class of conormal functions on $^{S}\overline{T^*X}$ is a subset of $S^\infty_{\frac{1}{2},0}(X)$, otherwise it is a subset of $S^\infty_{\frac{1}{2},\frac{1}{2}}(X)$.

Let T be a complementary subbundle for S and $T^\perp$ its annihilator. On a small open set U we can find sections, $Y_i \in \mathcal{C}^\infty(U; S^\perp), i = 1, \dots, k$ and $Z_j \in \mathcal{C}^\infty(U; T^\perp), j = 1, \dots, n-k$ which locally trivialize these vector bundles. Let $\xi_i = \sigma(Y_i)$ and $\eta_j = \sigma(Z_j)$ be the classical symbols of these vector fields. Note that sections of $S^\perp$ define symbols of order 1 whereas sections of $T^\perp$ define symbols of order 2. If $S^\perp$ is not integrable then we can find a pair of smooth sections Y, Y' such that $[Y, Y'] \neq 0$ is not a section of $S^\perp$. Thus $[Y, Y']$ defines a symbol of order 2 and therefore the polyhomogeneous symbols defined by the S-parabolic compactification do not belong to $S^*_{\rho,\delta}$ for any $0 \le \delta < \rho \le 1$.

An S–parabolic radial function is given locally by $R = (|\xi|^4 + |\eta|^2)^{\frac{1}{4}}$, the action of $\mathbb{R}^+$ is

$$M_\tau(x; \xi, \eta) = (x; \tau\xi, \tau^2\eta).$$

A function is smooth on the compact manifold with boundary $^{S}\overline{T^*X}$ if it can be expressed as a smooth function of the variables $R^{-1}, \xi/R, \eta/R^2$. A function is (locally) S-homogeneous of degree m if it is of the form, $f(x, \sigma(Y), \sigma(Z))$ where

$$f(x, \tau\xi, \tau^2\eta) = \tau^m f(x, \xi, \eta).$$

We can restate Lemma 5 and Lemma 12 in this case as follows:

[HHH2.1341]

LEMMA 15. *A function $a \in \mathcal{C}^\infty(T^*X)$ lifts to an element of $\rho^{-m}\mathcal{A}({}^S\overline{T^*X})$ if and only if for all triples of multi-indices, α, β, γ it satisfies the estimates*

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma a(x;\eta,\xi)| \leq C_{\alpha,\beta,\gamma}\langle \eta,\xi\rangle_S^{m-|\alpha,\beta|_S},\ |\alpha,\beta|_S = |\alpha| + 2|\beta| \tag{73}$$

[HHH2.1351]

for any local coordinates, x and symbols (η,ξ) as defined above and local fiber metric. The subspace $\dot{\mathcal{C}}^\infty({}^S\overline{T^*X})$ is dense in $\rho^{-m}\mathcal{A}({}^S\overline{T^*X})$ in the topology of $\rho^{-m'}\mathcal{A}({}^S\overline{T^*X})$ for any $m' > m$.

and

[HHH2.1411]

LEMMA 16. *A function $a \in \mathcal{C}^\infty(T^*X)$ lifts to an element of $\rho_R^{-m}\rho_S^{-k}\mathcal{A}({}^S\overline{T^*X})$ if and only if, in a neighborhood of each point in X it satisfies the estimates*

$$|\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma a| \leq C_{\alpha,\beta,\gamma}\langle \eta,\xi\rangle^{-m+k-|\beta|}\langle \eta,\xi\rangle_S^{2m-k-|\alpha|}. \tag{74}$$

[HHH2.1421]

If $S^\perp$ is not integrable then in general it is impossible to find sections of $S^\perp$ so that the symbols, ξ are independent of the base variables. In this case the symbol estimates, when expressed in terms of linear coordinates on the cotangent space defined by coordinate co-vectors, $\{dx_1, \dots dx_n\}$ appear quite different. In particular, differentiation in the base variables generally raises the order of a symbol.

There are infinitely many geometrically distinct possibilities for non-integrable $S^\perp$ but we are princiaplly interested in the special case where S has fiber dimension 1 and $S^\perp$ defines a contact structure. In this case the symbols $\{\xi_i\}$ cannot be made independent of the base variables, x. It can be seen that these symbol classes are contained in Hörmander's class $S^*_{\frac{1}{2},\frac{1}{2}}(X)$, see [**2**] and [**27**].

In the other extreme case, where $S^\perp$ is integrable we can find local coordinates (y,z) on an open set, U so that the integral manifolds of $S^\perp$ are given by $z =$constant. For the vector fields, $\{Y_i\}$ we use $\{\partial_{y_i}\}$ and for the vector fields $\{Z_j\}$ we use $\{\partial_{z_j}\}$. The coordinates, $(y,z;\xi,\eta)$ are simply dual linear coordinates for the cotangent space. In this case it is easy to see that the S-parabolic symbols belong to $S^*_{\frac{1}{2},0}(X)$. This case corresponds to the heat operator.