CANONICAL BASES AND REPRESENTATION CATEGORIES

COURSE GIVEN BY ROMAN BEZRUKAVNIKOV

These are notes from a course given by Roman Bezrukavnikov at MIT in the Fall 2013 and written up by Tina Kanstrup. The notes are work in progress. The parts in black are transcripts from my notes taken during the lectures. The parts in red are added by me to help my own understanding of the material. The parts in blue are Roman's original formulation in places where I expanded on the argument but couldn't get the exact sentence to fit into my expansion. Comments and corrections are very welcome. Please send an email to tina.kanstrup@mail.dk.

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1. Introduction

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$$K^0(\text{category of representations}) \simeq \text{ representation of something else}$$
$$\cup$$
$$\{\text{class of irreducible objects}\} \text{ gives a canonical basis.}$$

2. BGG category $\mathcal{O}$

2.1. **Flag variety and Borel-Weil Theorem.** Let G be a simply-connected reductive algebraic group over an algebraically closed field of char 0. Denote its Lie algebra by $\mathfrak{g}$. It is known that there is a 1-1 correspondence between representations of G and representations of $\mathfrak{g}$. Choose a Borel subgroup B and a Cartan subgroup T contained in B. The flag variety $\mathcal{B} \simeq G/B$ is the maximal projective homogenous space. The Weyl group is denoted by W. The set $\Lambda = \mathrm{Hom}(T, \mathbb{C}^*)$ is called the lattice of integral weights and the subset

$$\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all simple coroots } \alpha^\vee\}$$

is the set of positive weights. There is a 1-1 correspondence (see [Spr, Section 8.5.7]).

$$\Lambda \longleftrightarrow \{G-\text{equivariant line bundles on } \mathcal{B}\}$$
$$\lambda \mapsto \mathcal{O}(\lambda)$$

Example 2.1.1. For $G = SL_2$ we have $\mathcal{B} = \mathbb{P}^1$ and $\Lambda = \mathbb{Z}$. In this case $n \in \mathbb{Z}$ corresponds to the twisting sheaf $\mathcal{O}_{\mathbb{P}^1}(n)$.

The following theorem provides a geometric description of finite dimensional irreducible representations. It follows from the classification in terms of highest weights that all finite dimensional representations of $\mathfrak{g}$ arise this way. This classification will be recalled later.

Theorem 2.1.2 (Borel-Weil). *Let $\lambda \in \Lambda$. If $\lambda \in \Lambda^+$ then $\Gamma(\mathcal{B}, \mathcal{O}(\lambda))$ is an irreducible highest weight representation of G with highest weight λ. If $\lambda \notin \Lambda^+$ then $\Gamma(\mathcal{B}, \mathcal{O}(\lambda)) = 0$.*

Sketch of proof. Since G acts on $\mathcal{B} = G/B$ we get that $V := \Gamma(\mathcal{B}, \mathcal{O}(\lambda))$ is an algebraic representation of G. The representation V can be decomposed into irreducibles as

$$V = \bigoplus V_{\lambda_i}^{m_i},$$

where V_{λ_i} denotes the irreducible highest weight representation of G with highest weight λ_i. Recall that $B = TN$, where N is the unipotent elements of B. This correspond to a decomposition of the Lie algebras $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. Since T acts semisimply the N-invariant part of V splits into 1-dimensional representations with T acting by a character λ_i

$$V^N \simeq \bigoplus \mathbb{C}_{\lambda_i}^{m_i}.$$

Recall that the action of N on $\mathcal{B}$ has a unique open orbit $\mathcal{B}_0$. This orbit is dense in $\mathcal{B}$ so the restriction map is injective

$$\Gamma(\mathcal{B}, \mathcal{O}(\lambda)) \hookrightarrow \Gamma(\mathcal{B}_0, \mathcal{O}(\lambda))$$

Since $\mathcal{B}_0$ is a free N-orbit there exists a non-zero N-section for $\mathcal{O}(\lambda)$ on $\mathcal{B}_0$. Hence,

$$\dim V^N \leq \dim \Gamma(\mathcal{B}_0, \mathcal{O}(\lambda))^N = 1.$$

There is a unique T fixed point in $\mathcal{B}_0$ so T acts on $\Gamma(\mathcal{B}_0, \mathcal{O}(\lambda))^N$ by the character λ. Only the highest weight space in each V_{λ_i} is N-invariant so $\dim V_{\lambda_i}^N = 1$. Thus, $\dim V^N = 1$ implies that $V = V_\lambda$ and $\dim V^N = 0$ implies that $V = 0$.

We will now determine which case we are in. Consider the up to scaling unique N-invariant section in $\Gamma(\mathcal{B}_0, \mathcal{O}(\lambda))$. This is a rational section of $\mathcal{O}(\lambda)$ on $\mathcal{B}$. The invariance V^N is nonzero if and only if this section comes from a regular section in $\Gamma(\mathcal{B}, \mathcal{O}(\lambda))$. To check that this is the case we need to calculate the divisor and see that it is positive. Write $\mathcal{B}_w := BwB/B \subset G/B = \mathcal{B}$. Since $N \subset B$ each $\mathcal{B}_w$ lies in an N-orbit. In fact $\mathcal{B}_0 = \mathcal{B}_{w_0}$, where w_0 is the longest element in W. Consider the Bruhat decomposition

$$\mathcal{B} = \coprod_{w \in W} \mathcal{B}_w$$

Recall that $\dim(\mathcal{B}_w) = \ell(w)$. We look for w for which $\mathcal{B}_w$ has codimension 1.

$$\begin{aligned} \dim(\mathcal{B}_w) = \dim(\mathcal{B}) - 1 &\Leftrightarrow \ell(w) = \ell(w_0) - 1 \\ &\Leftrightarrow \ell(w^{-1} w_0) = 1 \\ &\Leftrightarrow w^{-1} w_0 = s_\alpha \qquad \text{simple reflection} \end{aligned}$$

Thus, the codimension 1 components $\mathcal{B} \setminus \mathcal{B}_0$ are $\overline{\mathcal{B}_{s_\alpha}} =: \mathcal{D}_\alpha$.

Claim 2.1.3. If σ is a non-zero N-invariant section of $\mathcal{O}(\lambda)$ then $(\sigma) = \sum \langle \alpha^\vee, \lambda \rangle \mathcal{D}_\alpha$

By the claim the divisor is effective if and only if $\lambda \in \Lambda^+$. Hence, $V^N \neq 0$ if and only if $\lambda \in \Lambda^+$. □

2.2. **Weyl character formula.** Let $\mu \in \mathfrak{t}^*$ and let V be a representation. We define the μ weight space of V to be

$$V[\mu] = \{v \in V \mid x(v) = \mu(x) v \ \ \forall x \in \mathfrak{t}\}$$

The character of V is defined as

$$\chi_V := \sum_\mu \dim(V[\mu]) e^\mu \quad \in \mathbb{Z}[\Lambda]$$

Exercise 2.2.1. Show that χ_V determines Tr(g,V) for $g \in G$.

Let ρ denote half the sum of all positive roots. The characters of the irreducible representations are given by the Weyl character formula.

Theorem 2.2.2 (Weyl character formula)**.** *For* $\lambda \in \Lambda^+$

$$\chi_{V_\lambda} = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}}$$

Note that the denominator can be rewritten as $e^\rho \prod_{\alpha \text{ pos. root}} (1 - e^{-\alpha})$

Example 2.2.3. For $\mathfrak{g} = sl_2$ the irreducible representation V_n corresponding to $n \in \mathbb{Z} = \Lambda$ is spanned by vectors $x^n, x^{n-1}y, \dots y^n$ with corresponding weights $n, n-2, \dots, -n$. Thus, $V_n \simeq \mathbb{C}[x,y]_n$, where the n indicates that we only consider degree n homogenous polynomials. Hence, the character is

$$\begin{aligned}\chi_{V_n} &= z^n + z^{n-2} + \dots z^{-n} \\ &= \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}},\end{aligned}$$

where $z := e^t$.

2.3. **Verma modules, and their simple quotients.** Let $\lambda \in \mathfrak{t}^*$ and let $\mathbb{C}_\lambda$ denote the 1-dimensional representation of $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ on which $t \in \mathfrak{t}$ acts by $t(v) = \lambda(t)v$ and $n(v) = 0$ for $n \in \mathfrak{n}$. The Verma module is defined as

$$\Delta_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda.$$

Note that for any $\mathfrak{g}$ module M we have

$$\mathrm{Hom}(\Delta_\lambda, M) = \{v \in M \mid n(v) = 0, t(v) = \lambda(t)v \ \ \forall n \in \mathfrak{n}, t \in \mathfrak{t}\}$$

Write $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_- = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$. By the PBW theorem Δ_λ is freely generated as a module over $\mathfrak{n}_-$ by a vector v_λ.

$$\mathcal{U}(\mathfrak{n}_-) \simeq \Delta_\lambda, \qquad x \mapsto x(v_\lambda)$$

By PBW we also have

$$\mathcal{U}(\mathfrak{n}_-) \simeq \mathrm{Sym}(\mathfrak{n}_-)$$

Consider the decomposition of Δ_λ into weight spaces

$$\Delta_\lambda = \bigoplus \Delta_\lambda[\mu].$$

By definition $\mathfrak{t}$ acts on $\Delta_\lambda[\mu]$ by $\mu \cdot \mathrm{Id}$. Thus,

$$\Delta_\lambda[\mu] \simeq \mathrm{Sym}(\mathfrak{n}_-)[\mu - \lambda]$$

and its dimension is equal to the number of ways to write $\lambda - \mu$ as a sum of positive roots.

If $\lambda \in \Lambda^+$ we have a morphism $\Delta_\lambda \to V_\lambda$ given by $x(v_\lambda) = 0 \ \ \forall x \in \mathcal{U}_+$ where v_λ is a highest weight vector in V_λ.

Lemma 2.3.1. *For any $\lambda \in \Lambda$, Δ_λ has a unique irreducible quotient L_λ.*

Proof. Irreducible quotients correspond to maximal proper submodules. Let N be such a maximal proper submodule. We can write it as $N = \oplus_\mu N[\mu] \subset \Delta_\lambda$. That $N \subsetneq \Delta_\lambda$ implies that $N[\lambda] = 0$ since otherwise we would have $v_\lambda \in N$ and then $N = \Delta_\lambda$. Let N_1, N_2 be two maximal proper submodules. Then $N_1 + N_2$ is also a submodule. Since $N_1[\lambda] = N_2[\lambda] = 0$ the sum is still a proper submodule. Thus we must have $N_1 = N_2$. □

We know that L_λ has finite dimensional weight components. We want to compute $\dim L_\lambda[\mu]$ for all μ.

2.4. **Dual Verma modules.** Since Δ_λ is infinite dimensional we only have

$$\bigoplus_\mu \Delta_\lambda[\mu]^* \subset (\Delta_\lambda)^*$$

as a $\mathfrak{g}$-submodule. We define the dual Verma module to be

$$\nabla_\lambda := \bigoplus_\mu \Delta'_\lambda[\mu]^*, \qquad \Delta'_\lambda = \mathrm{Ind}_{\mathfrak{b}_-}^{\mathfrak{g}}(\mathbb{C}_{-\lambda})$$

Remark 2.4.1. The reason for using the negative Borel and changing the sign of the weight will be clear once we have defined category $\mathcal{O}$.

The action of $\mathfrak{n}$ on Δ_λ is free, so the action of $\mathfrak{n}$ on ∇_λ is cofree. Notice that $\nabla_\lambda[\lambda] \simeq \mathbb{C}$. Choose a morphism

$$\phi : \nabla_\lambda \to \mathbb{C} \qquad \text{with } \phi \neq 0 \text{ and } \phi_{|\nabla_\lambda[\mu]} = 0 \text{ for } \mu \neq \lambda.$$

The pairing

$$\mathcal{U}(\mathfrak{n}) \times \nabla_\lambda \to \mathbb{C}, \qquad (x, v) \mapsto \phi(x(v))$$

is non-degenerate, so the graded dual ∇_λ is isomorphic to $\mathcal{U}(\mathfrak{n})^*$ as a $\mathfrak{n}$ module.

Claim 2.4.2. $\nabla_{\lambda=0} = \mathcal{O}(\mathcal{B}_0)$. For $\lambda \in \Lambda$ we have $\nabla_\lambda = \Gamma(\mathcal{B}_0, \mathcal{O}(\lambda))$.

For $\lambda \in \Lambda$ there is a unique (up to scaling) N-invariant nowhere-vanishing section for $\mathcal{O}(\lambda)|_{\mathcal{B}_0}$. Thus, $\mathcal{O}(\lambda)|_{\mathcal{B}_0}$ can be trivialized.

$$f_\lambda : \mathcal{O}(\lambda)|_{\mathcal{B}_0} \xrightarrow{\sim} \mathcal{O}|_{\mathcal{B}_0}$$

In particular,

$$\Gamma(\mathcal{B}_0, \mathcal{O}(\lambda)) \simeq \Gamma(\mathcal{B}_0, \mathcal{O}).$$

The isomorphism is invariant with respect to the N-action but not with respect to the $\mathfrak{g}$-action.

Since G acts on $\mathcal{B}$ we get a map $\mathfrak{g} \to \mathrm{Vect}(\mathcal{B})$. Restricting to the open set $\mathcal{B}_0$ we get $\mathfrak{g} \to \mathrm{Vect}(\mathcal{B}_0)$. $\mathrm{Vect}(\mathcal{B}_0)$ acts on $\mathcal{O}(\mathcal{B}_0)$ by the Lie derivative so we get a $\mathfrak{g}$-action on $\mathcal{O}(\mathcal{B}_0)$. For $\xi \in \mathrm{Vect}(\mathcal{B}_0)$ one can use f_λ to get a new Lie derivative on $\mathcal{O}(\lambda)$

$$f_\lambda^{-1} \mathrm{Lie}_\xi f_\lambda = \mathrm{Lie}_\xi + \varepsilon_\lambda(\xi),$$

where $\varepsilon_\lambda(\xi)$ is a linear function acting by multiplication. Thus, the isomorphism gives an action of $\mathfrak{g}$ on $\mathcal{O}(\mathcal{B}_0)$ depending on λ.

$$\alpha_\lambda : \mathfrak{g} \to \mathrm{End}(\mathcal{O}(\mathcal{B}_0)), \qquad x \mapsto \mathrm{Lie}_x + \varepsilon_\lambda(x).$$

Notice that $\varepsilon_{\lambda+\mu} = \varepsilon_\lambda + \varepsilon_\mu$ so we can extend $\{\varepsilon_\lambda\}$ to any $\lambda \in \mathfrak{t}^*$ by linearity.

Claim 2.4.3. The module $\mathcal{O}(\mathcal{B}_0)$ with $\mathfrak{g}$-action given by α_λ is isomorphic to ∇_λ.

Sketch of proof. Since N acts freely on $\mathcal{B}_0$ we have $N \simeq \mathcal{B}_0$. Let x_0 be the unique T fixed point in $\mathcal{B}_0$. We want to check that the pairing

$$\mathcal{O}(N) \times \mathcal{U}(\mathfrak{n}) \to \mathbb{C}, \qquad (f, \xi) \mapsto \xi(f)|_{x_0}$$

is non-degenerate. Here the action of $\mathcal{U}(\mathfrak{n})$ on $\mathcal{O}(N)$ is the extension of the Lie derivative from $\mathfrak{n}$ to all of $\mathcal{U}(\mathfrak{n})$ using the PBW filtration on $\mathcal{U}(\mathfrak{n})$. Let m_{x_0} be the maximal ideal of functions vanishing at x_0. The induced pairing

$$\mathcal{O}(N)/m_{x_0}^n \times \mathcal{U}(\mathfrak{n})^{\leq n} \to \mathbb{C}, \qquad (f,\xi) \mapsto \xi(f)|_{x_0},$$

is non-degenerate. Notice that a T-action is equivalent to a weight grading. The map is clearly T-invariant so we get

$$\mathcal{U}(\mathfrak{n})_\lambda^{\leq n} \simeq (\mathcal{O}(N)/m_{x_0}^n)^*_{-\lambda}.$$

Taking the limit we obtain

$$\mathcal{U}(\mathfrak{n}) \xrightarrow{\sim} \varinjlim_{n\to\infty} (\mathcal{O}(\mathcal{B}_0)/m_{x_0}^n)^*, \qquad \text{and} \quad \mathcal{U}(\mathfrak{g})_\lambda \simeq \mathcal{O}(\mathcal{B}_0)^*_{-\lambda}.$$

Hence, $\mathcal{U}(\mathfrak{n})$ is the graded dual of $\mathcal{O}(\mathcal{B}_0)$. The right hand side of the first isomorphism is called distributions at x_0 and is denoted by Dist_{x_0}. The exponential map gives an isomorphism $N \simeq \mathfrak{n}$. Since $\mathfrak{n}$ is just a vector space it is isomorphic to some $\mathbb{A}^n$ with a torus action with "positive" weights. For $\mathbb{A}^n$ we have

$$\mathcal{O}(\mathbb{A}^n) \xrightarrow{\sim} \text{graded dual of } \mathrm{Dist}^*_{x_0}.$$

This proves the claim. □

This also proves claim 2.4.2. In particular, for $\lambda \in \Lambda^+$ we have maps

$$\Delta_\lambda \twoheadrightarrow L_\lambda = \Gamma(\mathcal{B},\mathcal{O}(\lambda)) \xrightarrow{\mathrm{Res}^{\mathcal{B}}_{\mathcal{B}_0}} \Gamma(\mathcal{B}_0,\mathcal{O}(\lambda)) = \nabla_\lambda.$$

2.5. **Application of Verma-Harish Chandra isomorphisms.** There exists some canonical morphisms between Verma modules.

Claim 2.5.1. Let $\lambda \in \Lambda$ and let $\alpha^\vee$ be a simple coroot with $\langle \lambda, \alpha^\vee \rangle = n \in \mathbb{Z}_{\geq 0}$. Then there exists a morphism $\Delta_{s_\alpha(\lambda)-\alpha} \to \Delta_\lambda$.

Sketch of proof. Note that $s_\alpha(\lambda) - \alpha = \lambda - (n+1)\alpha$. Let v be the generator of Δ_λ and write f_α (resp. e_α) for the generator in $\mathfrak{n}_-$ (resp. $\mathfrak{n}$) corresponding to α. Set

$$v' := f_\alpha^{n+1} v$$

Then v' is a non-zero vector with weight $\lambda-(n+1)\alpha = s_\alpha(\lambda)-\alpha$. Since $[e_\beta, f_\alpha] = 0$ for $\beta \neq \alpha$ the fact that $e_\beta v = 0$ implies that $e_\beta v' = 0$. That $e_\alpha v' = 0$ follows from $\mathfrak{sl}_2$ representation theory. Thus, $v \mapsto v'$ induces the desired morphism. □

Theorem 2.5.2 (Harish-Chandra isomorphism)**.** *There is an isomorphism*

$$Z(\mathcal{U}(\mathfrak{g})) \simeq \{P \in \mathcal{O}(\mathfrak{t}^*) \mid P(s_\alpha(\lambda)-\alpha) = P(\lambda) \;\; \forall \textit{ simple roots } \alpha\}.$$

Sketch of proof. Since $\mathrm{End}(\Delta_\lambda) = \mathbb{C}$ the center $Z(\mathcal{U}(\mathfrak{g}))$ acts on each Δ_λ by scalars. Since $z \in Z(\mathcal{U}(\mathfrak{g}))$ commutes with the Cartan it has zero degree with respect to the natural grading on $\mathcal{U}(\mathfrak{g})$ by weights. This shows that z can be written as $z = z_{\mathfrak{t}} + z'$, where $z_{\mathfrak{t}} \in \mathcal{U}(\mathfrak{t})$ and $z' \in \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Hence, the scalar by which z acts on Δ_λ depends polynomially on λ, let $P_z(\lambda)$ denote the corresponding polynomial. The existence of a nonzero morphism $\Delta_{s_\alpha(\lambda)-\alpha} \to \Delta_\lambda$ for $\langle \lambda+\rho, \check{\alpha}\rangle \in \mathbb{Z}_{>0}$ shows that for such λ

$$P_z(s_\alpha(\lambda)-\alpha) = P_z(\lambda) \tag{1}$$

Since the set of such λ is Zariski dense in $\mathfrak{t}$, (1) holds as an identity of polynomials. Recall that ρ is the half sum of all the positive roots. Observe that $s_\alpha(\rho) = \rho - \alpha$. Hence,

$$s_\alpha(\lambda) - \alpha = s_\alpha(\lambda) + s_\alpha(\rho) - \rho = s_\alpha(\lambda+\rho) - \rho.$$

We define a new action of W called the dot action by

$$w \cdot \lambda := w(\lambda+\rho) - \rho.$$

Equality (1) shows that the polynomial P_z is invariant with respect to this action for all $z \in Z(\mathcal{U}(\mathfrak{g}))$. Thus, we have defined a map

$$Z(\mathcal{U}(\mathfrak{g})) \to \mathcal{O}(\mathfrak{t}^*)^{(W,\cdot)} \simeq \mathrm{Sym}(\mathfrak{t})^W, \qquad z \mapsto P_z.$$

Here the superscript $(W,\cdot)$ means taking invariants with respect to the dot action of W. To prove that it is an isomorphism we look at the associate graded. By PBW we have

$$\begin{aligned} \mathrm{gr}(Z(\mathcal{U}(\mathfrak{g}))) &= \mathrm{gr}(\mathcal{U}(\mathfrak{g})^G) = \mathrm{gr}(\mathcal{U}(\mathfrak{g}))^G \\ &\simeq \mathrm{Sym}(\mathfrak{g})^G \simeq \mathrm{Sym}(\mathfrak{t})^W \end{aligned}$$

The last isomorphism is the Chevalley isomorphism. □

2.6. **BGG category $\mathcal{O}$: definition.** We want to understand the characters of the L_λ by relating them to the Δ_λ. To do this we study a category containing these objects: the Bernstein-Gelfand-Gelfand category $\mathcal{O}$.

Definition 2.6.1 (BGG category $\mathcal{O}$)**.** The BGG category $\mathcal{O}$ is the full subcategory in $\mathfrak{g}$-mod where the modules satisfy the following axioms

(1) $\mathfrak{t}$ acts diagonalizably.
(2) $\mathfrak{n}$ acts locally nilpotently.
(3) The module is finitely generated as a $\mathfrak{g}$-module.

Note that (1) is equivalent to saying that any module $M \in \mathcal{O}$ splits up into a direct sum of its weight spaces

$$M = \bigoplus_\nu M[\nu]$$

Axiom (2) means that for any $x \in M$ there exists an n such that

$$e_1 \cdots e_n(x) = 0, \qquad \forall e_i \in \mathcal{U}(\mathfrak{n}_-).$$

Observe that $\Delta_\lambda, \nabla_\lambda \in \mathcal{O}$ and that for every module in $\mathcal{O}$ every quotient is also in $\mathcal{O}$. In particular, $L_\lambda \in \mathcal{O}$.

2.7. **First properties of category O: finiteness results and (generalized) central character decomposition.**

Lemma 2.7.1. *(1) Every $M \in \mathcal{O}$ has a finite filtration such that each $gr_i M$ is a quotient in a Verma module.*

(2) $Z(\mathcal{U}(\mathfrak{g}))$ acts on $M \in \mathcal{O}$ locally finitely so there is a decomposition

$$\mathcal{O} = \bigoplus_{\chi : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}} \mathcal{O}_\chi,$$

where $\mathcal{O}_\chi := \{M \in \mathcal{O} \mid \ker(\chi)^n M = 0 \text{ for some } n\}$.

Notice that a homomorphism $\xi : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ corresponds to a map

$$\mathrm{pt} = \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(Z(\mathcal{U}(\mathfrak{g}))$$

By the Harish-Chandra isomorphism $\mathrm{Spec}(Z(\mathcal{U}(\mathfrak{g})) \simeq \mathrm{Spec}(\mathcal{O}(\mathfrak{t}^*)^W) = \mathfrak{t}^*//W$. Hence, we can consider the direct sum to be over $\mathfrak{t}^*//W$.

Proof. (1) It follows from the definition that the set

$$\{\nu \mid M[\nu] \neq 0\}$$

has a maximal element with respect to the partial ordering

$$\lambda \leq \mu \quad \Leftrightarrow \quad \mu - \lambda \text{ is a sum of positive roots.}$$

Let λ be a maximal element. Then for all $v \in M[\lambda]$ we have $e_\alpha v \in M[\lambda + \alpha] = 0$. Thus, every element in $M[\lambda]$ defines a map $\Delta_\lambda \to M$. Set

$$M_1 := \mathrm{coker}(\Delta_\lambda \to M)$$

Repeating the same procedure for M_1 and continuing we get a sequence

$$M = M_0 \twoheadrightarrow M_1 \twoheadrightarrow M_2 \twoheadrightarrow \cdots,$$

where M_i is the cokernel of a map $\Delta_{\lambda_i} \to M_{i-1}$. We get a corresponding increasing chain of submodules $N_i := \ker(M \to M_i)$. Since M is Noetherian it must stabilize so $M_n = 0$ for some n. Thus, we have a filtration

$$M = N_n \supset N_{n-1} \supset \cdots \supset N_1 \supset 0.$$

By construction the subquotient $N_i/N_{i-1} \cong Im(N_i \to M_{i-1}) = Im(\Delta_{\lambda_i} \to M_{i-1})$ is a quotient of a Verma module. This proves part (1)

Using part (1), in order to prove part (2) it suffices to check that

$$K^0(\mathcal{O}) = \bigoplus_{\bar{\lambda} \in \mathfrak{t}^*//W} K^0(\mathcal{O}_{\bar{\lambda}}).$$

Let $\lambda \in \mathfrak{t}^*$ and set $\bar{\lambda} := \lambda \bmod W \in \mathfrak{t}^*//W \simeq \mathrm{Spec}(Z(\mathcal{U}(\mathfrak{g})))$. Notice that $\Delta_{w\cdot\lambda} \in \mathcal{O}_{\bar{\lambda}}$ for all $w \in W$. Hence,

$$K^0(\mathcal{O}_{\bar{\lambda}}) = \langle [\Delta_{w\cdot\lambda}] \rangle.$$

Part (2) now follows from part (1). $\square$

Remark 2.7.2. If λ is regular, i.e. $\mathrm{Stab}_{W\cdot\lambda}(\lambda) = \{e\}$, then this is a basis and so

$$K^0(\mathcal{O}_{\bar{\lambda}}) \simeq \Lambda^W, \qquad [\Delta_{w\cdot\lambda}] \leftarrow\!\shortmid \lambda.$$

Lemma 2.7.3. *1) Every $M \in \mathcal{O}$ has finite length. The simple module L_λ appears once in the Jordan-Hölder series $JH(\Delta_\lambda)$. If some L_μ appears twice then $\mu < \lambda$ and $\mu = w \cdot \lambda$ for some $w \in W$.*

2) $K^0(\mathcal{O})$ is freely generated by the classes of Verma modules.

Proof. 1) We know from the previous lemma that

(i) M has finite dimensional weight spaces.
(ii) Assuming without loss of generality that $M \in \mathcal{O}_{\bar{\lambda}}$ then M has a filtration by subquotients of $\Delta_{w\cdot\lambda}$ and the number of times $\Delta_{w\cdot\lambda}$ enters is $\leq \dim M[w \cdot \lambda]$.

It remains to show that Δ_λ has finite length. Consider the orbit $\mu \in W \cdot \lambda$. We have

$$\ker \hookrightarrow \Delta_\mu \twoheadrightarrow L_\mu.$$

Since $\ker \in \mathcal{O}_{\bar{\lambda}}$ it should have a filtration by subquotients of Δ_ν with $\nu \in W \cdot \mu$ and $\nu < \mu$. If $\mu \in W \cdot \lambda$ is minimal with respect to $\leq$ then no such ν exists so the filtration is empty meaning that $\ker = 0$ and $\Delta_\mu \simeq L_\mu$. Otherwise, by induction ker has a finite filtration by subquotients. Hence, Δ_μ has the finite filtration by subquotients

$$\Delta_\mu \supset \ker \supset (\text{finite filtration for} \ker).$$

2) Part 1 shows that

$$[\Delta_\lambda] = [L_\lambda] + \sum_{\mu<\lambda} m_{\lambda,\mu} L_\mu \text{ for some integers } m_{\lambda,\mu}.$$

Moreover $m_{\lambda,\mu} = 0$ unless λ, μ are in the same orbit of W. Since $[L_\lambda]$ freely generate the Grothendieck group and an upper triangular matrix with ones on the diagonal is invertible, we get statement (2). □

The Kazhdan-Lusztig problem reduces to relating the bases $\{[\Delta_{w\cdot\lambda}]\}$ and $\{[L_\lambda]\}$ in $K^0(\mathcal{O}_{\bar{\lambda}})$. This is equivalent to computing the Jordan-Hölder series for Δ_λ.

3. Highest weight categories

The category $\mathcal{O}_{\bar{\lambda}}$ is an example of a highest weight abelian category (alternative terms: quasi-hereditary, or cellular category). Let k be a field.

Definition 3.0.1 (Highest weight abelian category)**.** A k-linear abelian category $\mathcal{A}$ is a highest weight category if it satisfies the following axioms

(1) $\mathcal{A}$ is of finite type, i.e. every object has finite length.
(2) There exists only finitely many irreducible objects and $\mathrm{End}(L) = k$ for every irreducible object L.
(3) The set I of isomorphism classes of irreducible objects is equipped with a partial order.

(4) For every irreducible object L_i we have

$$\Delta_i \to L_i \to \nabla_i,$$

where L_i is the unique irreducible quotient of Δ_i and also the unique irreducible subobject in ∇_i. We call Δ_i a standard object and ∇_i a costandard object.

(5) $\ker(\Delta_i \to L_i)$ and $\operatorname{coker}(L_i \to \nabla_i)$ both lie in the subcategory

$$\begin{aligned}\mathcal{A}_{<i} :=& \langle L_j \mid j < i\rangle \\ =& \{M \in \mathcal{A} \mid \text{The Jordan-Hölder series of } M \text{ contain only these } L_j\}\end{aligned}$$

(6) $\operatorname{Ext}^n(\Delta_i, \nabla_j) = 0$ for all i, j with $n > 0$.

(7) $\operatorname{Hom}(\Delta_i, L_j) = 0$ for $i \nleq j$ and $\operatorname{Hom}(\Delta_i, L_i) = k$.

Remark 3.0.2. Replacing an order by a stronger order not changing the rest of the data turns a highest weight category into a highest weight category. So we can always replace the given partial order by a stronger complete order. Then $\nleq$ becomes $>$. We will sometimes write $>$ (respectively $<$) instead of $\nleq$ (respectively $\ngeq$) as we may in view of the above.

We will also use the notation $\mathcal{A}_{\leq i} := \langle L_j \mid j \leq i\rangle$. The following remark will be used repeatedly.

Remark 3.0.3. If $\operatorname{Ext}^n(N, L) = 0$ for all simple $L \in \mathcal{A}_{\leq k}$ then $\operatorname{Ext}^n(N, M) = 0$ for any $M \in \mathcal{A}_{\leq k}$. Same for $\mathcal{A}_{<k}$. This is proved by induction in the length on M using the long exact sequence for Ext's.

Corollary 3.0.4. We have $\operatorname{Ext}^n(\Delta_i, L_j) = 0$ and $\operatorname{Ext}^n(L_i, \nabla_j) = 0$ for all $j \leq i$ (or rather $i \nless j$) and $n > 0$, as well as for $j < i$ (or rather $i \nleq j$) and $n = 0$.

Proof. We use induction in j. When j is minimal then $\mathcal{A}_{<j} = 0$ and so $\ker(\Delta_j \to L_j) = \operatorname{coker}(L_j \to \nabla_j) = 0$. Hence,

$$\Delta_i \simeq L_i \simeq \nabla_i.$$

Thus, for j minimal the claim is a special case of Axioms 6 and 7. For a given j we use the short exact sequences

$$0 \to \ker_j \to \Delta_j \to L_j \to 0,$$

$$0 \to L_j \to \nabla_j \to \operatorname{coker} \to 0$$

and the corresponding long exact sequences of Ext's, then Axiom 5 together with Remark 3.0.3 yield the claim. □

Remark 3.0.5. We have $\operatorname{Hom}^\bullet(\Delta_i, \nabla_j) = 0$ for $j \neq i$. If $i \nless j$ then this follows from the first vanishing in Corollary 3.0.4 and previous Remark. Otherwise it follows from the second vanishing in Corollary 3.0.4.

Theorem 3.0.6. *(1) Δ_i is a projective cover of L_i in $\mathcal{A}_{\leq i}$.*

(2) ∇_i is an injective hull of L_i in $\mathcal{A}_{\leq i}$.

Thus Δ_i, ∇_i are uniquely defined once the partial order is given.

Proof. (1) Axiom 7 says that $\mathrm{Hom}(\Delta_i, L_j) = 0$ for $i \nleq j$ and $\mathrm{Hom}(\Delta_i, L_i) = k$. By Corollary 3.0.4 we have $\mathrm{Ext}^1(\Delta_i, L_j) = 0$ for $j \leq i$ (notice that $\mathrm{Ext}^1_{\mathcal{A}_{\leq i}}(M, N) = \mathrm{Ext}^1_{\mathcal{A}}(M, N)$ for $M, N \in \mathcal{A}_{\leq i}$ since $\mathcal{A}_{\leq i}$ is a Serre subcategory). This implies that $\mathrm{Ext}^1(\Delta_i, M) = 0$ for all $M \in \mathcal{A}_{\leq i}$ so Δ_i is a projective cover of L_i in $\mathcal{A}_{\leq i}$. The proof of (2) is similar, using the second half of corollary 3.0.4. □

Corollary 3.0.7. (1) $\mathrm{Ext}^\bullet_{\mathcal{A}}(\Delta_i, \Delta_j) = 0$ when $i > j$.

(2) $\mathrm{Ext}^n_{\mathcal{A}}(\Delta_i, \Delta_i) = \begin{cases} k & n = 0 \\ 0 & \text{otherwise} \end{cases}$.

Proof. Since $\Delta_j \in \mathcal{A}_{\leq j}$ part (1) and vanishing of higher Ext in (2) follows from Corollary 3.0.4 and Remark 3.0.3. Consider the long exact sequence

$$0 \to \mathrm{Hom}(\Delta_i, \ker_j) \to \mathrm{Hom}(\Delta_i, \Delta_j) \to \mathrm{Hom}(\Delta_i, L_j) \to \mathrm{Ext}^1(\Delta_i, \ker_j) = 0$$

Since $\ker_j \in \mathcal{A}_{<j}$, we see that $\mathrm{Hom}(\Delta_i, \ker_j) = 0$ so $\mathrm{Hom}(\Delta_i, \Delta_j) = \mathrm{Hom}(\Delta_i, L_j) = k$. □

Definition 3.0.8 (Exceptional collection)**.** (see [BK]) A partially order set of objects $\{X_i\}_{i \in I}$ in a triangulated category $\mathcal{C}$ is an exceptional collection if

(1) $\mathrm{Hom}(X_i, X_j[n]) = 0$ when $i \nleq j$.

(2) $\mathrm{Hom}(X_i, X_i[n]) = \begin{cases} k & n = 0 \\ 0 & \text{otherwise} \end{cases}$

For an exceptional collection Δ_i a dual collection is a set of objects ∇_i where $\mathrm{Ext}^\bullet(\Delta_i, \nabla_j) = k^{\delta_{ij}}$ ($k^{\delta_{ij}}$ sitting in degree 0 and 0's elsewhere). The dual collection ∇_i exists and it is an exceptional collection with the opposite partial order.

Example 3.0.9. Let $\mathcal{C} = D^b(\mathrm{Coh}(\mathbb{P}^n))$. Then we have an exceptional collection $\Delta_0, \ldots, \Delta_n$ with

$$\Delta_i = \mathcal{O}_{\mathbb{P}^n}(i).$$

It has a dual collection

$$\nabla_{-i} = \Omega^i(i)[i] = \Omega^i \otimes \mathcal{O}(i)[i] = \bigwedge^i T^*\mathbb{P}^n \otimes \mathcal{O}(i)[i],$$

where [ ] is homological shift. Note that in this case $\mathrm{Ext}^m(\Delta_i, \Delta_j) = 0$ for $m \neq 0$.

Example 3.0.10. Let X be a $\mathbb{C}$ algebraic variety which has a decomposition

$$X = \coprod_i X_i,$$

where each X_i is locally closed, $X_i \simeq \mathbb{C}^{n_i}$ and

$$\overline{X_i} = \coprod_{j \leq i} X_j.$$

Let $j_i : X_i \hookrightarrow X$ be the inclusion. Define

$$\Delta_i := j_{i!}(\underline{\mathbb{C}}), \qquad \nabla_i := Rj_{i*}(\underline{\mathbb{C}})$$

Set $\Sigma = \{X_i\}$. Let $\mathrm{Sh}(X)$ denote the category of sheaves on X. Define the full subcategory

$$\mathrm{Sh}_\Sigma(X) := \{M \in \mathrm{Sh}(X) \mid \mathrm{Ext}^{<0}(\Delta_i, M) = 0 = \mathrm{Ext}^{<0}(M, \nabla_i) \ \ \forall i\}.$$

Consider its derived category $D^b_\Sigma(X) \subset D^b(\mathrm{Sh}(X))$. Then the Δ_i is an exceptional collection in $D^b_\Sigma(X)$ and the ∇_i is its dual.

Proposition 3.0.11. *Let $\mathcal{A}$ be a highest weight category $\mathcal{A}$ with an exceptional collection Δ_i in $\mathcal{C} = D^b(\mathcal{A})$ and a dual collection ∇_i. Then $\mathcal{A}$ can be recovered from $D^b(\mathcal{A})$, Δ_i and ∇_i as*

$$\mathcal{A} = \{M \in D^b(\mathcal{A}) \mid \mathrm{Ext}^{<0}(\Delta_i, M) = 0 = \mathrm{Ext}^{<0}(M, \nabla_i)\ \forall i\},$$

where $\mathrm{Ext}^i(X,Y) := \mathrm{Hom}(X, Y[i])$.

Proof. If $B \notin \mathcal{A}$ then $H^i(B) \neq 0$ for some $i \neq 0$. If $H^i(B) \neq 0$ for some $i < 0$ choose i to be the minimal one for which $M := H^i(B) \neq 0$. Consider $M[-i]$ as a complex with M sitting in degree -i and 0's elsewhere. Then the identity map in degree $-i$ gives a non-zero map $M[-i] \to B$. Since M must contain some irreducible L_j we have maps $\Delta_j \twoheadrightarrow L_j \hookrightarrow M$. This gives a non-zero map

$$\Delta_j[-i] \to B.$$

Hence, we have found a non-zero element in $\mathrm{Hom}(\Delta_j, B[i]) = \mathrm{Ext}^i(\Delta_j, B)$ and so B is not contained in the right hand side. If $H^i(B) \neq 0$ for some $i > 0$ then we can make a similar argument with maps $B \to M[-i] \to \nabla$. □

Remark 3.0.12. If Δ_i is an exceptional collection then $\Delta_i[d_i]$ is also one for all $d_i \in \mathbb{Z}$ with dual collection $\nabla_i[d_i]$.

Theorem 3.0.13 (Beilinson, Bernstein, Deligne)**.** *Start with (X, Σ) as in the last example. Set $d_i := \dim(X_i)$. The exceptional collection in $Sh_\Sigma(X)$*

$$\Delta_i := j_{i!}(\underline{\mathbb{C}}[d_i])$$

comes from a highest weight subcategory

$$\mathcal{A} := \{\textit{perverse sheaves constructible with respect to } \Sigma\}.$$

We have $D^b_\Sigma(X) \simeq D^b(\mathcal{A})$.

Definition 3.0.14. A standard filtration on an object is a filtration with $gr_k \simeq \Delta_{i_k}$. A costandard filtration is a filtration with $gr_k \simeq \nabla_{i_k}$.

The following theorem will be useful in proving that category $\mathcal{O}_{\bar\lambda}$ is a highest weight category.

Theorem 3.0.15. *(cf.* [BGS, §3.2], [CPS]*) Replace the* Ext *vanishing condition in the axioms by*

$$\mathrm{Ext}^n(\Delta_i, \nabla_j) = 0 \qquad \forall i, j, n = 1, 2.$$

This still implies that $\mathcal{A}$ is a highest weight category.

For the proof we will mostly follow [BGS, §3.2]. In the process we will get some useful properties of highest weight categories. First we will prove the following.

Proposition 3.0.16. *Assume the* Ext *condition only for $n = 1, 2$.*

(1) An object $Q \in \mathcal{A}$ has a standard filtration iff

$$\mathrm{Ext}^1(Q, \nabla_i) = 0 \qquad \forall i$$

(2) L_i has a projective cover P_i, which has a standard filtration where Δ_i appear once and all other subquotients of this filtration are Δ_j with $j > i$.

$$\begin{tikzcd} P_i \arrow[r] \arrow[dr] & L_i \\ & \Delta_i \arrow[u] \end{tikzcd}$$

Proof of (1). "⇒": Given a standard filtration $0 \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n \subset Q$ we have short exact sequences

$$0 \to N_{i-1} \to N_i \to \Delta_{j_i} \to 0.$$

This gives the long exact sequence

$$\cdots \to \mathrm{Ext}^1(\Delta_{j_i}, \nabla_k) \to \mathrm{Ext}^1(N_i, \nabla_k) \to \mathrm{Ext}^1(N_{i-1}, \nabla_k) \to \mathrm{Ext}^2(\Delta_{j_i}, \nabla_k) \to \cdots$$

Since $\mathrm{Ext}^1(\Delta_{j_i}, \nabla_k) = 0 = \mathrm{Ext}^2(\Delta_{j_i}, \nabla_k)$ we get $\mathrm{Ext}^1(N_{i-1}, \nabla_k) \simeq \mathrm{Ext}^1(N_i, \nabla_k)$. By axiom $\mathrm{Ext}^1(\Delta_{j_1}, \nabla_k) = 0$ so using induction we get

$$\mathrm{Ext}^1(Q, \nabla_k) = 0.$$

"⇐": Pick a minimal i for which $\mathrm{Hom}(Q, L_i) \neq 0$, i.e. $\mathrm{Hom}(Q, L_j) = 0$ for all $j < i$. For $j < i$ consider the short exact sequence

$$0 \to L_j \to \nabla_j \to \mathrm{coker}_j \to 0$$

This gives the long exact sequence

$$\cdots \to \mathrm{Hom}(Q, \mathrm{coker}_j) \to \mathrm{Ext}^1(Q, L_j) \to \mathrm{Ext}^1(Q, \nabla_j) \to \cdots$$

Since $\mathrm{coker}_j \in \mathcal{A}_{<j}$ the choice of i gives $\mathrm{Hom}(Q, \mathrm{coker}_j) = 0$. By assumption $\mathrm{Ext}^1(Q, \nabla_j) = 0$ so $\mathrm{Ext}^1(Q, L_j) = 0$ for all $j < i$. Consider the short exact sequence

$$0 \to \ker_i \to \Delta_i \to L_i \to 0.$$

Since $\ker_i \in \mathcal{A}_{<i}$ we have $\mathrm{Ext}^1(Q, \ker_i) = 0$. Consider the long exact sequence corresponding to the first short exact sequence

$$\mathrm{Hom}(Q, \ker_i) \to \mathrm{Hom}(Q, \Delta_i) \to \mathrm{Hom}(Q, L_i) \to \mathrm{Ext}^1(Q, \ker_i)$$

All irreducible subquotients of $\ker_i$ are of the form $L_{j'}$ with $j' < i$ so a non-zero element in $\mathrm{Hom}(Q, \ker_i)$ would produce a non-zero element in $\mathrm{Hom}(Q, L_{j'})$ which would be a contradiction. Thus, $\mathrm{Hom}(Q, \Delta_i) \simeq \mathrm{Hom}(Q, L_i)$ and we can lift the map

$$\begin{tikzcd} Q \arrow[r] \arrow[dr] & L_i \\ & \Delta_i \arrow[u] \end{tikzcd}$$

We claim that the map from Q to Δ_i is onto. Indeed, assume that $\mathrm{im}(Q) \subsetneq \Delta_i$ then by axiom 4 the map $\Delta_i \to L_i$ would factor as

$$\Delta_i \twoheadrightarrow \Delta_i/\mathrm{im}(Q) \twoheadrightarrow L_i.$$

But then $Q \to L_i$ would be the zero map and by assumption it is not. Hence, we obtain a short exact sequence

$$0 \to Q' \to Q \to \Delta_i \to 0.$$

From this we get

$$\cdots \to \mathrm{Ext}^1(Q, \nabla_j) \to \mathrm{Ext}^1(Q', \nabla_j) \to \mathrm{Ext}^2(\Delta_i, \nabla_j)$$

By assumption $\mathrm{Ext}^1(Q, \nabla_j) = 0 = \mathrm{Ext}^2(\Delta_i, \nabla_j)$ so $\mathrm{Ext}^1(Q', \nabla_j) = 0$. Thus, Q' satisfies the assumption of the proposition so by induction it has a standard filtration

$$0 \subset Q_1 \subset \cdots \subset Q_m \subset Q'$$

Then Q has the standard filtration $0 \subset Q_1 \subset \cdots \subset Q_m \subset Q' \subset Q$. □

Before proving part (2) we prove the following lemma (still only assuming Ext vanishing for d=1,2).

Lemma 3.0.17. *(a)* $\mathrm{Ext}^1(A, B)$ *is finite dimensional for all* $A, B \in \mathcal{A}$.
(b) $\mathrm{Ext}^d(\Delta_i, L_j) = 0$ *if* $i \nleq j$, $d = 1, 2$.

Proof. (a) It is enough to check when A and B are irreducible. Assume that we have a short exact sequence

$$0 \to B' \to B \to C \to 0$$

this gives a long exact sequence

$$\cdots \to \mathrm{Hom}(A, C) \to \mathrm{Ext}^1(A, B') \to \mathrm{Ext}^1(A, B) \to \cdots$$

Now, Hom is always finite dimensional so if $\mathrm{Ext}^1(A, B)$ is finite dimensional this implies that $\mathrm{Ext}^1(A, B')$ is also finite dimensional. A similar argument shows that if $\mathrm{Ext}^1(A, B)$ is finite dimensional then so is $\mathrm{Ext}^1(A', B)$ if $A \twoheadrightarrow A'$. By axiom $\mathrm{Ext}^1(\Delta_i, \nabla_j) = 0$. In particular it is finite dimensional. Since $L_j \hookrightarrow \nabla_j$ this implies that $\mathrm{Ext}^1(\Delta_i, L_j)$ is finite dimensional. We also have $\Delta_i \twoheadrightarrow L_i$ so we get that $\mathrm{Ext}^1(L_i, L_j)$ is finite dimensional.

(b) Assume that it is true for $j' < j$. Consider the short exact sequence

$$0 \to L_j \to \nabla_j \to \mathrm{coker}_j \to 0$$

and the corresponding long exact sequence for $d = 1, 2$

$$\mathrm{Ext}^{d-1}(\Delta_i, \mathrm{coker}_j) \to \mathrm{Ext}^d(\Delta_i, L_j) \to \mathrm{Ext}^d(\Delta_i, \nabla_j) = 0.$$

Since $\mathrm{coker}_j \in \mathcal{A}_{<j}$ by induction we get $\mathrm{Ext}^1(\Delta_i, \mathrm{coker}_j) = 0$. If $i \nleq j$ then $\mathrm{Hom}(\Delta_i, L_{j'}) = 0$ for all $j' < j$ so $\mathrm{Hom}(\Delta_i, \mathrm{coker}_j) = 0$. Thus, $\mathrm{Ext}^{1,2}(\Delta_i, L_j) = 0$. □

Proof of part (2) in proposition 3.0.16. Recall that $\mathcal{A}_k = \langle L_{i_1}, \dots, L_{i_k} \rangle$ where the list is ordered such that $i_\ell < i_j$ implies that $\ell < j$. Since there are only finitely many irreducibles

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_n = \mathcal{A}$$

Notice that to prove that $Q \in \mathcal{A}_{<k}$ is projective it is enough to prove that $\mathrm{Ext}^1(Q, M) = 0$ for all irreducibles $M \in \mathcal{A}_{\leq k}$. By the proof of Corollary 3.0.4 we have $\mathrm{Ext}^1(\Delta_i, L_j) = 0$ for all $j \leq i$ and $n > 0$ so Δ_{i_k} is projective in $\mathcal{A}_{\leq k-1}$ (the Corollary relies on the strong form of the axioms and makes conclusion about vanishing of Ext^n for all n; we now only assume such vanishing for $n = 1, 2$, then the same argument shows vanishing of Ext^1). In particular, every L_{i_d} has the cover Δ_{i_d} which is projective in $\mathcal{A}_{\leq d-1}$. We construct a projective cover with a standard filtration of each irreducible in $\mathcal{A}_k$ by induction in k. I.e. starting from a cover which is projective in $\mathcal{A}_{\leq k-1}$ we want to construct a cover of which is also projective in $\mathcal{A}_{\leq k}$. Since L_{i_1} is minimal $\Delta_{i_1} \simeq L_{i_1} \simeq \nabla_{i_1}$. In particular, $\mathrm{Ext}^1(L_{i_1}, L_{i_1}) \simeq \mathrm{Ext}^1(\Delta_{i_1}, \nabla_{i_1}) = 0$ so L_{i_1} is projective in $\mathcal{A}_1$. This gives the base of the induction.

Assume that the statement is known for $k-1$. Let $L := L_{i_d}$ with $d \leq k$ and assume that P is a cover of L which is projective in $\mathcal{A}_{\leq k-1}$. Write $V := \mathrm{Ext}^1(P, \Delta_{i_k})^*$ and let $\tilde{P}$ be the universal extension

$$0 \to V \otimes \Delta_{i_k} \to \tilde{P} \to P \to 0. \tag{2}$$

To prove that $\tilde{P}$ is the projective cover of L in $\mathcal{A}_{\leq k}$ we need to check that

(i) $\mathrm{Hom}(\tilde{P}, L) = k$.
(ii) $\mathrm{Hom}(\tilde{P}, L') = 0$, when $L' \neq L$.
(iii) $\mathrm{Ext}^1(\tilde{P}, M) = 0$ for all irreducible $M \in \mathcal{A}_{\leq k}$.

Consider the long exact sequence

$$0 \to \mathrm{Hom}(P, L') \to \mathrm{Hom}(\tilde{P}, L') \to \mathrm{Hom}(V \otimes \Delta_{i_k}, L')$$

Since $\mathrm{Hom}(\Delta_{i_k}, L') = 0$ for $L' \in \mathcal{A}_{\leq k-1}$ we get $\mathrm{Hom}(V \otimes \Delta_{i_k}, L') = 0$. Hence, $\mathrm{Hom}(\tilde{P}, L') \simeq \mathrm{Hom}(P, L')$. This proves the conditions on Hom except if $L' = L_{i_k}$. In that case $\mathrm{Hom}(P, L_{i_k}) = 0$. $\tilde{P}$ is defined as the universal extension so

$$\mathrm{Hom}(V \otimes \Delta_{i_k}, L_{i_k}) \xrightarrow{\sim} \mathrm{Ext}^1(P, L_{i_k}).$$

Thus, the long exact sequence for $L' = L_{i_k}$

$$0 = \mathrm{Hom}(P, L_{i_k}) \to \mathrm{Hom}(\tilde{P}, L_{i_k}) \to \mathrm{Hom}(V \otimes \Delta_{i_k}, L_{i_k}) \xrightarrow{\sim} \mathrm{Ext}^1(P, L_{i_k})$$

shows that $\mathrm{Hom}(\tilde{P}, L_{i_k}) = 0$.

The only thing left to check is the Ext^1 vanishing. We start with the case $M \in \mathcal{A}_{\leq k-1}$. By the definition of P and part (b) of the lemma

$$0 = \mathrm{Ext}^1(P, M) \to \mathrm{Ext}^1(\tilde{P}, M) \to \mathrm{Ext}^1(V \otimes \Delta_{i_k}, M) = 0$$

so $\mathrm{Ext}^1(\tilde{P}, M) = 0$. The last case to check is $M = L_{i_k}$. Notice that

$$\mathrm{Hom}(V \otimes \Delta_{i_k}, L_{i_k}) \simeq V^* \otimes \mathrm{Hom}(\Delta_{i_k}, L_{i_k}) \simeq V^* = \mathrm{Ext}^1(P, \Delta_{i_k}).$$

Plugging this into the long exact sequence coming from (2)

$$\mathrm{Hom}(\Delta_{i_k}\otimes V,\Delta_{i_k}) \xrightarrow{\sim} \mathrm{Ext}^1(P,\Delta_{i_k}) \to \mathrm{Ext}^1(\tilde{P},\Delta_{i_k}) \to \mathrm{Ext}^1(\Delta_{i_k}\otimes V,\Delta_{i_k}) = 0$$

we get $\mathrm{Ext}^1(\tilde{P},\Delta_{i_k}) = 0$. Consider

$$0 \to \ker_{i_k} \to \Delta_{i_k} \to L_{i_k} \to 0$$

$\ker_{i_k} \in \mathcal{A}_{\le k-1}$ so $\mathrm{Ext}^1(\tilde{P},\ker_{i_k}) = 0$. Hence, $\mathrm{Ext}^1(\tilde{P},L_{i_k}) = 0$.

Notice that $V\otimes\Delta_{i_k} \simeq \Delta_{i_k}^{\dim V}$. Let

$$0 \subset P_1 \subset \cdots \subset P_n \subset P$$

be a standard filtration of P. Then $\tilde{P}$ has a standard filtration given by

$$0 \subset \Delta_{i_k} \subset \cdots \subset \Delta_{i_k}^{\dim V-1} \subset \Delta_{i_k}^{\dim V} \subset \Delta_{i_k}^{\dim V}\oplus P_1 \subset \cdots \subset \Delta_{i_k}^{\dim V}\oplus P \subset \tilde{P}. \qquad \square$$

Proof of theorem 3.0.15. We proved that the map $P_i \to L_i$ factors through Δ_i so there is a short exact sequence

$$0 \to \ker \to P_i \to \Delta_i \to 0$$

The projective P_i has a standard filtration where Δ_i occurs once and all the rest are Δ_j with $j < i$. Hence, we can apply descending induction in i to get $\mathrm{Ext}^d(\ker,\nabla_j) = 0$ for $d > 0$. Since P_i is projective $\mathrm{Ext}^d(P_i,\nabla_j) = 0$ for $d > 0$ so

$$0 \to \mathrm{Ext}^d(\Delta_i,\nabla_j) \xrightarrow{\sim} \mathrm{Ext}^{d+1}(\Delta_i,\nabla_j) \to 0 \quad \text{for } d > 0.$$

This finishes the proof since $\mathrm{Ext}^1(\Delta_i,\nabla_j) = 0$. $\square$

Corollary 3.0.18. Let $\mathcal{A}$ be a highest weight category. Then there exists a finite dimensional algebra A such that

$$\mathcal{A} \simeq A-\mathrm{mod}_{\text{f.d.}}.$$

Proof. By proposion 3.0.16 $\mathcal{A}$ has a set of projective generators $\{P_i\}$. Set

$$A := \mathrm{End}(\oplus_i P_i)^{op}$$

Then the functor $F : \mathcal{A} \to A-\mathrm{mod}_{\text{f.d.}}$ given by $M \mapsto \mathrm{Hom}(\oplus_i P_i, M)$ is an equivalence of categories. $\square$

4. $\mathcal{O}_{\bar{\lambda}}$ as a highest weight category.

Theorem 4.0.1. *Let $\lambda \in \mathfrak{t}^*$ and set $\bar{\lambda} := \lambda$ mod W. The category $\mathcal{O}_{\bar{\lambda}}$ is a highest weight category with the standard partial order on the weights*

$$\nu < \mu \quad \Leftrightarrow \quad \mu - \nu = \sum \textit{positive roots}$$

By theorem 3.0.15 the only thing to check is

$$\mathrm{Ext}^d_{\mathcal{O}_{\bar{\lambda}}}(\Delta_\mu,\nabla_\nu) = 0 \quad d = 1,2.$$

Lemma 4.0.2. *Let $\mathcal{A}$ be a full subcategory of $\mathcal{B}$ closed under extensions and subquotients. Then for all $M,N \in \mathcal{A}$*

(1) $\mathrm{Ext}^1_{\mathcal{A}}(M,N) \simeq \mathrm{Ext}^1_{\mathcal{B}}(M,N)$.

(2) $\mathrm{Ext}^2_{\mathcal{A}}(M,N)$ *maps injectively to* $\mathrm{Ext}^2_{\mathcal{B}}(M,N)$.

Proof. Part (1) is clear. For part (2) notice that given an element in $\mathrm{Ext}^2_{\mathcal{A}}(M,N)$ there exists a $\tilde{M} \in \mathcal{A}$ with $\tilde{M} \twoheadrightarrow M$ and $h \mapsto 0$ in $\mathrm{Ext}^2_{\mathcal{A}}(\tilde{M},N)$ (for example, if $\mathcal{A}$ has enough projectives a possible choice of $\tilde{M}$ is a projective cover of M).

Using the short exact sequence

$$0 \to \ker \to \tilde{M} \to M \to 0$$

We get long exact sequences

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{A}}(\tilde{M},N) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^1_{\mathcal{A}}(\ker,N) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^2_{\mathcal{A}}(M,N) \arrow[r] & \mathrm{Ext}^2_{\mathcal{A}}(\tilde{M},N) \\
\mathrm{Ext}^1_{\mathcal{B}}(\tilde{M},N) \arrow[r] & \mathrm{Ext}^1_{\mathcal{B}}(\ker,N) \arrow[r] & \mathrm{Ext}^2_{\mathcal{B}}(M,N) \arrow[r] & \mathrm{Ext}^2_{\mathcal{B}}(\tilde{M},N)
\end{tikzcd}$$

Since $h \mapsto 0$ it comes from an element in $\mathrm{Ext}^2_{\mathcal{A}}(M,N)$ so we get an injective map $\mathrm{Ext}^2_{\mathcal{A}}(M,N) \hookrightarrow \mathrm{Ext}^2_{\mathcal{B}}(M,N)$. □

Proof of theorem 4.0.1. By the lemma it is enough to check

$$\mathrm{Ext}^{1,2}_{\mathfrak{g}-\mathrm{mod}'}(\Delta_\lambda, \nabla_\mu) = 0,$$

where $\mathfrak{g}-\mathrm{mod}'$ is the category of $\mathfrak{g}$ modules with diagonalizable $\mathfrak{t}$ action. Define the universal Verma module

$$\tilde{\Delta} := \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{n}}(k) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(n)} k.$$

For $M \in \mathfrak{g} - \mathrm{mod}'$

$$\begin{aligned}
\mathrm{Hom}(\tilde{\Delta}, M) &\simeq \{v \in M \mid n(v) = 0 \ \ \forall n \in \mathcal{U}(\mathfrak{n})\} \\
&= \bigoplus_{\nu \in \mathfrak{t}^*} \{v \in M \mid n(v) = 0, x(v) = \nu(x)v \ \ \forall n \in \mathcal{U}(\mathfrak{n}), x \in \mathfrak{t}\} \\
&\simeq \bigoplus_{\nu \in \mathfrak{t}^*} \mathrm{Hom}(\Delta_\nu, M).
\end{aligned}$$

It follows that the same is true for higher derived functors, i.e.

$$\mathrm{Ext}(\tilde{\Delta}, M) = \bigoplus_{\nu \in \mathfrak{t}^*} \mathrm{Ext}(\Delta_\nu, M)$$

for $M \in \mathfrak{g} - \mathrm{mod}'$. The functor $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{n}}$ is left adjoint to $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}}$ and it sends projectives to projectives so

$$\mathrm{Ext}^i_{\mathfrak{g}-\mathrm{mod}}(\tilde{\Delta}, M) = \mathrm{Ext}^i_{\mathfrak{n}-\mathrm{mod}}(k, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}}(M)) = H^i(\mathfrak{n}, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}}(M)).$$

In the case $M = \nabla_\nu$ and $i > 0$ we have

$$\begin{aligned}
\mathrm{Ext}^i_{\mathfrak{g}-\mathrm{mod}}(\tilde{\Delta}, \nabla_\nu) &= H^i(\mathfrak{n}, \nabla_\nu) \\
&= H^i(\mathfrak{n}, \mathcal{O}(N)) \\
&= H^i_{DR}(\mathbb{A}^{\dim \mathfrak{n}}) = 0
\end{aligned}$$

So $\mathrm{Ext}^i_{\mathfrak{g}-\mathrm{mod}}(\Delta_\lambda, \nabla_\nu) = 0$ for $i = 1, 2$. This finishes the proof that $\mathcal{O}_{\bar{\lambda}}$ is a highest weight category. □

In proposition 3.0.16 we proved that P_i has a standard filtration. Let $[\Delta_j : P_i]$ be the multiplicity Δ_j in the filtration. The multiplicity of L_i in the Jordan-Hölder series of ∇_j is denoted by $[L_i : \nabla_j]$.

Proposition 4.0.3 (BGG reciprocity)**.** *For a highest weight category $\mathcal{A}$*

$$[L_i : \nabla_j] = [\Delta_j : P_i]$$

Sketch of proof. The idea is to prove that $[L_i : \nabla_j]$ and $[\Delta_j : P_i]$ are both equal to $\dim \mathrm{Hom}(P_i, \nabla_j)$. By axiom $\mathrm{Ext}^1(\Delta_i, \nabla_j) = 0$ so

$$\dim \mathrm{Hom}(P_i, \nabla_j) = \sum \dim \mathrm{Hom}(\Delta_{i_k}, \nabla_j),$$

where the sum is over all Vermas that occur in the standard filtration of P_i. □

Recall the duality on category $\mathcal{O}$

$$M^+ := \Big(\bigoplus_\nu M[\nu]^*\Big)',$$

where the $'$ indicates a twist of the $\mathfrak{g}$ action by the automorphism of $\mathfrak{g}$ which sends $\mathfrak{n}_\alpha$ to $\mathfrak{n}_{-\alpha}$ and fix $\mathfrak{t}$. For this duality

$$\Delta_\lambda^+ = \nabla_\lambda, \qquad L_\lambda^+ = L_\lambda.$$

so in category $\mathcal{O}_{\bar{\lambda}}$ we have $[L_i : \Delta_j] = [L_i : \nabla_j]$

Remark 4.0.4. Recall the Cartan matrix

$$c_{ij} := [L_i, P_j].$$

In category $\mathcal{O}$ we have the formula

$$[L_i : P_j] = \sum_k [\Delta_k : P_j][L_i : \Delta_k]$$

Set $m_{ij} := [L_i : \Delta_j] = [\Delta_j : P_i]$. Then

$$C = M^T M \quad \text{for } C := (c_{ij}),\ \ M := (m_{ij}).$$

Our goal is to describe the $m_{ij} = \dim \mathrm{Hom}(P_i, \Delta_j)$. The strategy is to equip this vector space with a $\mathbb{Z}$ grading to get a polynomial

$$Q_{ij}(q, q^{-1}) = \sum_s q^s \dim \mathrm{Hom}_s(P_i, \Delta_j),$$
$$Q_{ij}(1) = m_{ij}.$$

Up to a normalization this is the Kazhdan-Lusztig polynomials. By corollary 3.0.18 there exist a finite dimensional algebra A such that $\mathcal{O}_{\bar{\lambda}} = A\text{-mod}_{f.d.}$. We will construct such an algebra and define a grading on it.

4.1. **Irreducible Verma and its projective cover.** Restrict to the case $\lambda \in \Lambda$. Let $\lambda_{\min}$ denote the minimal element in $W \cdot \lambda$.

$$\Delta_1 := \Delta_{\min} \simeq L_{\min} \simeq \nabla_{\min}.$$

Proposition 4.1.1. *For all i, $\Delta_i \in \mathcal{O}_{\bar{\lambda}}$ contains Δ_1. This is the only irreducible submodule.*

For a commutative ring R and an R-module M the set-theoretic support of M is defined as

$$\operatorname{supp}_R(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

The support is a closed subset of $\operatorname{Spec}(R)$. One can also define a more refined notion of support, the scheme-theoretic support, as the closed subscheme corresponding to the ideal $Ann(M)$ (annihilator of M).

Definition 4.1.2 (Gelfand-Kirillov dimension)**.** Let M be a finitely generated $\mathfrak{g}$-module. Using the PBW filtration on $\mathcal{U}(\mathfrak{g})$ pick a compatible filtration on M such that $\operatorname{gr}(M) \in \operatorname{Coh}(\mathfrak{g}^*)$ is finitely generated as a $\operatorname{gr}(\mathcal{U}(\mathfrak{g})) = \operatorname{Sym}(\mathfrak{g})$-module. It is a fact that the set-theoretic (as opposed to scheme-theoretic) support is independent of the choice of filtration. The Gelfand-Kirillov dimension of M is defined to be

$$GK \dim(M) := \dim \operatorname{supp}_{\operatorname{Sym}(\mathfrak{n})}(\operatorname{gr}(M)).$$

Proposition 4.1.3. *(a) For every non-zero submodule $M \subset \Delta_\lambda$.*

$$GK \dim(M) = GK \dim(\Delta_\lambda) = \dim \mathfrak{n}.$$

(b) For all $\mu \in W \cdot \lambda$ with $\mu \neq \lambda_{\min}$

$$GK \dim(L_\mu) < \dim \mathfrak{n}$$

Proof. (a) Choosing the obvious filtration on Δ_λ we have $\operatorname{gr}(\Delta_\lambda) \simeq \operatorname{Sym}(\mathfrak{g}/\mathfrak{b}) \simeq \mathcal{O}_{\mathfrak{b}^\perp}$.

$$\operatorname{supp}(\operatorname{gr}(\Delta_\lambda)) \simeq \operatorname{supp}(\mathcal{O}_{\mathfrak{b}^\perp}) = \mathfrak{b}^\perp \simeq \mathfrak{n}.$$

Let $M \subset \Delta_\lambda$ have the induced filtration

$$0 \neq \operatorname{gr}(M) \subset \mathcal{O}_{\mathfrak{b}^\perp}$$

$\operatorname{Sym}(\mathfrak{n})$ is a free $\mathfrak{n}$-module so the submodule $\operatorname{gr}(M)$ is torsion free. Therefore $\operatorname{gr}(M)$ has full support

$$\operatorname{supp}(\operatorname{gr}(M)) = \operatorname{supp}(\operatorname{gr}(\Delta_\lambda)) = \mathfrak{b}^\perp.$$

(b) Let L_μ be irreducible with $\mu \in W \cdot \lambda$ and $\mu \neq \lambda_{\min}$. Then L_μ is not isomorphic to Δ_μ

$$0 \neq \ker_\mu \hookrightarrow \Delta_\mu \twoheadrightarrow L_\mu.$$

$$0 \neq \operatorname{gr}(\ker_\mu) \hookrightarrow \operatorname{gr}(\Delta_\mu) \twoheadrightarrow \operatorname{gr}(L_\mu).$$

Let $f \in \operatorname{gr}(\ker_\mu)$. Then $\operatorname{gr}(L_\mu)_f = 0$ so $\operatorname{supp}(\operatorname{gr}(L_\mu)) \subseteq V(f)$ where $V(f)$ is the set of prime ideals containing f. For $f \neq 0$ this is a proper closed subset so it has codimension ≥ 1.

$$GK \dim(L_\mu) \leq \dim V(f) < \dim \mathfrak{n}. \qquad \square$$

Proof of proposition 4.1.1. Δ_μ has a Jordan-Hölder series so in particular it contains some irreducible submodule L. Then $GK\dim(L) = \dim\mathfrak{n}$. The only possible irreducible submodules are L_μ with $\mu \in W\cdot\lambda$ but all of them except $L_{\lambda_{\min}} \simeq \Delta_1$ has too small GK dimension. Hence, Δ_1 is the unique irreducible submodule. □

Corollary 4.1.4. We have $[\Delta_\mu : P_1] = 1$ for all μ. When λ is regular we also have $[L_1 : P_1] = |W|$ and $\dim\operatorname{End}(P_1) = |W|$.

Proof. By BGG reciprocity

$$[\Delta_\mu : P_1] = [L_1 : \nabla_\mu] = [L_1 : \Delta_\mu] \qquad \forall\mu$$

We just proved that L_1 appears as the first term in $\mathrm{JH}(\Delta_\mu)$. We need to check that it only occurs once. Consider the short exact sequence from the start of $\mathrm{JH}(\Delta_\mu)$

$$0 \to L_1 \hookrightarrow \Delta_\mu \to \mathrm{coker} \to 0$$

The remaining simple quotients in $\mathrm{JH}(\Delta_\mu)$ are the simple quotients in JH(coker). The same argument as in the proof of proposition 4.1.3 shows that

$$GK\dim\mathrm{coker} < \dim\mathfrak{n}.$$

Thus, since $GK\dim\mathrm{coker} < GK\dim L_1$ the module L_1 cannot appear in JH(coker).

We have the formula

$$[L_1 : P_1] = \sum_k [\Delta_k : P_1][L_1 : \Delta_k] = \sum_k 1,$$

where the sum is over all Verma module which appear in the standard filtration of P_1. By proposition 3.0.16 each Verma appears once in this filtration. If λ is regular then $\dim\operatorname{End}(P_1) = [L_1 : P_1] = |W|$. □

Exercise 4.1.5. Let M be an object in category $\mathcal{O}$ with integral central character.

$$d_\lambda := \dim \bigoplus_{-\nu<\lambda} M[\nu].$$

Show that if λ is deep in Λ^+, i.e. $\langle\lambda, \alpha_i\rangle >> 0$, and λ in a fixed coset of the root lattice then d_λ is a polynomial in λ of degree $GK\dim$.

The category $\mathcal{O}_{\bar{\lambda}}$ with $\lambda \in \Lambda$ contains the minimal irreducible $L_1 := L_{\lambda_{\min}}$ (By proposition 4.1.3 this irreducible has maximal GK dimension). Let Ξ be its projective cover.

$$L_1 \simeq \Delta_{\lambda_{\min}} \simeq \nabla_{\lambda_{\min}} \twoheadleftarrow P_1 =: \Xi.$$

Set $W_\lambda := \mathrm{Stab}_{W,\cdot}(\lambda)$. The projective cover Ξ has a filtration with

$$\mathrm{gr}(\Xi) = \bigoplus_{w\in W/W_\lambda} \Delta_{w\cdot\lambda}$$

Hence,

$$\begin{aligned}\dim \operatorname{End}(\Xi) &= \dim \operatorname{Hom}(\Xi,\Xi) = \sum_{w\in W/W_\lambda} \dim \operatorname{Hom}(\Xi, \Delta_{w\cdot\lambda}) \\ &= \sum_{w\in W/W_\lambda} [L_1 : \Delta_{w\cdot\lambda}] = |W/W_\lambda| \\ &= \operatorname{rank} K^0(\mathcal{O}_{\bar\lambda}).\end{aligned}$$

Theorem 4.1.6. *Let λ be regular. Then*

$$\operatorname{End}(\Xi) \simeq \operatorname{Sym}(\mathfrak{t})/(\operatorname{Sym}(\mathfrak{t})_+^W).$$

Here the + indicates polynomials without constant term.

A proof of this theorem will be given later. A map

$$\operatorname{Sym}(\mathfrak{t})/(\operatorname{Sym}(\mathfrak{t})_+^W) \to \operatorname{End}(\Xi)$$

can be constructed in the following way. Let $m_{\bar\lambda}$ be the maximal ideal of functions vanishing at $\bar\lambda$. There is an isomorphism of completions

$$Z_{\hat{\bar\lambda}} := \varprojlim \mathcal{O}(\mathfrak{t}^*)^W/m_{\bar\lambda}^n \simeq \varprojlim \mathcal{O}(\mathfrak{t}^*)/m_{\bar\lambda}^n =: \mathcal{O}(\mathfrak{t}^*)_{\hat\lambda}.$$

Here $Z \simeq \mathcal{O}(\mathfrak{t}^*)^(W,\cdot)$ acts on all of $\mathcal{O}$ and $Z/m_{\bar\lambda}^n$ acts on $\mathcal{O}_{\bar\lambda}$ for any n. After a change of coordinates

$$Z_{\hat{\bar\lambda}} \simeq \mathcal{O}(\mathfrak{t}^*)_{\hat\lambda} \simeq \mathcal{O}(\mathfrak{t}^*)_{\hat 0} \simeq \operatorname{Sym}(\mathfrak{t})_{\hat 0}.$$

Z surjects onto each of the terms $Z/m_{\bar\lambda}^n$ in the inverse limit and each term in $\operatorname{Sym}(\mathfrak{t})_{\hat 0}$ surjects onto $\operatorname{Sym}(\mathfrak{t})/(\operatorname{Sym}(\mathfrak{t})_+^W)$. Hence, we obtain a map $\phi : Z \to \operatorname{Sym}(\mathfrak{t})/(\operatorname{Sym}(\mathfrak{t})_+^W)$. We claim (and will prove later) that the action of Z on Ξ factors through ϕ.

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "\phi"'] & & \operatorname{End}(\Xi) \\ & \operatorname{Sym}(\mathfrak{t})/(\operatorname{Sym}(\mathfrak{t})_+^W) \arrow[ur] & \end{tikzcd}$$

5. Translation functors

Translation functors provide a way of moving between blocks with different central character.

Lemma 5.0.1. *Let $V = V_\mu$ be a finite dimensional representation with highest weight μ. Then $\Delta_\lambda \otimes V$ has a standard filtration with*

$$\operatorname{gr}(\Delta_\lambda \otimes V) = \bigoplus_\nu \Delta_{\lambda+\nu} \otimes V[\nu].$$

Proof. By the tensor identity we have

$$\Delta_\lambda \otimes V = \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda) \otimes V \simeq \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda \otimes V|_{\mathfrak{b}}).$$

Here $V|_{\mathfrak{b}}$ is V considered as a $\mathfrak{b}$-module. When an element of b_0 acts on an element in a weight space one only gets terms sitting in higher weight spaces. Thus, the direct sum of the $\mathbb{C}$-span of a vector sitting in one weight space with all higher weight spaces is a $\mathfrak{b}$-submodule of M. In particular, $V|_{\mathfrak{b}}$ has a filtration with $\mathrm{gr}(V|_{\mathfrak{b}}) = \oplus_\nu V[\nu]$. The functor $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda \otimes \cdot)$ is exact so applying it to each term in the filtration on $V|_{\mathfrak{b}}$ produces a filtration on $\Delta_k \otimes V$ with

$$\mathrm{gr}_i(\Delta_\lambda \otimes V) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_{\lambda+\nu} \otimes V[\nu]) = \bigoplus_\nu \Delta_{\lambda+\nu} \otimes V[\nu]$$

This finishes the proof. □

Corollary 5.0.2. Let M be a $\mathfrak{g}$-module on which the center acts by generalized central character $\bar{\lambda}$. Then the center acts on $V \otimes M$ by the generalized central characters $\overline{\lambda + \nu}$ for which $V[\nu] \neq 0$.

Remark 5.0.3. For M in category $\mathcal{O}$ one can see this directly as follows. When $M = \Delta_\lambda$ this follows from the previous lemma. Hence, it is true for any M with a standard filtration. In particular, it is true for projectives in $\mathcal{O}$. Since category $\mathcal{O}$ has enough projectives by proposition 3.0.16 it is true for all M in $\mathcal{O}$.

The proof of the corollary uses the following lemma.

Lemma 5.0.4. *The is an inclusion* $\mathcal{U}_{\bar{\lambda}} \hookrightarrow \mathrm{End}_k(\nabla_\lambda)$.

Proof of lemma. To prove that the map is injective it is enough to check it for the associated graded

$$\mathrm{gr}(\mathcal{U}_{\bar{\lambda}}) \simeq \mathcal{O}(\mathfrak{g}^*)/\mathcal{O}(\mathfrak{g}^*)^G_+.$$

It is known that $\mathcal{O}(\mathfrak{g}^*)/\mathcal{O}(\mathfrak{g}^*)^G_+ \simeq \mathcal{O}(\mathcal{N})$ where $\mathcal{N}$ is the nilpotent cone. Recall that $\nabla_\lambda \simeq \mathcal{O}(\mathcal{B}_0) \simeq \mathcal{O}(N)$. Consider the differential operators on N

$$\mathrm{Diff}(N) \simeq \mathcal{O}(N) \otimes \mathcal{U}(\mathfrak{n}) \subset \mathrm{End}_k(\mathcal{O}(N)) \simeq \mathrm{End}_k(\nabla_\lambda),$$

where elements in $\mathcal{O}(N)$ act by multiplication and elements in $\mathcal{U}(\mathfrak{n})$ act by derivation. It is also known that

$$\mathrm{gr}(\mathrm{Diff}(N)) \simeq \mathcal{O}(T^*\mathcal{B}_0).$$

Notice that

$$T^*(\mathcal{B}_0) \subset T^*(\mathcal{B}) = \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g}^* \mid x|_{\mathfrak{b}} = 0\}.$$

The projection $T^*\mathcal{B} \to \mathcal{N}$ given by $(\mathfrak{b}, x) \mapsto x$ is surjective and when restricted to $\mathcal{B}_0$ its image is dense. Thus, we have an inclusion

$$\mathrm{gr}(\mathcal{U}_{\bar{\lambda}}) \simeq \mathcal{O}(\mathcal{N}) \hookrightarrow \mathcal{O}(T^*\mathcal{B}_0) \simeq \mathrm{gr}(\mathrm{Diff}(\mathcal{B}_0)).$$

This inclusion shows that we have an injective map

$$\mathcal{U}_{\bar{\lambda}} \hookrightarrow \mathrm{Diff}(\mathcal{B}_0) \hookrightarrow \mathrm{End}_k(\nabla_\lambda).$$

This finishes the proof. □

Proof of Corollary. It is enough to prove this for

$$M = \mathcal{U}(\mathfrak{g}) \otimes_Z \mathbb{C}_{\bar{\lambda}} =: \mathcal{U}_{\bar{\lambda}}.$$

By the lemma it suffices to show this for $\mathrm{End}_k(\nabla_\lambda)$. The result is known for ∇_λ and $\mathrm{End}_k(\nabla_\lambda)$ is an infinite product of copies of ∇_λ so this finishes the proof. □

Definition 5.0.5 (Translation functor)**.** Let λ, μ be integral weights with $\langle \lambda + \rho, \alpha_i^\vee \rangle \geq 0$ and $\langle \mu + \rho, \alpha_i^\vee \rangle \geq 0$ for all simple coroots $\alpha_i^\vee$. Let V be the module with extreme weight $\mu - \lambda$, i.e. $V = V_{w(\mu-\lambda)}$ for a w for which $w(\mu - \lambda)$ is positive. The translation functor is defined as

$$T_{\lambda \to \mu} : \mathfrak{g}\text{-mod}_{\hat{\bar{\lambda}}} \to \mathfrak{g}\text{-mod}_{\hat{\bar{\mu}}}, \qquad M \mapsto (M \otimes V)_{\hat{\bar{\mu}}}.$$

Here $\mathfrak{g}\text{-mod}_{\hat{\bar{\lambda}}}$ denotes the subcategory of $\mathfrak{g}$-modules with generalized central character $\bar{\lambda}$ and the functor $(-)_{\hat{\bar{\mu}}}$ takes the summand on which the center acts by generalized central character $\bar{\mu}$.

Lemma 5.0.6. *The functor $T_{\lambda \to \mu}$ is exact.*

Proof. The functors $- \otimes V$ and $- \otimes V^*$ are adjoint and $V_\lambda^* \simeq V_{-\lambda}$ so $T_{\lambda \to \mu}$ is left and right adjoint to $T_{\mu \to \lambda}$. In particular, $T_{\lambda \to \mu}$ is exact. □

Definition 5.0.7. Let $\lambda, \mu \in \Lambda^+$ be positive integral weights. Write $\lambda \downarrow \mu$ if $\mathrm{Stab}_W(\lambda) \subset \mathrm{Stab}_W(\mu)$. The weights λ and μ are on the same face if $\mathrm{Stab}_W(\lambda) = \mathrm{Stab}_W(\mu)$.

Proposition 5.0.8. *(a) Assume that $\lambda \downarrow \mu$ then*

$$T_{\lambda \to \mu}(\Delta_{w \cdot \lambda}) = \Delta_{w \cdot \mu}.$$

(b) When λ and μ are on the same face the functor $T_{\lambda \to \mu}$ is an equivalence.
(c) If $\lambda_1 \downarrow \lambda_2 \downarrow \lambda_3$ or $\lambda_3 \downarrow \lambda_2 \downarrow \lambda_1$ then

$$T_{\lambda_2 \to \lambda_3} \circ T_{\lambda_1 \to \lambda_2} \simeq T_{\lambda_1 \to \lambda_3}.$$

We only prove the proposition for translation functors restricted to category $\mathcal{O}$.

Proof. (a) By lemma 5.0.1

$$\mathrm{gr}(\Delta_\lambda \otimes V) = \bigoplus_\nu \Delta_{\lambda+\nu} \otimes V[\nu].$$

We need to check that only one term has central character $\bar{\mu}$ and that the weight space is 1-dimensional. Assume first that $(\lambda - \mu, \mu) \geq 0$. Then we have (see [Hum, Lemma 7.5])

$$\lambda + \{\text{weights of } V\} \cap W \cdot \mu = \{\mu\}.$$

Thus, only one representative for $\bar{\mu}$ occurs in the sum. The center only acts by generalized central character on the term for which $\lambda + \nu = \mu$. Since $\nu = \mu - \lambda$ is an extremal weight in V we have $\dim V[\mu - \lambda] = 1$. This shows that

$$T_{\lambda \to \mu}(\Delta_\lambda) = \Delta_\mu.$$

Replace λ and μ by $w \cdot \lambda$ and $w \cdot \mu$. This does not change V since $\lambda - \mu$ is still an extremal weight. Hence,

$$T_{\lambda \to \mu}(\Delta_{w \cdot \lambda}) = \Delta_{w \cdot \mu}.$$

(b) By (a) $T_{\lambda \to \mu}$ induces an isomorphism.

$$K^0(\mathcal{O}_{\bar{\lambda}}) \xrightarrow{\sim} K^0(\mathcal{O}_{\bar{\mu}}).$$

To finish the proof we need the following lemma.

Lemma 5.0.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories of finite type with a pair of biadjoint functors*

$$F : \mathcal{A} \to \mathcal{B},$$
$$G : \mathcal{B} \to \mathcal{A}.$$

Assume that they are exact and induces isomorphisms on K^0. Then F and G are equivalences of categories.

Proof of lemma. Since F induces an isomorphism on K^0 we have $F(M) \neq 0$ for $M \neq 0$. The functor F does not kill morphisms so the map

$$GF(M) \to M$$

induced by $\mathrm{id}_{F(M)}$ by adjunction is injective (see [Gait, Lemma 4.27]). Since $[GF(M)] = [M]$ it must be an isomorphism. □

By the lemma $T_{\lambda \to \mu} : \mathcal{O}_{\bar{\lambda}} \to \mathcal{O}_{\bar{\mu}}$ is an equivalence of categories when λ, μ are on the same face and $(\lambda - \mu, \mu) \geq 0$. Hence, it is also true when $(\lambda - \mu, \mu) \leq 0$.

A proof of (c) will be given in section 5.2.1. □

Proposition 5.0.10. *For λ integral $P_1 = T_{-\rho \to \lambda}(\Delta_{-\rho})$.*

Proof. By adjunction

$$\mathrm{Hom}(T_{-\rho \to \lambda}(\Delta_{-\rho}), M) = \mathrm{Hom}(\Delta_{-\rho}, T_{\lambda \to -\rho}(M)).$$

Since $T_{\lambda \to -\rho}$ is exact and $\Delta_{-\rho}$ is projective in $\mathcal{O}_{-\rho}$, the functor in the right hand side is exact. Hence, $T_{-\rho \to \lambda}(\Delta_{-\rho})$ is projective in $\mathcal{O}_{\lambda}$. It remains to show that

$$\mathrm{Hom}(T_{-\rho \to \lambda}(\Delta_{-\rho}), L_\mu) = k^{\delta_{\mu, \lambda_{\min}}}.$$

By adjunction we have

$$\mathrm{Hom}(T_{-\rho \to \lambda}(\Delta_{-\rho}), L_\mu) = \mathrm{Hom}(\Delta_{-\rho}, T_{\lambda \to -\rho}(L_\mu)).$$

Thus, it is enough to show that for L irreducible

$$T_{\lambda \to -\rho}(L) = \begin{cases} 0 & \text{if } L \neq L_1 \\ \Delta_{-\rho} = L_{-\rho} & \text{if } L = L_1 \end{cases}$$

Notice that

$$GK \dim T_{\lambda \to \mu}(L) \leq GK \dim(L \otimes V) = GK \dim(L).$$

The only nonzero irreducible module in $\mathcal{O}_{-\rho}$ is $\Delta_{-\rho}$ so every nonzero object in this category has GK dimension $\dim \mathfrak{n}$. By proposition 4.1.3 we have $GK\dim(L) < \dim \mathfrak{n}$ for $L \neq L_1$. Thus, $T_{\lambda\to-\rho}(L) = 0$ for $L \neq L_1$. By proposition 5.0.8

$$T_{\lambda\to-\rho}(L_1) = T_{\lambda\to-\rho}(\Delta_1) = \Delta_{-\rho} = L_{-\rho}.$$

This finishes the proof. □

5.1. **Extended translation functors.** Recall that $Z(\mathcal{U}(\mathfrak{g})) \simeq \mathcal{O}(\mathfrak{t}^*)^{W,\cdot} \simeq \mathrm{Sym}(\mathfrak{t})^W$. Define

$$\tilde{\mathcal{U}} := \mathcal{U} \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t}).$$

For $\lambda \in \mathfrak{t}^*$ let $\mathcal{I}_{\bar{\lambda}}$ denote the maximal ideal of functions in $\mathrm{Sym}(\mathfrak{t})^W$ which vanish at $\bar{\lambda}$. Let $I_{\bar{\lambda}}$ be the preimage of $\mathcal{I}_{\bar{\lambda}}$ under the map $\mathfrak{t}^* \twoheadrightarrow \mathfrak{t}^*//W$. Define the completions

$$\mathcal{U}_{\hat{\bar{\lambda}}} := \mathcal{U} \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \varprojlim (\mathrm{Sym}(\mathfrak{t})^W/\mathcal{I}^n_{\bar{\lambda}}),$$
$$\tilde{\mathcal{U}}_{\hat{\lambda}} := \mathcal{U} \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \varprojlim (\mathrm{Sym}(\mathfrak{t})/I^n_{\bar{\lambda}}).$$

Notice that

$$\mathcal{U}_{\hat{\bar{\lambda}}} - \mathrm{mod} \supset \mathcal{U} - \mathrm{mod}_{\hat{\bar{\lambda}}},$$
$$\tilde{\mathcal{U}}_{\hat{\lambda}} - \mathrm{mod} \supset \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}}\,.$$

In the case where λ is regular, i.e. $\mathrm{Stab}_W(\lambda) = \{e\}$, we have

$$\tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}} \simeq \mathcal{U} - \mathrm{mod}_{\hat{\bar{\lambda}}}.$$

In fact, $\tilde{\mathcal{U}}_{\hat{\lambda}} \simeq \mathcal{U}_{\hat{\bar{\lambda}}}$. In the most singular case

$$\tilde{\mathcal{U}}_{-\hat{\rho}} \simeq \mathcal{U}_{-\hat{\rho}} \otimes_{\widehat{\mathrm{Sym}(\mathfrak{t})^W}_{\hat{0}}} \widehat{\mathrm{Sym}(\mathfrak{t})}_{\hat{0}}.$$

Theorem 5.1.1 (Beilinson, Ginzburg)**.** *If the face of μ is in the closure of the face of λ then the $T_{\lambda\to\mu}$ lifts to a functor $\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}} \to \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\mu}}$ called the extended translation functor which satisfies the following*

(a) The following diagram is commutative

$$\begin{tikzcd} \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}} \arrow[r, "\tilde{T}_{\lambda\to\mu}"] \arrow[d, "\mathrm{Res}"] & \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\mu}} \arrow[d, "\mathrm{Res}"] \\ \mathcal{U} - \mathrm{mod}_{\hat{\bar{\lambda}}} \arrow[r, "T_{\lambda\to\mu}"] & \mathcal{U} - \mathrm{mod}_{\hat{\bar{\mu}}} \end{tikzcd}$$

(b) The functor $\tilde{T}_{\mu\to\lambda}$ is fully faithful.

The construction of the functors $\tilde{T}_{\lambda\to\mu}$, proof of the Theorem and further properties of these functors will appear in section 5.3.

Proof of theorem 4.1.6. Assume that λ is regular. Then the theorem implies that

$$\Xi = P_1 = T_{-\rho\to\lambda}(\Delta_{-\rho}) = \tilde{T}_{-\rho\to\lambda}(\mathrm{Ind}_{\mathcal{U}}^{\tilde{\mathcal{U}}}(\Delta_{-\rho}))$$

Using this we get

$$\begin{aligned}\mathrm{End}(\Xi) &\simeq \mathrm{End}\big(\mathrm{Ind}_{\mathcal{U}}^{\tilde{\mathcal{U}}}(\Delta_{-\rho})\big)\\ &\simeq \mathrm{End}_{\mathrm{Sym}(\mathfrak{t})}\big(\mathrm{Ind}_{\mathrm{Sym}(\mathfrak{t})_+^W}^{\mathrm{Sym}(\mathfrak{t})}(\mathbb{C})\big)\\ &\simeq \mathrm{Sym}(\mathfrak{t})\Big/(\mathrm{Sym}(\mathfrak{t})_+^W).\end{aligned}$$

This finishes the proof. □

5.2. **Harish-Chandra modules.**

Definition 5.2.1 (Harish-Chandra modules)**.** Consider the diagonal embedding $G \hookrightarrow G\times G$ and observe that $\mathfrak{g}\oplus\mathfrak{g} = \mathrm{Lie}(G\times G)$. A Harish-Chandra module is a $(\mathfrak{g}\oplus\mathfrak{g})$-module for which the diagonal action extends to an action of G. The category of Harish-Chandra modules is denoted by H.Ch.- mod and also by $(\mathfrak{g}\oplus\mathfrak{g}, G)$-mod.

Forgetting the second $\mathfrak{g}$ action

$$(\mathfrak{g}\oplus\mathfrak{g}, G) - \mathrm{mod} \xrightarrow{\sim} \left\{ \begin{array}{c} \mathfrak{g}\text{-modules } M \text{ with an algebraic } G \text{ action s.t.} \\ \mathrm{g(x(v))=Ad(g)(x)(g(v))} \;\; \forall g\in G, x\in\mathfrak{g}, v\in M \end{array} \right\}$$

Example 5.2.2. Consider $\mathcal{U}$ as a $\mathfrak{g}\oplus\mathfrak{g}$ module with the action

$$(\mathfrak{g}\oplus\mathfrak{g})\times\mathcal{U}\to\mathcal{U}, \qquad (x,y,u)\mapsto xu - uy.$$

Let V be any finite dimensional $\mathfrak{g}$-module. Then $\mathcal{U}\otimes V$ is a Harish-Chandra-module with the action

$$(\mathfrak{g}\oplus\mathfrak{g})\times(\mathcal{U}\otimes V)\to\mathcal{U}\otimes V, \qquad (x,y,u\otimes v)\mapsto xu\otimes v - uy\otimes v + u\otimes yv.$$

Any bimodule B gives rise to a functor

$$\mathcal{U} - \mathrm{mod}\to\mathcal{U} - \mathrm{mod}, \qquad M\mapsto B\otimes_{\mathcal{U}} M.$$

In the case where $B = V\otimes\mathcal{U}$ this is just a tensor product of $\mathfrak{g}$ modules $M\mapsto V\otimes M$. Let B be a Borel.

$$(\mathfrak{g}, B) - \mathrm{mod} := \left\{ \begin{array}{c} \mathfrak{g} - \text{modules } M \text{ with an algebraic } B \text{ action s.t.} \\ \mathrm{g(x(v))=Ad(g)(x)(g(v))} \;\; \forall g\in B, x\in\mathfrak{g}, v\in M \end{array} \right\}$$

The natural restriction functor

$$\mathrm{H.Ch.} - \mathrm{mod} \xrightarrow{\mathrm{Res}_B^G} (\mathfrak{g}, B) - \mathrm{mod}$$

has a right adjoint

$$\mathrm{CoInd} : (\mathfrak{g}, B) - \mathrm{mod}\to \mathrm{H.Ch.} - \mathrm{mod}$$

which commutes with forgetting the second $\mathfrak{g}$ action. Notice that an algebraic B-module M defines a G-equivariant quasi-coherent sheaf $\mathcal{F}_M$ on $\mathcal{B} = G/B$. The coinduction functor is taking global sections

$$\mathrm{CoInd}(M) := \Gamma(\mathcal{F}_M).$$

Define $\Delta_\lambda(-\lambda) \in (\mathfrak{g}, B)-\mathrm{mod}$ to be Δ_λ as a $\mathfrak{g}$-module with a $\mathfrak{b}$-action defined by tensoring the ordinary $\mathfrak{b}$ action by $(-\lambda)$. This $\mathfrak{b}$-action integrates to a B-action for which the highest weight vector is B-invariant. This makes the action compatible with the torus action so with this choice of B-action the map

$$\mathcal{U} \twoheadrightarrow \Delta_\lambda(-\lambda)$$

is a map of $(\mathfrak{g}, B)$-modules.

Lemma 5.2.3. *There is an isomorphism* $\mathcal{U}_{\bar{\lambda}} \xrightarrow{\sim} \mathrm{CoInd}(\Delta_\lambda(-\lambda))$.

Proof. It is enough to equip both sides with a filtration such that the map on the associated graded is an isomorphism. Recall that $\mathrm{gr}(\mathcal{U}_{\bar{\lambda}}) \simeq \mathcal{O}(\mathcal{N})$. For the sheaf

$$\begin{aligned}\mathrm{gr}(\mathcal{F}_{\Delta_\lambda}) &= \mathrm{Sym}(\mathfrak{g}/\mathfrak{b}) \times^B G \\ &\simeq \mathcal{O}((\mathfrak{g}/\mathfrak{b})^* \times^B G) \\ &= \mathcal{O}(\{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathrm{rad}(\mathfrak{b})\})\end{aligned}$$

Exercise 5.2.4. Show that $\mathcal{O}((\mathfrak{g}/\mathfrak{b})^* \times^B G) \simeq \mathcal{O}(T^*\mathcal{B})$.

Note that $(\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{b}^\perp \simeq \mathfrak{n}$ (the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ coming from the Killing form) so

$$\mathrm{gr}(\Delta_\lambda) \simeq \mathrm{Sym}(\mathfrak{g}/\mathfrak{b}).$$

Since $H^1(T^*\mathcal{B}, \mathcal{O}) = 0$ in this case the global sections functor is exact and so it commutes with taking associate graded

$$\mathrm{gr}(\Gamma(\mathcal{F}_{\Delta_\lambda})) \simeq \Gamma(\mathrm{gr}(\mathcal{F}_{\Delta_\lambda}))$$

Consider the projection $T^*\mathcal{B} \to \mathcal{N}$ given by $(\mathfrak{b}, x) \mapsto x$. It is known that the induced map

$$\mathrm{gr}(\mathcal{U}_{\bar{\lambda}}) \simeq \mathcal{O}(\mathcal{N}) \xrightarrow{\sim} \Gamma(\mathcal{O}(T^*\mathcal{B})) \simeq \mathrm{gr}(\mathrm{CoInd}(\Delta_\lambda))$$

is an isomorphism. □

Remark 5.2.5. Fix $\lambda \in \Lambda^+$. Let $\mathrm{H.Ch}-\mathrm{mod}_{\hat{\bar{\lambda}},\bar{\lambda}}$ denote the full subcategory of H.Ch-mod for which the right action of $\mathfrak{g}$ factors through $\bar{\lambda}$ and the left action factors through a power of $\bar{\lambda}$. Consider the functor

$$C_\lambda : \mathcal{O}_{\bar{\lambda}} \to \mathrm{H.\,Ch.}-\mathrm{mod}_{\hat{\bar{\lambda}},\bar{\lambda}}, \qquad M \mapsto \mathrm{CoInd}(M(-\lambda)),$$

where $(-\lambda)$ indicates that the B-module structure is the one coming from $\mathfrak{g}$ twisted by $-\lambda$. The functor is an equivalence of Abelian categories (This follows from the localization theorem using $B\backslash G/B = (G/B \times G/B)/G$).

Proposition 5.2.6. *(1) For M with diagonalizable action*

$$\mathrm{CoInd}(T_{\lambda\to\mu}(M)(-\mu)) \simeq T_{\lambda\to\mu}(\mathrm{CoInd}(M(-\lambda))),$$

where the $(-\lambda)$ indicates that the natural $\mathfrak{g}$ action is twisted by $-\lambda$.
(2) For $M \in \mathcal{U}_\lambda - \mathrm{mod}$

$$T_{\lambda\to\mu}(M) \simeq T_{\lambda\to\mu}(\mathcal{U}_\lambda) \otimes_{\mathcal{U}_\lambda} M.$$

Proof. Exercise. □

Assume that $\lambda \downarrow \mu$. Then

$$\begin{aligned} T_{\lambda\to\mu}(\mathcal{U}_{\bar\lambda}) &\simeq T_{\lambda\to\mu}(\mathcal{U}_\lambda) \otimes_{\mathcal{U}_\lambda} \mathcal{U}_\lambda \simeq T_{\lambda\to\mu}(\mathrm{CoInd}(\Delta_\lambda(-\lambda))) \\ &\simeq \mathrm{CoInd}(T_{\lambda\to\mu}(\Delta_\lambda)(-\mu)) \simeq \mathrm{CoInd}(\Delta_\mu(-\mu)) \\ &\simeq \mathcal{U}_{\bar\mu}. \end{aligned}$$

Since $T_{\lambda\to\mu}$ is determined by $T_{\lambda\to\mu}(\mathcal{U}_\lambda)$ this also proves part (c) of proposition 5.0.8 in the case $\lambda_1 \downarrow \lambda_2 \downarrow \lambda_3$ as

$$T_{\lambda_2\to\lambda_3} T_{\lambda_1\to\lambda_2}(\mathcal{U}_{\lambda_1}) \simeq \mathcal{U}_{\lambda_3}(\lambda_3 - \lambda_1) \simeq T_{\lambda_1\to\lambda_3}(\mathcal{U}_{\lambda_1}).$$

The case $\lambda_3 \downarrow \lambda_2 \downarrow \lambda_1$ follows by taking the adjoint functors.

Combining the proposition with lemma 5.2.3 we get that for λ, μ on the same face

$$\begin{aligned} T_{\mu\to\lambda} T_{\lambda\to\mu}(\mathcal{U}_{\bar\lambda}) &\simeq T_{\mu\to\lambda} T_{\lambda\to\mu}(\mathrm{CoInd}(\Delta_\lambda(-\lambda))) \\ &\simeq \mathrm{CoInd}(T_{\mu\to\lambda} T_{\lambda\to\mu}(\Delta_\lambda(-\lambda))) \\ &\simeq \mathrm{CoInd}(\Delta_\lambda(-\lambda)) \\ &\simeq \mathcal{U}_{\bar\lambda}. \end{aligned}$$

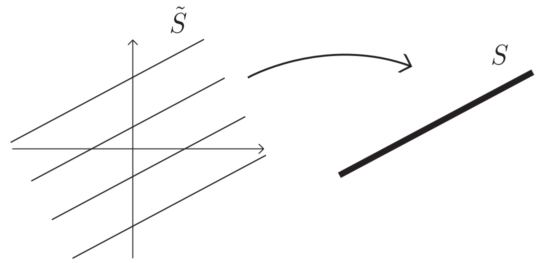
This proves that $T_{\mu\to\lambda} T_{\lambda\to\mu} \simeq \mathrm{Id}$.

5.3. **Construction of extended translation functors.** To define extended translation functors $\tilde{T}_{\lambda\to\mu}$ consider the module $\mathcal{U} \otimes V$, where $V = V_\mu$ is a finite dimensional $\mathfrak{g}$ module with highest weight μ. It is a Harish-Chandra bimodule with the left $\mathfrak{g}$-action being the action on the first coordinate and the right action being the diagonal action.

Lemma 5.3.1. *The scheme $S = \mathrm{supp}_{Z\otimes Z}(\mathcal{U} \otimes V)$ is the image of the subscheme*

$$\tilde{S} := \bigcup_{\nu, V[\nu]\neq 0} \{(\lambda, \mu) \mid \mu - \lambda = \nu\}$$

under the map $(\mathfrak{t}^*)^2 \to (\mathfrak{t}^*//W)^2$.



Proof. The second $\mathfrak{g}$-action is integrable so

$$\mathrm{End}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathcal{U}\otimes V) \simeq (\mathrm{End}_{\mathfrak{g}}(\mathcal{U}\otimes V))^G \simeq (\mathcal{U}\otimes \mathrm{End}(V))^G \subseteq \mathcal{U}\otimes \mathrm{End}(V).$$

Hence, $\mathrm{End}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathcal{U}\otimes V)$ is a torsion free module over the left copy of Z so nothing vanishes when we localize.

Claim 5.3.2. $\mathrm{End}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathcal{U}\otimes V)$ is finite over Z.

Proof of claim. Passing to the associated graded $\mathrm{Sym}(\mathfrak{t})^W$ acts on

$$\mathrm{gr}((\mathcal{U}\otimes \mathrm{End}(V))^G) \simeq (\mathrm{Sym}(\mathfrak{g})\otimes \mathrm{End}(V))^G \simeq \mathrm{Maps}(\mathfrak{g}, \mathrm{End}(V))^G.$$

The projection

$$\mathfrak{g} \twoheadrightarrow \mathfrak{g}/G \simeq \mathfrak{t}/W$$

has a section. Denote its image by S. Then restriction to S provides a map from $\mathrm{Maps}(\mathfrak{g}, \mathrm{End}(V))^G$ to $\mathrm{Maps}(S, \mathrm{End}(V))$.

Now, $S \subset \mathfrak{g}^{reg}$ and $\mathrm{Ad}_G(S) = \mathfrak{g}^{reg}$. It is clear that for $\sigma \in \mathrm{Maps}(\mathfrak{g}, \mathrm{End}(V))^G$ the restriction $\sigma|_S$ determines $\sigma|_{G(S)}$. We have $G(S) = \mathfrak{g}^{reg}$ which is Zariski dense in $\mathfrak{g}$ so

$$\mathrm{Maps}(\mathfrak{g}, \mathrm{End}(V))^G \hookrightarrow \mathrm{Maps}(S, \mathrm{End}(V)).$$

Since $\mathrm{Maps}(S, \mathrm{End}(V))$ is finite over $\mathcal{O}(\mathfrak{t}/W)$ we see that $\mathrm{Maps}(\mathfrak{g}, \mathrm{End}(V))^G$ is also finite. □

Thus, $\mathcal{U}\otimes V$ has full support with respect to the first action. Now consider the action of the second copy of Z. It is enough to check that if λ is generic then $\mathcal{U}_\lambda \otimes V$ as a module over the second copy of Z is supported on

$$\{\overline{\lambda+\nu} \mid V[\nu] \neq 0\}.$$

The set of integral weights is Zariski dense so we may assume that λ is integral. Since $\mathcal{U}_\lambda = \mathrm{CoInd}(\Delta_\lambda(-\lambda))$ it is enough to check this for $\Delta_\lambda \otimes V$. This was proved in corollary 5.0.2. □

Recall that $V\otimes\mathcal{U}$ is a Harish-Chandra bimodule for any finite dimensional representation V.

$$\widetilde{V\otimes\mathcal{U}} := (V\otimes\mathcal{U}) \otimes_{\mathcal{O}(S)} \mathcal{O}(\tilde{S})$$

is a $\tilde{\mathcal{U}} \otimes \tilde{\mathcal{U}}$-bimodule. Suppose $\lambda, \mu \in \Lambda^+$ with $\lambda \downarrow \mu$. A calculation shows that

$$(\widetilde{V \otimes \mathcal{U}})_{\hat{\lambda},\hat{\mu}} = (V \otimes \mathcal{U})_{\hat{\hat{\lambda}},\hat{\hat{\mu}}} \otimes_{\mathcal{O}(\mathfrak{t}^*//\mathrm{Stab}(\lambda))_{\hat{0}}} \mathcal{O}(\mathfrak{t}^*)_{\hat{0}}$$

As before let V be the module with extremal weight $\mu - \lambda$. Use this to define the extended translation functor.

$$\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}} \to \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\mu}}, \qquad M \mapsto ((\widetilde{V \otimes \mathcal{U}}) \otimes_{\tilde{\mathcal{U}}} M)_{\hat{\mu}},$$

where the subscript $\hat{\mu}$ indicates taking the direct summand with this generalized central character. From the definition we get the commutative diagram from part (a) of theorem 5.1.1

$$\begin{tikzcd}
\tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}} \arrow[r, "\tilde{T}_{\lambda\downarrow\mu}"] \arrow[d, "\mathrm{Res}"] & \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\mu}} \arrow[d, "\mathrm{Res}"] \\
\mathcal{U} - \mathrm{mod}_{\hat{\hat{\lambda}}} \arrow[r, "T_{\lambda\downarrow\mu}"] & \mathcal{U} - \mathrm{mod}_{\hat{\hat{\mu}}}
\end{tikzcd}$$

In particular, $\tilde{T}_{\mu\downarrow\lambda}$ is exact. Taking left adjoints we get the diagram

$$\begin{tikzcd}
\tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\lambda}} & \tilde{\mathcal{U}} - \mathrm{mod}_{\hat{\mu}} \arrow[l, "\tilde{T}_{\mu\uparrow\lambda}"'] \\
\mathcal{U} - \mathrm{mod}_{\hat{\hat{\lambda}}} \arrow[u, "\mathrm{Ind}"] & \mathcal{U} - \mathrm{mod}_{\hat{\hat{\mu}}} \arrow[l, "T_{\mu\uparrow\lambda}"'] \arrow[u, "\mathrm{Ind}"]
\end{tikzcd}$$

For λ regular Ind and Res are equivalences of categories. The last part of theorem 5.1.1 is to prove that $\tilde{T}_{\lambda\to\mu}$ is fully faithful. Recall the following lemma from Morita theory

Lemma 5.3.3. *Let $F : A - \mathrm{mod} \to B - \mathrm{mod}$ be an exact functor coming from tensoring with a bimodule M which is projective over A. Assume that $B \tilde{\to} End_A(M)$. Then the left adjoint is faithful.*

Proof of part (b) of theorem 5.1.1. By the lemma it is enough to show that

$$\tilde{\mathcal{U}}_{\hat{\mu}} \simeq \mathrm{End}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}))$$

Recall the universal Verma $\tilde{\Delta} := \mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}(\mathbb{C}) \simeq \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathrm{Sym}(\mathfrak{t}))$. View it as a $(\mathfrak{g}, B)$-module in a natural way. We proved that

$$\tilde{\mathcal{U}} \simeq \mathrm{CoInd}(\tilde{\Delta})$$

Define $\tilde{\Delta}_{\hat{\lambda}} := \tilde{\Delta} \otimes_{\mathrm{Sym}(\mathfrak{t})} \mathrm{Sym}(\mathfrak{t})_{\hat{\lambda}}$. With this definition

$$\tilde{\mathcal{U}}_{\hat{\lambda}} \simeq \mathrm{CoInd}(\tilde{\Delta}_{\hat{\lambda}})$$

Assume that $\lambda \downarrow \mu$. By Nakayama's lemma $T_{\lambda\to\mu}(\Delta_\lambda) = \Delta_\mu$ implies that

$$\tilde{T}_{\lambda\to\mu}(\tilde{\Delta}_{\hat{\lambda}}) \simeq \tilde{\Delta}_{\hat{\mu}}.$$

Using this we get

$$\begin{aligned}\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}) &\simeq \tilde{T}_{\lambda\to\mu}(\mathrm{CoInd}(\tilde{\Delta}_{\hat{\lambda}}))\\ &\simeq \mathrm{CoInd}(\tilde{T}_{\lambda\to\mu}(\tilde{\Delta}_{\hat{\lambda}})(\mu-\lambda))\\ &\simeq \mathrm{CoInd}(\tilde{\Delta}_{\hat{\mu}}(\mu-\lambda)).\end{aligned}$$

Thus, the proof is reduced to showing

$$\tilde{\mathcal{U}}_{\hat{\mu}} \simeq \mathrm{End}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\mathrm{CoInd}(\tilde{\Delta}_{\bar{\mu}}(\mu-\lambda)).$$

This would follow if we can show

$$\tilde{\mathcal{U}} \simeq \mathrm{End}_{\tilde{\mathcal{U}}}\mathrm{CoInd}(\tilde{\Delta}(\mu-\lambda)) = \mathrm{End}_{\tilde{\mathcal{U}}}(\Gamma(\mathcal{B},\mathcal{F}_{\tilde{\Delta}(\mu-\lambda)})).$$

For this it is enough to prove the isomorphism for the associate graded. The sheaf has a filtration with

$$\mathrm{gr}(\mathcal{F}_{\tilde{\Delta}(\mu-\lambda)}) \simeq \mathcal{O}(\tilde{\mathfrak{g}}) \otimes \mathcal{O}(\mu-\lambda),$$

where

$$\tilde{\mathfrak{g}} := (\mathfrak{g}/\mathfrak{n})^* \times^B G \simeq \{(x,\mathfrak{b}) \in \mathfrak{g}\times\mathcal{B} \mid x \in \mathfrak{b}\}.$$

We have

$$\begin{aligned}\mathrm{gr}\Gamma(\mathcal{B},\mathcal{F}_{\tilde{\Delta}(\mu-\lambda)}) &\subset \Gamma(\mathcal{B},\mathrm{gr}(\mathcal{F}_{\tilde{\Delta}(\mu-\lambda)}))\\ &\simeq \Gamma(\mathcal{B},p_{1*}(\mathcal{O}(\mu-\lambda)))\\ &\simeq \Gamma(\tilde{\mathfrak{g}},\mathcal{O}(\mu-\lambda)).\end{aligned}$$

Consider the maps

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "x\mapsto x \bmod [\mathfrak{b},\mathfrak{b}]"] & \mathfrak{g} \\ \mathcal{B} & \mathfrak{t} \end{tikzcd}$$

where p_1 and p_2 are the projection maps. By the Harish-Chandra isomorphism $\mathfrak{g}^*//G \simeq \mathfrak{t}^*//W$. The maps are compatible with this isomorphism so there is a map

$$\tilde{\mathfrak{g}}^* \to \mathfrak{g}^* \times_{\mathfrak{t}^*//W} \mathfrak{t}^*.$$

This map is generically an isomorphism.

Lemma 5.3.4. *Let A be a filtered ring and M a filtered A-module then*

$$\mathrm{gr}(\mathrm{End}_A(M)) \subset \mathrm{End}_{\mathrm{gr}(A)}(\mathrm{gr}M).$$

Remark 5.3.5. We will only apply this lemma to M which is finitely generated with a separated and exhaustive filtration.

Lemma 5.3.6. *Let $\mathcal{L}$ be a line bundle on $\tilde{\mathfrak{g}}$ and let M be a nonzero submodule of* $\Gamma(\tilde{\mathfrak{g}},\mathcal{L})$. *Then* $\mathrm{End}(M) \simeq \mathcal{O}(\mathfrak{g}\times_{\mathfrak{t}//W}\mathfrak{t})$.

Applying the lemmas to $M := \mathrm{gr}\Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}(\mu-\lambda)}) \subset \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}(\mu-\lambda))$. We get

$$\mathrm{gr}(\mathrm{End}_{\tilde{\mathcal{U}}}(\Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}(\mu-\lambda)}))) \subset \mathrm{End}_{\mathrm{gr}(\tilde{\mathcal{U}})}(\mathrm{gr}(\Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}(\mu-\lambda)}))) \simeq \mathcal{O}(\mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}).$$

Observe that

$$\mathcal{O}(\mathfrak{g}^* \times_{\mathfrak{t}^*//W} \mathfrak{t}^*) \simeq \mathrm{Sym}(\mathfrak{g}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t}) = \mathrm{gr}(\tilde{\mathcal{U}}).$$

Since $\tilde{\mathcal{U}} \simeq \Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}})$ there is an inclusion

$$\tilde{\mathcal{U}} \hookrightarrow \mathrm{End}_{\tilde{\mathcal{U}}}(\Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}})).$$

The composition of maps

$$\mathrm{gr}(\tilde{\mathcal{U}}) \hookrightarrow \mathrm{gr}(\mathrm{End}_{\tilde{\mathcal{U}}}(\Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}}))) \hookrightarrow \mathcal{O}(\mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}) \simeq \mathrm{gr}(\tilde{\mathcal{U}})$$

is an isomorphism so all maps are isomorphisms. In particular, $\mathrm{gr}(\tilde{\mathcal{U}}) \simeq \mathrm{gr}(\mathrm{End}_{\tilde{\mathcal{U}}}(\Gamma(\mathcal{B}, \mathcal{F}_{\tilde{\Delta}})))$ which is what we wanted. □

Proposition 5.3.7. *For λ regular $\tilde{T}_{\mu\to\lambda}$ is left adjoint to $\tilde{T}_{\lambda\to\mu}$.*

Notice that this implies that the right adjoint $\tilde{T}_{\lambda\to\mu}$ is left exact. Since it is defined as a tensor product it is also right exact. In fact, we already know this exactness from the commutative diagram in theorem 5.1.1(a).

In the proof of the proposition we will need the following fact.

Proposition 5.3.8. *Let A, B be rings and F a right exact functor*

$$F : A - \mathrm{mod} \to B - \mathrm{mod}.$$

Then F is equivalent to the functor $M \mapsto F(A) \otimes_A M$.

From the above proposition we get

$$\tilde{T}_{\lambda\to\mu} \simeq \tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}) \otimes_{\tilde{\mathcal{U}}_{\hat{\lambda}}} -$$
$$\tilde{T}_{\mu\to\lambda} \simeq \tilde{T}_{\mu\to\lambda}(\tilde{\mathcal{U}}_{\hat{\mu}}) \otimes_{\tilde{\mathcal{U}}_{\hat{\mu}}} -$$

Proof of proposition 5.3.7. Using the above observation it is enough to show that

$$\tilde{T}_{\mu\to\lambda}(\tilde{\mathcal{U}}_{\hat{\mu}}) \simeq \mathrm{Hom}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}), \tilde{\mathcal{U}}_{\hat{\lambda}})$$

as $\tilde{\mathcal{U}}_{\hat{\lambda}} - \tilde{\mathcal{U}}_{\hat{\mu}}$ bimodules. Recall that for λ regular $\tilde{\mathcal{U}}_{\hat{\lambda}} \simeq \mathcal{U}_{\hat{\hat{\lambda}}}$

$$\begin{tikzcd} \mathcal{U}_{\hat{\hat{\lambda}}} - \mathrm{mod} \arrow[r, "\tilde{T}_{\lambda\to\mu}"] \arrow[dr, "T_{\lambda\to\mu}"'] & \tilde{\mathcal{U}}_{\hat{\mu}} - \mathrm{mod} \arrow[d, "\mathrm{Res}"] \\ & \mathcal{U}_{\hat{\hat{\mu}}} - \mathrm{mod} \end{tikzcd}$$

We already know that $T_{\mu\to\lambda}$ is right adjoint to $T_{\lambda\to\mu}$ so

$$\begin{aligned} T_{\mu\to\lambda}(\mathcal{U}_{\hat{\hat{\mu}}}) &\simeq \mathrm{Hom}_{\mathcal{U}_{\hat{\hat{\lambda}}}}(T_{\lambda\to\mu}(\mathcal{U}_{\hat{\hat{\lambda}}}), \mathcal{U}_{\hat{\hat{\lambda}}}) \\ &\simeq \mathrm{Hom}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}), \tilde{\mathcal{U}}_{\hat{\lambda}}) \end{aligned}$$

Since $\operatorname{Res}\tilde{T}_{\mu\to\lambda} = T_{\mu\to\lambda}$ we have

$$\tilde{T}_{\mu\to\lambda}(\tilde{\mathcal{U}}_{\hat{\mu}}) \simeq \operatorname{Hom}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}), \tilde{\mathcal{U}}_{\hat{\lambda}})$$

as $\mathcal{U}_{\hat{\mu}} - \mathcal{U}_{\hat{\lambda}}$ bimodules. On both sides the action of $\widehat{\operatorname{Sym}}(\mathfrak{t})$ comes from

$$\mathcal{U}_{\hat{\lambda}} \simeq \tilde{\mathcal{U}}_{\hat{\lambda}} \simeq \mathcal{U}_{\bar{\lambda}} \otimes_{\widehat{\operatorname{Sym}}(\mathfrak{t})} \widehat{\operatorname{Sym}}(\mathfrak{t}) \supset \widehat{\operatorname{Sym}}(\mathfrak{t})$$

so they are also isomorphic as $\tilde{\mathcal{U}}_{\hat{\mu}} - \tilde{\mathcal{U}}_{\hat{\lambda}}$ bimodules. □

Corollary 5.3.9. $\tilde{T}_{\lambda\to\mu}\tilde{T}_{\mu\to\lambda} \simeq \operatorname{Id}$ for $\lambda \downarrow \mu$.

Proposition 5.3.10. *Let B be a ring and P a finitely generated projective B module. Then*

$$\operatorname{End}_B(P) \simeq P \otimes_B \operatorname{Hom}_B(P, B)$$

Proof. See [Cohn, Lemma 4.5.3]. □

Proof of corollary 5.3.9. It is enough to show that $\tilde{T}_{\lambda\to\mu}\tilde{T}_{\mu\to\lambda}(\tilde{\mathcal{U}}_{\hat{\mu}}) \simeq \tilde{\mathcal{U}}_{\hat{\mu}}$. By proposition 5.3.7 we have

$$\begin{aligned}\tilde{T}_{\lambda\to\mu}\tilde{T}_{\mu\to\lambda}(\tilde{\mathcal{U}}_{\hat{\mu}}) &\simeq \tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}) \otimes_{\tilde{\mathcal{U}}_{\hat{\lambda}}} \tilde{T}_{\mu\to\lambda}(\tilde{\mathcal{U}}_{\hat{\mu}}) \\ &\simeq \tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}) \otimes_{\tilde{\mathcal{U}}_{\hat{\lambda}}} \operatorname{Hom}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}), \tilde{\mathcal{U}}_{\hat{\lambda}}).\end{aligned}$$

Now, $\tilde{\mathcal{U}}_{\hat{\lambda}}$ is projective so $\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}})$ is also projective and the proposition above states that

$$\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}) \otimes_{\tilde{\mathcal{U}}_{\hat{\lambda}}} \operatorname{Hom}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}), \tilde{\mathcal{U}}_{\hat{\lambda}}) \simeq \operatorname{End}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}))$$

As part of the proof of theorem 5.1.1 part (b) we proved that

$$\tilde{\mathcal{U}}_{\hat{\mu}} \simeq \operatorname{End}_{\tilde{\mathcal{U}}_{\hat{\lambda}}}(\tilde{T}_{\lambda\to\mu}(\tilde{\mathcal{U}}_{\hat{\lambda}}))$$

This proves the result. □

Definition 5.3.11. An exact functor $F : \mathcal{A} \to \mathcal{B}$ is called a Serre factorization if $\mathcal{B} \simeq \mathcal{A}/\ker F$, where $\mathcal{A}/\ker F$ is the Serre quotient category.

Lemma 5.3.12. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor and $G : \mathcal{B} \to \mathcal{A}$ its left adjoint. Assume that $F \circ G \simeq \operatorname{Id}_{\mathcal{B}}$. Then $\bar{F} : \mathcal{A}/\ker\mathcal{A} \simeq \mathcal{B}$, i.e. F is a Serre factorization.*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, shift left, "F"] \arrow[dr] & & \mathcal{B} \arrow[ll, shift left, "G"] \\ & \mathcal{A}/\ker\mathcal{A} \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

Proof. We need to show that $\bar{F}$ is essentially surjective and that it is an isomorphism on Hom's. The first part follows from $F \circ G \simeq \operatorname{Id}$. Since F is exact, $\bar{F}$ is also exact; since $\bar{F}$ does not kill nonzero objects, we see that $\bar{F}$ is injective on Hom's. To prove that $\bar{F}$ is surjective on Hom's it is enough to show that for all $\bar{M}, \bar{N} \in \mathcal{A}/\ker\mathcal{A}$ with representatives $M, N \in \mathcal{A}$ and any morphism $x : F(M) \to F(N)$ there exists a morphism $y : \bar{M} \to \bar{N}$ such that $\bar{F}(y) = x$. Consider the adjunction arrow $GF(M) \to M$. The condition $FG \simeq \operatorname{Id}_{\mathcal{B}}$

implies that $\ker, \operatorname{coker} \in \ker(F)$ so it becomes an isomorphism in $\mathcal{A}/\ker\mathcal{A}$. Thus, we get the following diagram and so a choice of y

$$\begin{tikzcd} \overline{GF(M)} \arrow[r, "G(x)"] \arrow[d, "\wr"] & \overline{GF(N)} \arrow[d, "\wr"] \\ \bar{M} \arrow[r, "y"] & \bar{N} \end{tikzcd}.$$

This finishes the proof. □

In particular, we have the following corollary

Corollary 5.3.13. For $\lambda \downarrow \mu$ the functor

$$\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{U}} - \operatorname{mod}_{\hat{\lambda}} \to \tilde{\mathcal{U}} - \operatorname{mod}_{\hat{\mu}}$$

is a Serre factorization.

Define $\tilde{\mathcal{O}}_\mu$ to be the preimage of $\mathcal{O}_\mu$ under the restriction map $\tilde{\mathcal{U}} - \operatorname{mod}_{\hat{\mu}} \to \mathcal{U} - \operatorname{mod}_{\hat{\bar{\mu}}}$. Then

$$\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{O}}_\lambda \to \tilde{\mathcal{O}}_\mu$$

is also a Serre factorization for $\lambda \downarrow \mu$.

Corollary 5.3.14. Assume that λ is regular. Then

$$T_{\lambda\to\mu}T_{\mu\to\lambda}(M) \simeq M \otimes_Z \operatorname{Sym}(\mathfrak{t}) \simeq M \otimes_{\hat{Z}_{\hat{\mu}}} \operatorname{Sym}(\mathfrak{t})_{\hat{0}}.$$

Proof. If λ is regular then $\lambda \downarrow \mu$ for any μ and

$$\tilde{\mathcal{U}} - \operatorname{mod}_{\hat{\lambda}} \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} \mathcal{U} - \operatorname{mod}_{\hat{\bar{\lambda}}}$$

are equivalences of categories. Hence,

$$T_{\lambda\to\mu} \simeq \operatorname{Res}_\mu \circ \tilde{T}_{\lambda\to\mu},$$
$$T_{\mu\to\lambda} \simeq \tilde{T}_{\mu\to\lambda} \circ \operatorname{Ind}_\mu.$$

By the corollary

$$\begin{aligned} T_{\lambda\to\mu}T_{\mu\to\lambda} &\simeq \operatorname{Res}_\mu \circ \tilde{T}_{\lambda\to\mu} \circ \tilde{T}_{\mu\to\lambda} \circ \operatorname{Ind}_\mu \\ &\simeq \operatorname{Res}_\mu \circ \operatorname{Ind}_\mu. \end{aligned}$$

This proves the result. □

5.4. **Example:** $\mathfrak{g} = \mathfrak{sl}_2$**.** Recall that $\operatorname{End}(P_1) \simeq \operatorname{Sym}(\mathfrak{t})/\operatorname{Sym}(\mathfrak{t})^W_+$ and

$$\Delta_{-\rho} \otimes (\operatorname{Sym}(\mathfrak{t})/\operatorname{Sym}(\mathfrak{t})^W_+) = \Delta_{-\rho} \otimes_{\mathcal{O}(\mathfrak{t}^*//W)_{\hat{0}}} \mathcal{O}(\mathfrak{t}^*)_{\hat{0}}.$$

It follows that $\tilde{\mathcal{O}}_{-\rho} \simeq \operatorname{Sym}(\mathfrak{t})/\operatorname{Sym}(\mathfrak{t})^W_+ - \operatorname{mod}$.

If $\mathcal{A}$ is an Abelian category with irreducible objects $L_1, \dots, L_n$ and projective covers $P_1, \dots, P_n$. Then

$$\mathcal{A} \simeq \operatorname{End}(\oplus P_i)^{\text{opp}} - \operatorname{mod} \qquad \text{and} \qquad \mathcal{A}/\langle L_1, \dots, L_n\rangle \simeq \operatorname{End}(P_1)^{\text{opp}} - \operatorname{mod}.$$

We now calculate all projectives in category $\mathcal{O}$ for $\mathfrak{g} = \mathfrak{sl}_2$.

The weight $\lambda = 0$ is regular and the $W = \Sigma_2$-orbit is $\{0, -2\}$. Thus, the irreducible modules in $\mathcal{O}_0$ are L_0 and $L_{-2} \simeq \Delta_{-2}$. Consider the short exact sequence

$$0 \to \ker_0 \to \Delta_0 \to L_0 \to 0.$$

Since $\ker \in \langle L_{-2}\rangle$ and $[L_{-2}, \Delta_0] = 1$ by corollary 4.1.4 the above short exact sequence is

$$0 \to L_{-2} \to \Delta_0 \to L_0 \to 0.$$

By proposition 3.0.16 P_{-2} has a standard filtration in which Δ_{-2} appears once and any other Δ_j in the filtration has $j > -2$. The only possibility in this case is Δ_0 and since $[\Delta_0, P_1] = 1$ by corollary 4.1.4 we get the short exact sequence

$$0 \to \Delta_0 \to P_{-2} \to \Delta_{-2} \to 0.$$

Putting these two exact sequences together we get a Jordan Hölder filtration $L_0 \subset \Delta_0 \subset P_{-2}$ with quotients L_{-2}, L_0, L_{-2}. This is written as

$$\operatorname{gr} P_{-2} = \begin{bmatrix} L_{-2} \\ L_0 \\ L_{-2} \end{bmatrix}$$

Notice that $P_0 = \Delta_0$

$$\operatorname{Hom}(P_{-2}, P_0) = k = \operatorname{Hom}(P_0, P_{-2})$$

$$\operatorname{End}(P_{-2}) \simeq \operatorname{Sym}(\mathfrak{t})/\operatorname{Sym}(\mathfrak{t})^W_+ = k[t]/t^2.$$

For $\lambda = -\rho = -1$ the only irreducible is L_{-1} so $\mathcal{O}_{-1} \simeq$Vect and as noted above

$$\tilde{\mathcal{O}}_{-1} \simeq k[t]/t^2 - \operatorname{mod}.$$

The image of $k[t]/t^2$ under this equivalence is $\tilde{\Delta}_{-1}$.

Recall that $P_{-2} \simeq T_{-1\to 0}(\Delta_{-1}) \simeq \tilde{T}_{-1\to 0}(\tilde{\Delta}_{-1})$. We claim that $\tilde{T}_{-1\to 0}(\Delta_{-1}) = \nabla_0$.

To check this, consider the exact sequence

$$k[t]/(t^2) \xrightarrow{t} k[t]/(t^2) \to k \to 0,$$

from which we get:

$$\tilde{\Delta}_{-1} \to \tilde{\Delta}_{-1} \to \Delta_{-1} \to 0.$$

Applying $\tilde{T}_{-1\to 0}$ we obtain

$$P_{-2} \to P_{-2} \to \tilde{T}_{-1\to 0}(\Delta_{-1}) \to 0$$

The first map is nilpotent so it factors through the submodule L_{-2}. Thus, its image is L_{-2} and its cokernel is $P_{-2}/L_{-2} \simeq \nabla_0$. Here the last isomorphism follows from the short exact sequence

$$0 \to \Delta_0 \to P_{-2} \to L_{-2} \to 0$$

by duality.

5.5. **Wall-crossing functors.** Our goal is to describe $\mathcal{O}_\lambda \simeq \tilde{\mathcal{O}}_\lambda$ for λ regular integral. One tool for doing this is Wall-crossing functors for each simple root α.

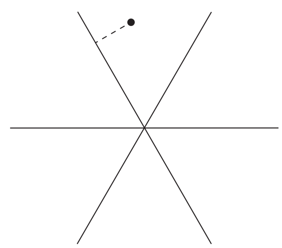
Fix a simple root α. Choose μ such that

$$\langle \mu + \rho, \alpha^\vee \rangle = 0$$
$$\langle \mu + \rho, \beta^\vee \rangle > 0 \qquad \text{for } \beta^\vee \neq \alpha^\vee \text{ simple coroot.}$$

If $\langle \alpha^\vee, \lambda \rangle \in 2\mathbb{Z}$ then a possible choice of μ is

$$\mu = \lambda - \frac{\langle \alpha^\vee, \lambda \rangle}{2} \alpha$$

This choice correspond to orthogonal projection to the wall.



Define the wall-crossing functor

$$R_\alpha := T_{\mu \to \lambda} T_{\lambda \to \mu}$$

Note the R_α does not depend on the choice of μ because of the composition rule

$$T_{\mu' \to \lambda} T_{\lambda \to \mu'} \simeq T_{\mu \to \lambda} T_{\mu' \to \mu} T_{\mu \to \mu'} T_{\lambda \to \mu} \simeq T_{\mu \to \lambda} T_{\lambda \to \mu}.$$

If λ' is another regular integral weight then there is an equivalence of categories

$$T_{\lambda \to \lambda'} : \mathfrak{g} - \mathrm{mod}_{\hat{\lambda}} \xrightarrow{\sim} \mathfrak{g} - \mathrm{mod}_{\hat{\lambda}'} .$$

and the wall-crossing functor fit into the commutative diagram.

$$\begin{tikzcd}
\mathfrak{g} - \mathrm{mod}_{\hat{\lambda}} \arrow[r, "\sim", "T_{\lambda \to \lambda'}"'] \arrow[d, "R_\alpha"'] & \mathfrak{g} - \mathrm{mod}_{\hat{\lambda}'} \arrow[d, "R_\alpha"] \\
\mathfrak{g} - \mathrm{mod}_{\hat{\lambda}} \arrow[r, "\sim", "T_{\lambda \to \lambda'}"'] & \mathfrak{g} - \mathrm{mod}_{\hat{\lambda}'}
\end{tikzcd}$$

5.5.1. *Effect of R_α on extended translation to $-\rho$.* We still assume that λ is regular integral so that $\mathcal{O}_\lambda \simeq \tilde{\mathcal{O}}_\lambda$. Set $A := \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})_+^W$ and let A^{s_α} denote the subalgebra $\mathrm{Sym}(\mathfrak{t})^{s_\alpha}/\mathrm{Sym}(\mathfrak{t})_+^W$. There is an action

$$S_\alpha \circlearrowright A-\mathrm{mod}, \qquad M \mapsto A \otimes_{A^{s_\alpha}} M = \mathrm{Ind}_{A^{s_\alpha}}^A (M|_{A^{s_\alpha}}).$$

Lemma 5.5.1. *The action of S_α corresponds to the action of R_α under $\tilde{T}_{\lambda\to-\rho}$*

$$R_\alpha \circlearrowright \mathcal{O}_{\bar{\lambda}} \xrightarrow{\tilde{T}_{\lambda\to-\rho}} \tilde{\mathcal{O}}_{-\rho} \simeq A-\mathrm{mod} \circlearrowleft S_\alpha$$

More generally,

$$R_\alpha \circlearrowright \mathfrak{g}-\mathrm{mod}_{\hat{\bar{\lambda}}} \xrightarrow{\tilde{T}_{\lambda\to-\rho}} \tilde{\mathcal{U}}_{-\rho}-\mathrm{mod}_{-\hat{\rho}} \circlearrowleft \Sigma_\alpha$$

where $\Sigma_\alpha(M) = \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} M$.

Proof. Using the composition rule we can write

$$\tilde{T}_{\lambda\to-\rho} = \tilde{T}_{\mu\to-\rho}\tilde{T}_{\lambda\to\mu}.$$

By corollary 5.3.14 we have $T_{\lambda\to\mu}T_{\mu\to\lambda} \simeq \mathrm{Res}\circ\mathrm{Ind}$ so

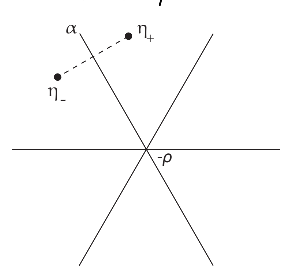
$$\tilde{T}_{\lambda\to\mu}T_{\mu\to\lambda}(M) \simeq \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} M.$$

Using this we get

$$\begin{aligned}
\tilde{T}_{\lambda\to-\rho}R_\alpha(M) &\simeq \tilde{T}_{\mu\to-\rho}\tilde{T}_{\lambda\to\mu}T_{\mu\to\lambda}T_{\lambda\to\mu}(M) \\
&\simeq \tilde{T}_{\mu\to-\rho}\Big(\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} T_{\lambda\to\mu}(M)\Big) \\
&\simeq \tilde{T}_{\mu\to-\rho}\Big(\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \tilde{T}_{\lambda\to\mu}(M)|_{\mathcal{U}_{\hat{\bar{\mu}}}}^{\tilde{\mathcal{U}}_{\hat{\bar{\mu}}}}\Big) \\
&\simeq \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \tilde{T}_{\mu\to-\rho}\tilde{T}_{\lambda\to\mu}(M)|_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}}^{\mathrm{Sym}(\mathfrak{t})} \\
&\simeq \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \tilde{T}_{\lambda\to-\rho}(M)|_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}}^{\mathrm{Sym}(\mathfrak{t})} \\
&\simeq S_\alpha\tilde{T}_{\lambda\to-\rho}(M).
\end{aligned}$$

Which is what we wanted. □

5.5.2. *Effect of R_α on Δ and ∇.* Let λ be a positive integral weight and let $\eta \in W\cdot\lambda$. Let $s_\alpha\eta$ denote the right action of W, i.e. $s_\alpha(w\cdot\lambda) = ws_\alpha\cdot\lambda$ where $\cdot$ is the usual action. Geometrically, s_α maps η to the reflection of η in the face of its chamber of type α.



Lemma 5.5.2. *There is a short exact sequence*

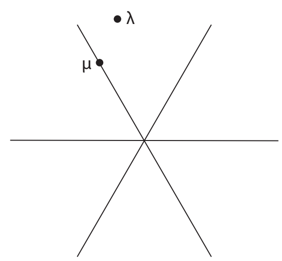
$$0 \to \Delta_{\eta_+} \to R_\alpha(\Delta_\eta) \to \Delta_{\eta_-} \to 0,$$

where $\{\eta_+, \eta_-\} = \{\eta, s_\alpha \eta\}$ *with* $\eta_+ > \eta_-$. *I.e. if* $\langle \alpha^\vee, \eta + \rho \rangle > 0$ *then* $\eta_+ = \eta$ *and* $\eta_- = s_\alpha \eta$. *Otherwise,* $\eta_+ = s_\alpha \eta$ *and* $\eta_- = \eta$.

Proof. Since λ is regular $\lambda \downarrow \mu$ so

$$T_{\lambda \to \mu}(\Delta_\eta) = \Delta_{\eta'}$$

for $\eta = w \cdot \lambda$ and $\eta' = w \cdot \mu$.



Consider $R_\alpha(\Delta_\eta) = T_{\lambda \to \mu}(\Delta_{\eta'})$. Without loss of generality we may assume that $\lambda = \mu + \alpha$ so the highest weight of V is a root conjugate to α. In this case $\eta_+ = \eta' + \alpha$ and $\eta_- = \eta' - \alpha$. By lemma 5.0.1 $\Delta_{\eta'} \otimes V$ has a filtration with terms of the form

$$\Delta_{\eta' + \nu} \otimes V[\nu]$$

Claim 5.5.3. The only weights of the form $\eta' + \nu$ with $V[\nu] \neq 0$ in $W \cdot \lambda$ are $\eta' + \alpha$ and $\eta' - \alpha$.

Proof of claim. It is enough to show that

$$\{\mu + \text{weights of } V\} \cap W \cdot \lambda = \{\mu + \alpha, \mu - \alpha\}.$$

A nonzero weight of V is a root β with $(\beta)^2 \leq (\alpha)^2$.

If $\beta > 0$ and $\beta \neq \alpha$ then $\mu + \beta \nleq \mu + \alpha = \lambda$ so $\mu + \beta \notin W \cdot \lambda$.

If $\beta < 0$ and $\beta \neq -\alpha$ then

$$\begin{aligned}(\mu + \beta)^2 &= \mu^2 + 2(\mu, \beta) + (\beta)^2 \\ &< \mu^2 + (\beta)^2 \leq \mu^2 + (\alpha)^2 \\ &= \lambda^2.\end{aligned}$$

Thus, $\mu + \beta \notin W \cdot \lambda$. □

Since $\dim V[\alpha] = 1 = \dim V[-\alpha]$ the standard filtration of $R_\alpha(\Delta_\eta)$ only have Δ_{η_+} and Δ_{η_-}. The lowest one is the submodule so we get the short exact sequence. □

Remark 5.5.4. Dualizing the short exact sequence we get

$$0 \to \nabla_{\eta_+} \to R_\alpha(\nabla_\eta) \to \nabla_{\eta_-} \to 0.$$

Notice also that

$$\begin{aligned}\operatorname{End}(R_\alpha(\Delta_\eta)) &= \operatorname{End}(\operatorname{Ind}_{\mathcal{U}}^{\tilde{\mathcal{U}}}\Delta_{\eta'})\\ &= \operatorname{End}_{k[t]/t^2}(k[t]/t^2 \otimes \Delta_{\eta'})\\ &= k[t]/t^2.\end{aligned}$$

Corollary 5.5.5. $S_\alpha := [R_\alpha] - 1$ induces an action of W on $K^0(\mathcal{O}_\lambda)$.

Proof. From the short exact sequence we get

$$[R_\alpha]([\Delta_\eta]) = [\Delta_{\eta_-}] + [\Delta_{\eta_+}] = [\Delta_\eta] + [\Delta_{s_\alpha\cdot\eta}].$$

Thus,

$$([R_\alpha] - 1)([\Delta_\eta]) = [\Delta_{s_\alpha\cdot\eta}].$$

The composition of such R_α satisfy Weyl group relations. □

5.5.3. *Translation to* $-\rho$.

Lemma 5.5.6. *For λ positive integral $\tilde{T}_{-\rho\to\lambda}(\Delta_{-\rho}) = \nabla_\lambda$.*

Proof. Recall that $T_{-\rho\to\lambda}(L_{-\rho}) \simeq P_1 =: \Xi$ and P_1 has a standard filtration with

$$\operatorname{gr}(P_1) = \bigoplus \Delta_{w\cdot\lambda}.$$

Since $\Delta_{-\rho} \simeq L_{-\rho}$ is self-dual and $T_{-\rho\to\lambda}$ commutes with duality we get that Ξ is also self-dual. Hence, Ξ has a costandard filtration with

$$\operatorname{gr}(\Xi) = \bigoplus \nabla_{w\cdot\lambda}.$$

In particular, there is a surjection $\Xi \twoheadrightarrow \nabla_\lambda$. Recall that $A = \operatorname{Sym}(\mathfrak{t})/\operatorname{Sym}(\mathfrak{t})^W_+$. Consider the exact sequence

$$\mathfrak{t}\otimes A \xrightarrow{t\otimes f\mapsto tf} A \to k \to 0$$

We have $\tilde{L}_{-\rho} = L_{-\rho}\otimes A$ so tensoring with $L_{-\rho}$ we obtain

$$\mathfrak{t}\otimes \tilde{L}_{-\rho} \to \tilde{L}_{-\rho} \to L_{-\rho} \to 0$$

Notice that

$$\Xi \simeq T_{-\rho\to\lambda}(L_{-\rho}) \simeq \tilde{T}_{-\rho\to\lambda}(\tilde{L}_{-\rho}).$$

Since $\tilde{T}_{-\rho\to\lambda}$ is right exact applying it to the exact sequence we get

$$\mathfrak{t}\otimes\Xi \to \Xi \to \tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \to 0.$$

There exists a map $\tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \to \nabla_\lambda$. The space $\operatorname{Hom}(\Xi,\nabla_\lambda)$ is 1-dimensional and a module over $\operatorname{End}(\Xi)$. The subspace $\mathfrak{t}\subset\operatorname{End}(\Xi)$ acts nilpotently so it acts by 0 on the 1-dimensional module. Therefore, the surjection $\Xi\twoheadrightarrow\nabla_\lambda$ factors as

$$\begin{tikzcd} \mathfrak{t}\otimes\Xi \arrow[r] & \Xi \arrow[r] \arrow[dr] & \tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \arrow[r] \arrow[d, dashed] & 0 \\ & & \nabla_\lambda & \end{tikzcd}$$

Hence, we get a surjection

$$\ker(\Xi \to \Delta_\lambda) \twoheadrightarrow \ker(\tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \to \nabla_\lambda).$$

Here $\ker(\tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \to \nabla_\lambda)$ maps to 0 under $T_{\lambda\to-\rho}$ but

$$T_{\lambda\to-\rho}\tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \simeq L_{-\rho} \xrightarrow{\neq 0} T_{\lambda\to-\rho}(\nabla_\lambda)$$

For any ∇_μ the only irreducible quotient is $L_1 = L_{w_0\cdot\lambda}$ and $T_{\lambda\to-\rho}(L_1) \simeq L_{-\rho} \neq 0$ so any quotient of ∇_μ which gets sent to zero under $T_{\lambda\to-\rho}$ must be zero itself. The same holds for any non-zero quotient of ∇_μ. Hence, it holds for any module with a costandard filtration.

Notice that $\ker(\Xi \to \nabla_\lambda)$ has a costandard filtration so in particular we get

$$\ker(\tilde{T}_{-\rho\to\lambda}(L_{-\rho}) \to \nabla_\lambda) = 0.$$

This finishes the proof. □

5.6. **Intertwining functors.** In this section λ is always regular. Recall that $R_\alpha = T_{\mu\to\lambda}T_{\lambda\to\mu}$. These functors are biadjoint so there are adjunction maps $\mathrm{Id} \to R_\alpha$ and $R_\alpha \to \mathrm{Id}$.

Definition 5.6.1 (Intertwining functors)**.** The intertwining functor (also called shuffling functor) is the functor

$$\Theta_\alpha : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda), \qquad M \mapsto \mathrm{Cone}(M \to R_\alpha(M)).$$

Remark 5.6.2. This comes from an exact functor on the category of complexes.

We also define the functor

$$\Theta_\alpha : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda), \qquad M \mapsto \mathrm{Cone}(R_\alpha(M) \to M)[-1].$$

Lemma 5.6.3. *In the following all the actions are the right W action.*

$$\Theta_\alpha : \Delta_\eta \mapsto \Delta_{s_\alpha\eta} \quad \text{if } s_\alpha\lambda > \lambda.$$
$$\Theta'_\alpha : \Delta_\eta \mapsto \Delta_{s_\alpha\eta} \quad \text{otherwise.}$$

Viewing the first functor as a functor $D^{\le 0} \to D^{\le 0}$ and the second functor as a functor $D^{\ge 0} \to D^{\ge 0}$

$$\Theta_\alpha(M) = (M \to R_\alpha(M)), \quad \text{with } M \text{ in degree -1 and } R_\alpha(M) \text{ in degree 0.}$$
$$\Theta'_\alpha(M) = (R_\alpha(M) \to M), \quad \text{with } R_\alpha(M) \text{ in degree 0 and } M \text{ in degree 1.}$$

Proof. Consider the short exact sequence from lemma 5.5.2

$$0 \to \Delta_{\eta_+} \to R_\alpha(\Delta_\eta) \to \Delta_{\eta_-} \to 0.$$

In the first case $\eta_+ = \eta$ and $\eta_- = s_\alpha\eta$ and in the second case $\eta_+ = s_\alpha\eta$ and $\eta_- = \eta$.

Exercise 5.6.4. (i) In case 1 the second map $R_\alpha(\Delta_\eta) \to \Delta_\eta$ is the adjunction map.
(ii) In case 2 the first map $\Delta_\eta \to R_\alpha(\Delta_\eta)$ is the adjunction map.

In the derived category the exact sequence corresponds to an exact triangle. The cone of a map is the term completing the exact triangle so by the exercise in case 1

$$\Theta_\alpha(\Delta_\eta) = \Delta_{s_\alpha \eta},$$

and in case 2

$$\Theta'_\alpha(\Delta_\eta) = \Delta_{s_\alpha \eta}.$$

Which is what we wanted. □

Corollary 5.6.5. The Θ'_α's satisfy braid group relations so they define an action of the braid group on $D^b(\mathcal{O}_\lambda)$. Moreover,

$$\Theta_\alpha \circ \Theta'_\alpha \simeq \mathrm{Id} \simeq \Theta'_\alpha \circ \Theta_\alpha.$$

Proof. The main step is to prove the following claim.

Claim 5.6.6. A functor coming from a complex of Harish-Chandra bimodules is determined by the image of (extended) Verma modules.

Proof. The functor is of the form

$$F(X) = M \otimes^L_{\mathcal{U}} X, \qquad \text{for some Harish-Chandra bimodule } M.$$

For $M \in \mathfrak{g} - \mathrm{mod}_{\hat{\lambda}}$ there exists a $n \in \mathbb{N}$ such that $\mathcal{I}^n_{\hat{\lambda}} M = 0$ so $M \otimes_{\mathcal{U}} \mathcal{U}_{\hat{\lambda}} \simeq M$.

$$(M \otimes^L_{\mathcal{U}} \mathcal{U}_{\hat{\lambda}}) \otimes^L_{\mathcal{U}} X \simeq M \otimes^L_{\mathcal{U}} (\mathcal{U}_{\hat{\lambda}} \otimes^L_{\mathcal{U}} X) \simeq M \otimes^L_{\mathcal{U}} X.$$

Thus, $M \simeq F(\mathcal{U}_{\hat{\lambda}})$. Observe that F commutes with the derived functor

$$\mathrm{CoInd} : D^b((\mathfrak{g}, B) - \mathrm{mod}) \to D^b((\mathfrak{g}, G) - \mathrm{mod})$$

For λ regular we have previously shown that

$$\mathrm{CoInd}(\Delta_\lambda(-\lambda)) \simeq \mathcal{U}_{\bar{\lambda}}$$
$$\mathrm{CoInd}(\tilde{\Delta}_{\hat{\lambda}}(-\lambda)) \simeq \mathcal{U}_{\hat{\lambda}}.$$

Hence, F is determined by $F(\tilde{\Delta}_{\hat{\lambda}}(-\lambda))$. The module $\tilde{\Delta}_{\hat{\lambda}}$ is characterized by

$$\tilde{\Delta}_{\hat{\lambda}} \otimes_{\mathcal{O}(\mathfrak{t})_{\hat{0}}} k = \Delta_\lambda.$$

This proves the claim. □

By the claim it is enough to check that

$$\Theta_\alpha \Theta'_\alpha(\Delta_\eta) \simeq \Delta_\eta \simeq \Theta'_\alpha \Theta_\alpha(\Delta_\eta) \qquad \text{for some } \eta \in W \cdot \lambda.$$

The same is true for the braid relations. Both statements follow from lemma 5.6.3. □

5.6.1. *Group actions on categories.*

Definition 5.6.7 (Group action on a category)**.** A weak action of a group Γ on a category $\mathcal{C}$ is a collection of functors $\{F_\gamma\}_{\gamma\in\Gamma}$ satisfying that for all $\gamma_1,\gamma_2\in\Gamma$ there exists isomorphisms.

$$\phi_{\gamma_1,\gamma_2}: F_{\gamma_1\gamma_2}\simeq F_{\gamma_1}F_{\gamma_2}.$$
$$\phi_0:\mathrm{Id}\simeq F_e$$

An action is called strong if we fix the isomorphism for all γ_1,γ_2 in such a way that for all $\gamma_1,\gamma_2,\gamma_3\in\Gamma$ we have commutative diagrams.

$$\begin{tikzcd} F_{\gamma_1\gamma_2\gamma_3} \arrow[r, "\phi_{\gamma_1\gamma_2,\gamma_3}F_{\gamma_3}", "\sim"'] \arrow[d, "\phi_{\gamma_1,\gamma_2\gamma_3}"', "\wr"] & F_{\gamma_1\gamma_2}F_{\gamma_3} \arrow[d, "\phi_{\gamma_1,\gamma_2}F_{\gamma_3}", "\wr"'] \\ F_{\gamma_1}F_{\gamma_2\gamma_3} \arrow[r, "\sim", "F_{\gamma_1}\phi_{\gamma_2,\gamma_3}"'] & F_{\gamma_1}F_{\gamma_2}F_{\gamma_3} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Id}\,F_\gamma = F_\gamma = F_\gamma\,\mathrm{Id} \arrow[r, "F_\gamma\phi_0", "\sim"'] \arrow[d, "\phi_0 F_\gamma"', "\wr"] \arrow[dr, equal] & F_\gamma F_e \arrow[d, "\phi_{\gamma,e}"', "\wr"] \\ F_eF_\gamma \arrow[r, "\sim", "\phi_{e,\gamma}"'] & F_\gamma \end{tikzcd}$$

Deligne in [De] explains a practical way to check that a given collection of functors generates a strong action of a braid group on a category. It can be used to check that actions considered in the course are in fact strong actions; we will not get into this issue and only use the weak actions instead.

6. Description of category $\mathcal{O}$ a la Soergel: Struktursatz

We already know that

$$\mathcal{O}_\lambda\simeq A-\mathrm{mod}_{f.d.}$$

for $A=\mathrm{End}(P)^{opp}$ with $P=\oplus_{w\in W}P_{w\cdot\lambda}$ (or any other projective generator).

Proposition 6.0.1. *The functor $\tilde{T}_{\lambda\to-\rho}$ is fully faithful on projectives.*

For the proof we need the following lemma

Lemma 6.0.2. *If P is projective then*

$$P\hookrightarrow T_{-\rho\to\lambda}T_{\lambda\to-\rho}(P)$$

and the cokernel has a standard filtration.

The lemma will be proved in section 6.2.

Proof of proposition 6.0.1. We want to prove that if P_1 and P_2 are projective

$$\mathrm{Hom}(P_1,P_2)\xrightarrow{\sim}\mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1),\tilde{T}_{\lambda\to-\rho}(P_2))$$

We will use the following claim

Claim 6.0.3. (1) If Y has a standard filtration or X has a costandard filtration then

$$\mathrm{Hom}(X,Y)\hookrightarrow\mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(X),\tilde{T}_{\lambda\to-\rho}(Y))$$

(2) If $\mathcal{A}$ is an Abelian category, $\mathcal{B}$ a Serre subcategory and I an injective hull of an irreducible $L \notin \mathcal{B}$ then
$$\mathrm{Hom}_{\mathcal{A}}(X, I) \simeq \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, I).$$

Proof of part (1) of claim. Let $f : X \to Y$ be a non-zero map. Then
$$\mathrm{Im}(\tilde{T}_{\lambda\to-\rho}(f)) = \tilde{T}_{\lambda\to-\rho}(\mathrm{Im}(f)).$$
Any non-zero submodule in Δ_η contains $L_1 = L_{w_0\cdot\lambda}$. Hence, so does any submodule in Y if Y has a standard filtration. In particular, $\mathrm{Im}(f)$ contains L_1 so
$$0 \neq \tilde{T}_{\lambda\to-\rho}(L_1) \subset \tilde{T}_{\lambda\to-\rho}(\mathrm{Im}(f)).$$
Thus, $\tilde{T}_{\lambda\to-\rho}(f)$ is non-zero. □

By the lemma there is a short exact sequence
$$0 \to P_2 \to T_{-\rho\to\lambda}T_{\lambda\to-\rho}(P_2) \to C \to 0,$$
where C has a standard filtration. This gives a short exact sequence
$$0 \to \mathrm{Hom}(P_1, P_2) \to \mathrm{Hom}(P_1, T_{-\rho\to\lambda}T_{\lambda\to-\rho}(P_2)) \to \mathrm{Hom}(P_1, C) \to 0.$$
The only indecomposable projective in $\mathcal{O}_{-\rho}$ is $\Delta_{-\rho} \simeq L_{-\rho}$ and since $T_{\lambda\to-\rho}$ sends projectives to projectives we have
$$T_{-\rho\to\lambda}T_{\lambda\to-\rho}(P_2) \simeq T_{-\rho\to\lambda}(\Delta_{-\rho}^{\oplus d}) \simeq \Xi^{\oplus d}.$$
The functor $\tilde{T}_{\lambda\to-\rho}$ is exact so we get an exact sequence
$$0 \to \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(P_2)) \to \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(\Xi^{\oplus d})) \to \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(C))$$
Since Ξ is the projective cover of L_1 and it is self-dual it is also the injective hull of L_1. Hence, by part (2) of the claim
$$\mathrm{Hom}(X, \Xi) \simeq \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(X), \tilde{T}_{\lambda\to-\rho}(\Xi^{\oplus d})).$$
Using part 1 of the claim
$$\begin{tikzcd}
0 \to \mathrm{Hom}(P_1, P_2) \arrow[r] \arrow[d, hook] & \mathrm{Hom}(P_1, \Xi^{\oplus d}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}(P_1, C) \to 0 \arrow[d, hook] \\
0 \to \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(P_2)) \arrow[r] & \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(\Xi^{\oplus d})) \arrow[r] & \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(C))
\end{tikzcd}$$
Applying the 5 lemma proves that $\mathrm{Hom}(P_1, P_2) \simeq \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(P_2))$. □

6.1. **Recap on Serre quotient categories.**

Definition 6.1.1. Let $\mathcal{A}$ be an Abelian category and $\mathcal{B}$ a Serre subcategory, i.e. a full subcategory closed under subquotients and extensions. The Serre quotient category $\mathcal{A}/\mathcal{B}$ has Ob($\mathcal{A}/\mathcal{B}$)=Ob($\mathcal{A}$). Write $\bar{X}$ for $X \in \mathcal{A}$ considered as an object in $\mathcal{A}/\mathcal{B}$. A map $\bar{X} \to \bar{Y}$ is represented by

$$\begin{tikzcd} & X' \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \qquad \text{for some } X' \in \mathcal{A} \text{ with } \ker, \operatorname{coker}(X' \to X) \in \mathcal{B}$$

or

$$\begin{tikzcd} & Y' & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd} \qquad \text{for some } Y' \in \mathcal{A} \text{ with } \ker, \operatorname{coker}(Y \to Y') \in \mathcal{B}$$

up to the equivalence relation

$$\begin{tikzcd} & Y' & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd} \sim \begin{tikzcd} & Y'' & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

if there exists another triple

$$\begin{tikzcd} & Y''' & \\ X \arrow[ur] & & Y \arrow[ul] \end{tikzcd}$$

and maps $Y' \to Y'''$ and $Y'' \to Y'''$ such that the following diagram is commutative

$$\begin{tikzcd} & Y' \arrow[d] & \\ X \arrow[ur] \arrow[r] \arrow[dr] & Y''' & Y \arrow[ul] \arrow[l] \arrow[dl] \\ & Y'' \arrow[u] & \end{tikzcd}$$

A similar equivalence relation is imposed for the triple with the arrows in the opposite direction.

There is a natural map $\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(\bar{X}, \bar{Y})$ given by

$$f : X \to Y \qquad \mapsto \qquad \begin{tikzcd} & Y & \\ Y \arrow[ur, "\mathrm{Id}"] & & X \arrow[ul, "f"'] \end{tikzcd}$$

It is universal (in the appropriate sense) among exact functors from $\mathcal{A}$ to an abelian category sending $\mathcal{B}$ to zero. In particular, an object in $\mathcal{A}$ goes to zero in $\mathcal{A}/\mathcal{B}$ iff it lies in $\mathcal{B}$; an arrow in $\mathcal{A}$ goes to zero iff its image lies in $\mathcal{B}$.

$$
\begin{aligned}
{}^{\perp}\mathcal{B} &:= \{X \in \mathcal{A} \mid \mathrm{Hom}(X, B) = 0 \ \ \forall B \in \mathcal{B}\} \\
\mathcal{B}^{\perp} &:= \{X \in \mathcal{A} \mid \mathrm{Hom}(B, X) = 0 \ \ \forall B \in \mathcal{B}\}
\end{aligned}
$$

Lemma 6.1.2. *There map* $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\bar{X}, \bar{Y})$ *is an isomorphism if either*

i) $X \in {}^{\perp}\mathcal{B}$ *and* X *is projective in* $\mathcal{A}$ *or* $Y \in \mathcal{B}^{\perp}$ *and* Y *is injective in* $\mathcal{A}$.
ii) $X \in {}^{\perp}\mathcal{B}$ *and* $Y \in \mathcal{B}^{\perp}$.

Proof. i) First we show injectivity. Assume that $f : X \to Y$ goes to 0. This is equivalent to $\mathrm{Im} f \in \mathcal{B}$. By assumption $X \in {}^{\perp}\mathcal{B}$ so the map $X \to \mathrm{Im} f$ is 0 and so $f = 0$.

Next, we show surjectivity. Let $\Phi \in \mathrm{Hom}(\bar{X}, \bar{Y})$ with representative

$$
\begin{tikzcd}
& Y' & \\
X \arrow[ur, "f"] & & Y \arrow[ul]
\end{tikzcd}
$$

Consider the exact sequence

$$0 \to \ker(Y \to Y') \to Y \to Y' \to \mathrm{coker} \to 0$$

Since X is projective this gives an exact sequence of Hom's; since $X \in {}^{\perp}\mathcal{B}$ and ker, coker $\in \mathcal{B}$, we have $\mathrm{Hom}(X, \ker(Y \to Y')) = 0 = \mathrm{Hom}(X, \mathrm{coker})$, so $\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(X, Y')$. Therefore, f factors through Y

$$
\begin{tikzcd}
& Y' & \\
X \arrow[ur, "f"] \arrow[rr, dashed, "\tilde{f}"'] & & Y \arrow[ul]
\end{tikzcd}
$$

Thus, we get a map $\tilde{f} \in \mathrm{Hom}(\bar{X}, \bar{Y})$. The other part of i) follows by dualizing.

ii) The proof of part i) shows that the map is injective. Let $\Phi \in \mathrm{Hom}(\bar{X}, \bar{Y})$ with representative

$$
\begin{tikzcd}
& X' \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}
$$

By assumption $X \in {}^{\perp}\mathcal{B}$ so $\mathrm{Hom}(X, \mathrm{coker}(X' \to X)) = 0$ and hence, $X' \twoheadrightarrow X$. Likewise, $Y \in \mathcal{B}^{\perp}$ so $\mathrm{Hom}(\ker(X' \to X), Y) = 0$. Thus, the map $X' \to Y$ factors through

$$X' / \ker(X' \to X) \simeq X.$$

This finishes the proof. $\square$

6.2. **Back to category $\mathcal{O}$.** We now prove lemma 6.0.2. For this we need the following

Claim 6.2.1. The functor $T_{\lambda \to -\rho}$ does not kill any nonzero submodules of a module with a standard filtration.

Proof. Let N be a nonzero submodule in M and $0 \subset M_1 \subset \cdots \subset M_n \subset M$ a standard filtration for M. Choose m such that $N \subset M_{m+1}$ but $N \not\subset M_m$. Then

$$0 \neq N/(N \cap M_k) \hookrightarrow \Delta_{\mu_k}.$$

Using proposition 4.1.1 we get that $\Delta_1 \subset N/(N \cap M_k)$ and since $T_{\lambda \to -\rho}$ is exact $\Delta_{-\rho} \simeq T_{\lambda\to-\rho}(\Delta_{\mu_k}) \hookrightarrow T_{\lambda\to-\rho}(N)/T_{\lambda\to-\rho}(N \cap M_k)$. □

Proof of lemma 6.0.2. Recall that $A := \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})^W_+$ and $R_\alpha := T_{-\rho\to\lambda}T_{\lambda\to-\rho}$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_\lambda \arrow[r, "R_\alpha"] \arrow[d, "\tilde{T}_{\lambda\to-\rho}"'] & \mathcal{O}_\lambda \arrow[d, "\tilde{T}_{\lambda\to-\rho}"] \\ A-\mathrm{mod} \arrow[r, "S_\alpha"'] & A-\mathrm{mod} \end{tikzcd}$$

The map $\tilde{T}_{\lambda\to-\rho}(M \to R_\alpha(M))$ is injective so $T_{\lambda\to-\rho}(\ker(M \to R_\alpha(M))) = 0$. By the claim this implies that $\ker(M \to R_\alpha(M)) = 0$.

It remains to show that coker has a standard filtration. By proposition 3.0.16 it is enough to prove that

$$\mathrm{Ext}^1(\mathrm{coker}, \nabla_\mu) = 0 \qquad \forall \mu.$$

Using the short exact sequence

$$0 \to P \to R_\alpha(P) \to \mathrm{coker} \to 0$$

we get

$$\mathrm{Hom}(R_\alpha(P), \nabla_\mu) \to \mathrm{Hom}(P, \nabla_\mu) \to \mathrm{Ext}^1(\mathrm{coker}, \nabla_\mu) \to 0.$$

We want to prove that the map $\mathrm{Hom}(R_\alpha(P), \nabla_\mu) \to \mathrm{Hom}(P, \nabla_\mu)$ is surjective. By adjunction we have

$$\begin{tikzcd} \mathrm{Hom}(R_\alpha(P), \nabla_\mu) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(P, \nabla_\mu) \\ \mathrm{Hom}(P, R_\alpha(\nabla_\mu)) \arrow[ur] & \end{tikzcd}$$

Since P is projective to show that $\mathrm{Hom}(P, R_\alpha(P)) \twoheadrightarrow \mathrm{Hom}(P, \nabla_\mu)$ it is enough to show that $R_\alpha(\nabla_\mu) \twoheadrightarrow \nabla_\mu$. This is true because $T_{\lambda\to-\rho}(R_\alpha(\nabla_\mu) \to \nabla_\mu)$ is onto and $T_{\lambda\to-\rho}$ does not kill any quotient of ∇_μ. □

6.2.1. *A second proof of proposition 6.0.1.*

Definition 6.2.2 (Tilting)**.** An object in $\mathcal{O}_\lambda$ is tilting if it has both a standard and a costandard filtration.

Example 6.2.3. The modules $\Delta_1 \simeq \nabla_1$ and $\Xi \simeq T_{-\rho\to\lambda}(\Delta_{-\rho})$ are tilting.

The group W acts on $A = \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})^W_+$ so it acts by autoequivalences on the category $A-\mathrm{mod} \simeq \tilde{\mathcal{O}}_{-\rho}$ and on $D^b(\tilde{\mathcal{O}}_{-\rho})$.

Lemma 6.2.4. *a) If T_1, T_2 are tilting then*

$$\mathrm{Hom}_{\tilde{\mathcal{O}}_\lambda}(T_1, T_2) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{\mathcal{O}}_{-\rho}}(\tilde{T}_{\lambda\to-\rho}(T_1), \tilde{T}_{\lambda\to-\rho}(T_2)).$$

b) Let $w_0 = s_{\alpha_1} \cdots s_{\alpha_n}$ be a minimal expression for the longest element in W. Then

$$\Theta_{w_0} := \Theta_{s_{\alpha_1}} \circ \cdots \circ \Theta_{s_{\alpha_n}}$$

sends projectives to tiltings.

c) The following diagram is commutative

$$\begin{tikzcd} D^b(\mathcal{O}_\lambda) \arrow[r, "\Theta_{s_\alpha}"] \arrow[d, "\tilde{T}_{\lambda\to-\rho}"'] & D^b(\mathcal{O}_\lambda) \arrow[d, "\tilde{T}_{\lambda\to-\rho}"] \\ D^b(\tilde{\mathcal{O}}_{-\rho}) \arrow[r, "s_\alpha"] & D^b(\tilde{\mathcal{O}}_{-\rho}) \end{tikzcd}$$

Proof of proposition 6.0.1 assuming the lemma. Let P_1, P_2 be projective. Since Θ_{w_0} is an equivalence,

$$\mathrm{Hom}(P_1, P_2) \simeq \mathrm{Hom}(\Theta_{w_0}(P_1), \Theta_{w_0}(P_2)).$$

By part b) $T_1 := \Theta_{w_0}(P_1)$ and $T_2 := \Theta_{w_0}(P_2)$ are tilting so by part a)

$$\mathrm{Hom}(T_1, T_2) \simeq \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(T_1), \tilde{T}_{\lambda\to-\rho}(T_2)).$$

Using part c) we get

$$\begin{aligned} \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(T_1), \tilde{T}_{\lambda\to-\rho}(T_2)) &\simeq \mathrm{Hom}(w_0(\tilde{T}_{\lambda\to-\rho}(P_1)), w_0(\tilde{T}_{\lambda\to-\rho}(P_2))) \\ &\simeq \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(P_2)). \end{aligned}$$

Thus, we have shown that

$$\mathrm{Hom}(P_1, P_2) \simeq \mathrm{Hom}(\tilde{T}_{\lambda\to-\rho}(P_1), \tilde{T}_{\lambda\to-\rho}(P_2)).$$

Which is what we wanted. □

Proof of lemma. a) From corollary 5.3.13 we have

$$\begin{tikzcd} \mathcal{O}_{\bar{\lambda}} \arrow[rr, "\tilde{T}_{\lambda\to-\rho}"] \arrow[dr] & & \tilde{\mathcal{O}}_{-\rho} \\ & \mathcal{O}_{\bar{\lambda}}/\ker(T_{\lambda\to-\rho}) \arrow[ur, "\sim"'] & \end{tikzcd}$$

In the proof of proposition 5.0.10 we proved that

$$T_{\lambda\to-\rho}(L) = \begin{cases} 0 & \text{if } L \neq L_1 \\ \Delta_{-\rho} = L_{-\rho} & \text{if } L = L_1 \end{cases}$$

so $\ker(T_{\lambda\to-\rho}) = \langle L_i \mid L_i \neq L_1 \rangle$.

Claim 6.2.5. We have $T_1 \in {}^\perp\ker(T_{\lambda\to-\rho})$ and $T_2 \in \ker(T_{\lambda\to-\rho})^\perp$.

Proof of claim. Let $N \in \ker(T_{\lambda\to-\rho}) = \langle L_i \mid L_i \neq L_1 \rangle$. Using induction on the length of the Jordan Hölder series of N we reduce the first part to showing that

$$\mathrm{Hom}(T_1, L_\mu) = 0 \qquad \forall L_\mu \neq L_1.$$

For this part of the claim we only need the costandard filtration on T_1. Using induction on the length of the costandard filtration we get that it is enough to show that

$$\mathrm{Hom}(\nabla_\lambda, L_\mu) = 0 \qquad \forall L_\mu \neq L_1.$$

Notice that $\mathrm{Hom}(\nabla_\lambda, L_\mu) = \mathrm{Hom}(L_\mu, \Delta_\lambda)$. If $\mathrm{Hom}(L_\mu, \Delta_\lambda) \neq 0$ this would mean that $L_\mu \hookrightarrow \Delta_\lambda$, but by proposition 4.1.1 L_1 is the only irreducible submodule. This proves the first part of the claim. The second part follows from the first by dualizing. □

part a) now follows from lemma 6.1.2.

c) From lemma 5.5.1 we have the commutative diagram

$$\begin{tikzcd} \mathcal{O}_\lambda \arrow[r, "R_\alpha"] \arrow[d, "\tilde{T}_{\lambda\to-\rho}"'] & \mathcal{O}_\lambda \arrow[d, "\tilde{T}_{\lambda\to-\rho}"] \\ A-\mathrm{mod} \simeq \tilde{\mathcal{O}}_{-\rho} \arrow[r, "M\mapsto A\otimes_{A^{s_\alpha}} M"'] & \tilde{\mathcal{O}}_{-\rho} \end{tikzcd}$$

We need to calculate $\mathrm{Cone}(M \to A \otimes_{A^{s_\alpha}} M)$.

$$\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \mathrm{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*//\{1,s_\alpha\}} \mathfrak{t}^*)$$

There are short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(\{(x, s_\alpha(x)) \mid x \in \mathfrak{t}^*\}) \arrow[r] & \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*//\{1,s_\alpha\}} \mathfrak{t}^*) \arrow[r] \arrow[d, no head, "\wr"] & \mathcal{O}(\{(x, x) \mid x \in \mathfrak{t}^*\}) \arrow[r] \arrow[d, no head, "\wr"] & 0 \\ & & \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \mathrm{Sym}(\mathfrak{t}) & \mathrm{Sym}(\mathfrak{t}) & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(\{(x, x) \mid x \in \mathfrak{t}^*\}) \arrow[r] \arrow[d, no head, "\wr"] & \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*//\{1,s_\alpha\}} \mathfrak{t}^*) \arrow[r] \arrow[d, no head, "\wr"] & \mathcal{O}(\{(x, s_\alpha(x)) \mid x \in \mathfrak{t}^*\}) \arrow[r] & 0 \\ & \mathrm{Sym}(\mathfrak{t}) & \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \mathrm{Sym}(\mathfrak{t}) & & \end{tikzcd}$$

Tensoring with k over $\mathrm{Sym}(\mathfrak{t})^W$ we get

$$0 \to A \to A \otimes_{A^{s_\alpha}} A \to A_{s_\alpha} \to 0\ ,$$

where

$$A_{s_\alpha} := \mathcal{O}(\{(x, s_\alpha(x)) \mid x \in \mathfrak{t}^*\}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} k \simeq \mathcal{O}(\{(x, s_\alpha(x)) \mid x \in \mathfrak{t}^*\}) / \mathrm{Sym}(\mathfrak{t})^W.$$

Thus, $\mathrm{Cone}(M \to A \otimes_{A^{s_\alpha}} M) \simeq A_{s_\alpha} \otimes_A M$. Tensoring with the bimodule A_{s_α} changes the A-module structure on M. We obtain

$$A_{s_\alpha} \otimes_A M \simeq \mathcal{O}(\{(x, s_\alpha(x)) \mid x \in \mathfrak{t}^*\}) / (\mathrm{Sym}(\mathfrak{t})^W_+) \otimes_{\mathcal{O}(\{(x,x) \mid x \in \mathfrak{t}^*\})/(\mathrm{Sym}(\mathfrak{t})^W_+)} M \simeq s_\alpha(M).$$

Which is what we wanted.

b) To prove b) we need the following lemma

Lemma 6.2.6. *Let $\{standard\}$ be the set of modules with a standard filtration and $\{costandard\}$ the set of modules with a costandard filtration. Then we have a bijection*

$$\Theta_{w_0} : \{standard\} \xrightarrow{\sim} \{costandard\}.$$

For $w \in W$ write $\Delta_w := \Delta_{w(-\lambda^\vee)}$ using the right W action and $-\lambda^\vee = w_0(\lambda)$. Then the Bruhat order on W corresponds to the standard order on weights.

Proof of lemma 6.2.6. Since Θ is exact it is enough to prove that it takes Vermas to dual Vermas. Using lemma 5.6.3 we get

$$\begin{aligned}\Delta_w &= \Theta'_{w_0}(\Delta_1).\\ \nabla_w &= \Theta_{w_0}(\nabla_1).\end{aligned}$$

We want to show that $\Theta_{w_0}(\Delta_w) = \nabla_{ww_0}$. Set $w_1 := ww_0$. Then $w_0 = w^{-1}w_1$ and $\ell(w_0) = \ell(w^{-1}) + \ell(w_1)$ so

$$\Theta_{w_0} = \Theta_{w_1} \circ \Theta_{w^{-1}}.$$

Using this and corollary 5.6.5 we get

$$\begin{aligned}\Theta_{w_0}(\Delta_w) &\simeq \Theta_{w_1} \circ \Theta_{w^{-1}}(\Theta'_w(\Delta_1))\\ &\simeq \Theta_{w_1}(\Delta_1) \simeq \Theta_{w_1}(\nabla_1)\\ &\simeq \nabla_{w_1}.\end{aligned}$$

Which is what we wanted. □

A projective P has a standard filtration so by the lemma $\Theta_{w_0}(P)$ is in degree 0 and has a costandard filtration. By proposition 3.0.16 the module $\Theta_{w_0}(P)$ has a standard filtration iff

$$\operatorname{Ext}^1(\Theta_{w_0}(P), \nabla_\mu) = 0 \qquad \forall \mu.$$

By lemma 6.2.6 one can write ∇_μ as $\Theta_{w_0}(\Delta_\nu)$ for some ν so

$$\operatorname{Ext}^1(\Theta_{w_0}(P), \nabla_\mu) \simeq \operatorname{Ext}^1(\Theta_{w_0}(P), \Theta_{w_0}(\Delta_\nu)) \simeq \operatorname{Ext}^1(P, \Delta_\nu) = 0.$$

This finishes the proof. □

Exercise 6.2.7.

(1) Prove that there is bijection $\Theta_{w_0} : \text{Projectives} \xrightarrow{\sim} \text{Tiltings}$. I.e. prove that

$$\Theta'_{w_0} : \text{Tiltings} \xrightarrow{\sim} \text{Projectives}$$

is the inverse.

Remark 6.2.8. In particular, the indecomposable tilting objects are in bijection with the irreducibles so they also generate $D^b(\mathcal{O}_\lambda)$. This is true in any highest weight category.

(2) Let $w, v \in W$. The character of $\Theta_w(\Delta_v) \in \mathcal{O}_\lambda$ is the same as the character of Δ_{wv}. E.g.

$$\begin{aligned}\Theta_w(\Delta_e) &= \nabla_w.\\ \Theta_w(\Delta_{w_0}) &= \Delta_{ww_0}.\end{aligned}$$

These are called shuffled Vermas.

6.2.2. *A projective generator for $\mathcal{O}_\lambda$ for λ regular.*

Lemma 6.2.9. *Assume that $s_\alpha \cdot \mu > \mu$ with the right action. Then in $K^0(\mathcal{O}_\lambda)$*

$$[R_\alpha(L_\mu)] = [L_{s_\alpha\cdot\mu}] + \sum_{\nu < s_\alpha\cdot\mu} d_\nu [L_\nu],$$

where d_ν is some coefficient.

Proof. The proof goes by induction. For $\mu = \lambda_{\min}$ we have $L_\mu \simeq \Delta_\mu$ so the short exact sequence

$$0 \to \Delta_{\mu_+} \to R_\alpha(\Delta_\mu) \to \Delta_{\mu_-} \to 0.$$

gives

$$[R_\alpha(L_\mu)] = [L_\mu] + [\Delta_{s_\alpha\cdot\mu}] = [L_{s_\alpha\cdot\mu}] + 2[L_\mu].$$

For μ arbitrary with $s_\alpha \cdot \mu > \mu$ we still have

$$[R_\alpha(\Delta_\mu)] = [\Delta_{s_\alpha\cdot\mu}] + [\Delta_\mu].$$

By exactness of R_α the short exact sequence

$$0 \to \ker_\mu \to \Delta_\mu \to L_\mu \to 0$$

implies

$$[R_\alpha(\Delta_\mu)] = [R_\alpha(L_\mu)] + [R_\alpha(\ker_\mu)].$$

Hence, we get

$$[R_\alpha(L_\mu)] = [\Delta_\mu] + [\Delta_{s_\alpha\cdot\mu}] - [R_\alpha(\ker_\mu)].$$

By induction

$$[R_\alpha(\ker_\mu)] = \sum_{\nu < s_\alpha\cdot\mu} d'_\nu [L_\nu].$$

Recall that for any λ

$$[\Delta_\lambda] = [L_\lambda] + \sum_{\nu<\lambda} a_\nu [L_\nu].$$

Thus,

$$[R_\alpha(L_\mu)] = [L_{s_\alpha\cdot\mu}] + \sum_{\nu < s_\alpha\cdot\mu} d_\nu [L_\nu].$$

Which is what we wanted. □

Corollary 6.2.10. Every indecomposable projective is a direct summand in a module $R_{\alpha_1}\cdots R_{\alpha_n}(\Delta_{w_0})$. More precisely, if P_w is the projective cover of L_w and we have a minimal decomposition $w_0 w = s_{\alpha_1}\cdots s_{\alpha_n}$ then P_w is a direct summand in $R_{\alpha_1}\cdots R_{\alpha_n}(\Delta_{w_0})$.

Proof. The R_α are exact and self-adjoint so they send projectives to projectives. $\Delta_{w_0} = \Delta_{w_0\cdot w_0(\lambda)} = \Delta_\lambda$ is projective, so we only need to find $\alpha_1, \dots, \alpha_n$ such that $R_{\alpha_1}\cdots R_{\alpha_n}(\Delta_{w_0})$ contains the indecomposable projective P_w. Pick a minimal decomposition $w_0 w = s_{\alpha_1}\cdots s_{\alpha_n}$. Then it is enough show that

$$\mathrm{Hom}(R_{\alpha_1}\cdots R_{\alpha_n}(\Delta_{w_0}), L_w) \neq 0.$$

By adjunction

$$\mathrm{Hom}(R_{\alpha_1}\cdots R_{\alpha_n}(\Delta_{w_0}), L_w) = \mathrm{Hom}(\Delta_{w_0}, R_{\alpha_n}\cdots R_{\alpha_1}(L_w)).$$

The proof of a more general version of BGG reciprocity ([Hum, Theorem 3.11]) shows that
$$\dim \mathrm{Hom}(\Delta_{w_0}, R_{\alpha_n}\cdots R_{\alpha_1}(L_w)) = [L_{w_0} : R_{\alpha_n}\cdots R_{\alpha_1}(L_w)].$$
Since $ws_{\alpha_n}\cdots s_{\alpha_1} = w_0$ the lemma shows that
$$\begin{aligned}[R_{\alpha_1}\cdots R_{\alpha_n}(L_w)] &= [L_{ws_{\alpha_n}\cdots s_{\alpha_1}}] + \text{lower terms}\\ &= [L_{w_0}] + \text{lower terms}.\end{aligned}$$
Hence,
$$\dim \mathrm{Hom}(R_{\alpha_1}\cdots R_{\alpha_n}(\Delta_{w_0}), L_w) = [L_{w_0}, R_{\alpha_1}\cdots R_{\alpha_n}(L_w)] = 1.$$
This finishes the proof. $\square$

Corollary 6.2.11. (1) Fix a minimal decomposition $w_0 w = s_{\alpha_1^w}\cdots s_{\alpha_n^w}$ for each $w \in W$. Then
$$P := \bigoplus_w R_{\alpha_1^w}\cdots R_{\alpha_n^w}(\Delta_{w_0})$$
is a projective generator for $\tilde{\mathcal{O}}_\lambda \simeq \mathcal{O}_\lambda$.
(2) Set $M := \bigoplus_w A \otimes_{A^{s_{\alpha_1^w}}} A \otimes_{A^{s_{\alpha_2^w}}} \cdots \otimes_{A^{s_{\alpha_{n-1}^w}}} A \otimes_{A^{s_{\alpha_n^w}}} k$ and $\mathcal{A} := \mathrm{End}(M)^{opp}$. Then
$$\mathcal{O}_\lambda \simeq \mathrm{End}(P)^{opp} - \mathrm{mod}_{f.g.} \simeq \mathcal{A} - \mathrm{mod}_{f.g.}$$

Proof. Part (1) follows directly from the lemma.
2) By lemma 5.5.1
$$\tilde{T}_{\lambda\to-\rho}(P) \simeq \bigoplus_w A \otimes_{A^{s_{\alpha_1^w}}} A \otimes_{A^{s_{\alpha_2^w}}} \cdots \otimes_{A^{s_{\alpha_{n-1}^w}}} A \otimes_{A^{s_{\alpha_n^w}}} \tilde{T}_{\lambda\to-\rho}(\Delta_{w_0}).$$
Under the equivalence $\tilde{\mathcal{O}}_{-\rho} \xrightarrow{\sim} A - \mathrm{mod}$ the module $\Delta_{-\rho} \simeq \tilde{T}_{\lambda\to-\rho}(\Delta_{w_0})$ gets send to k so
$$\tilde{T}_{\lambda\to-\rho}(P) \simeq \bigoplus_w A \otimes_{A^{s_{\alpha_1^w}}} A \otimes_{A^{s_{\alpha_2^w}}} \cdots \otimes_{A^{s_{\alpha_{n-1}^w}}} A \otimes_{A^{s_{\alpha_n^w}}} k.$$
Since $\tilde{T}_{\lambda\to-\rho}$ is fully faithful on projectives we get
$$\mathrm{End}(P) \simeq \mathrm{End}(\tilde{T}_{\lambda\to-\rho}(P)) \simeq \mathrm{End}(M)$$
This finishes the proof. $\square$

6.3. **Other versions of category $\mathcal{O}_\lambda$.** Recall that the usual category $\mathcal{O}_\lambda$ is defined as
$$\mathcal{O}_\lambda := \left\{\begin{matrix}\mathfrak{n}-\text{integrable}\\ \text{modules } M\end{matrix} \,\middle|\, \begin{matrix}\text{The action of } \mathfrak{t} \text{ on } M \text{ is diagonalizable}\\ Z \text{ acts by generalized central character } \bar{\lambda}\end{matrix}\right\}$$
Write $\mathcal{O}_\lambda^\wedge := \mathcal{O}_\lambda$. We can also consider the following related categories
$$^\wedge\mathcal{O}_\lambda := \left\{\begin{matrix}\mathfrak{n}-\text{integrable}\\ \text{modules } M\end{matrix} \,\middle|\, \begin{matrix}M = \oplus\text{generalized eigenspaces of } \mathfrak{t}\\ Z \text{ acts by character } \bar{\lambda}\end{matrix}\right\}$$
$$^\wedge\mathcal{O}_\lambda^\wedge := \left\{\begin{matrix}\mathfrak{n}-\text{integrable}\\ \text{modules } M\end{matrix} \,\middle|\, \begin{matrix}M = \oplus\text{generalized eigenspaces of } \mathfrak{t}\\ Z \text{ acts by generalized central character } \bar{\lambda}\end{matrix}\right\}$$
$$\bar{\mathcal{O}}_\lambda := \left\{\begin{matrix}\mathfrak{n}-\text{integrable}\\ \text{modules } M\end{matrix} \,\middle|\, \begin{matrix}\text{The action of } \mathfrak{t} \text{ on } M \text{ is diagonalizable}\\ Z \text{ acts by character } \bar{\lambda}\end{matrix}\right\}$$

These categories have descriptions similar to the one for $\mathcal{O}_\lambda$. Consider the modules for

$$\tilde{A} := \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*//W} \mathfrak{t}^*) = \mathcal{O}\Big(\bigcup_w \Gamma_w\Big),$$

where Γ_w is the graph of w acting on $\mathfrak{t}^*$. Notice that $A \simeq \tilde{A} \otimes_{\mathrm{Sym}(\mathfrak{t})} k$ and consider the functor

$$R_\alpha : \tilde{A} - \mathrm{mod} \to \tilde{A} - \mathrm{mod}, \qquad M \mapsto \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} M.$$

Starting with $\mathcal{O}_\Delta = \mathcal{O}(\Gamma_1)$ and applying R_α's we get

$$\tilde{M} := \bigoplus_{w = s^w_{\alpha_1} \cdots s^w_{\alpha_n}} R_{\alpha_1} \cdots R_{\alpha_n}(\mathcal{O}_\Delta)$$

Set $\tilde{\mathcal{A}} := \mathrm{End}(\tilde{M})^{opp}$

$$^\wedge\mathcal{O}^\wedge_\lambda \simeq \tilde{\mathcal{A}} - \mathrm{mod}^{f.d.}_{\mathrm{nil}} \subset \tilde{\mathcal{A}} - \mathrm{mod}\,.$$

Here the subscript nil means the modules on which $\mathcal{O}(\mathfrak{t}^*//W)_+$ acts nilpotently. The other variations of $\mathcal{O}_\lambda$ have similar descriptions.

$$\mathcal{O}^\wedge_\lambda \simeq k \otimes_{\mathrm{Sym}(\mathfrak{t})} \tilde{\mathcal{A}} - \mathrm{mod}_{f.d.},$$
$$^\wedge\mathcal{O}_\lambda \simeq \tilde{\mathcal{A}} \otimes_{\mathrm{Sym}(\mathfrak{t})} k - \mathrm{mod}_{f.d.},$$
$$\bar{\mathcal{O}}_\lambda \simeq \tilde{\mathcal{A}} \otimes_{\tilde{A}} k - \mathrm{mod}_{f.d.}\,.$$

Note that $k \otimes_{\mathrm{Sym}(\mathfrak{t})} \tilde{\mathcal{A}} \simeq \tilde{\mathcal{A}} \otimes_{\mathrm{Sym}(\mathfrak{t})} k$ so $\mathcal{O}^\wedge_\lambda \simeq {}^\wedge\mathcal{O}_\lambda$.

7. A graded version of category $\mathcal{O}_\lambda$ for regular λ

The ring $\mathrm{Sym}(\mathfrak{t})$ is a graded and $\mathrm{Sym}(\mathfrak{t})^W_+$ is a homogenous ideal so A is graded and we obtain a grading on M. Notice that a grading is equivalent to an algebraic $\mathbb{C}^*$-action and that a $\mathbb{C}^*$-action on M induces a $\mathbb{C}^*$-action on $\mathrm{End}(M)$. Hence, we get a grading on $\mathcal{A}$

Definition 7.0.1 (Graded category $\mathcal{O}_\lambda$)**.** Define the graded version of $\mathcal{O}_\lambda$ to be

$$\mathcal{O}^{\mathrm{gr}}_\lambda := \mathcal{A} - \mathrm{mod}^{\mathrm{gr}}_{f.g.}\,.$$

For any finite dimensional graded algebra B there is an automorphism

$$q \circlearrowright K^0(B - \mathrm{mod}^{\mathrm{gr}}_{f.g.}), \qquad [M] \mapsto [M[1]],$$

where $M[i]_n := M_{i+n}$. The forgetful functor factor as

$$\begin{tikzcd} K^0(B - \mathrm{mod}^{\mathrm{gr}}_{f.g.}) \arrow[rr] \arrow[dr] & & K^0(B - \mathrm{mod}_{f.g.}) \\ & K^0(B - \mathrm{mod}^{\mathrm{gr}}_{f.g.})/(q-1) \arrow[ur] & \end{tikzcd}$$

This can be thought of as a q deformation.

Lemma 7.0.2. *There is an isomorphism* $K^0(B - \mathrm{mod}^{\mathrm{gr}}_{f.g.})/(q-1) \simeq K^0(B - \mathrm{mod}_{f.g.})$.

Proof. It is enough to check that any irreducible $L \in B - \mathrm{mod}_{f.g.}$ can be equipped with a grading and that the isomorphism class of the graded module $\tilde{L}$ is unique up to a grading shift. A grading is equivalent to an algebraic $\mathbb{C}^*$-action. It is a general fact that

$$B/\mathrm{Rad}(B) \simeq \bigoplus_{L_i \text{ irred. rep.}} \mathrm{End}_k(L_i).$$

Hence, a $\mathbb{C}^*$-action on B gives rise to a $\mathbb{C}^*$-action on each $\mathrm{End}_k(L_i) \simeq \mathrm{Mat}_{n_i}(k)$ where $n_i = \dim L_i$. This lifts to a $\mathbb{C}^*$-action on L_i.

$$\begin{tikzcd} \mathbb{C}^* \arrow[r] \arrow[dr, dashed] & PGL_{n_i} \simeq \mathrm{Aut}(\mathrm{End}_k(L_i)) \\ & GL_{n_i} \simeq \mathrm{End}_k(L_i) \arrow[u] \end{tikzcd}$$

The lift is unique up to a scalar so the corresponding grading is unique up to a shift. □

The action of $R_\alpha, \Theta_\alpha, \Theta'_\alpha$ on $\mathcal{O}_\lambda$ and $D^b(\mathcal{O}_\lambda)$ lifts to an action on $\mathcal{O}_\lambda^{\mathrm{gr}}$ and $D^b(\mathcal{O}_\lambda^{\mathrm{gr}})$. Indeed, consider the full subcategories

$$\mathrm{Proj}(\mathcal{O}_\lambda) \hookrightarrow \mathcal{A} - \mathrm{mod},$$
$$\mathrm{Proj}(\mathcal{O}_\lambda^{\mathrm{gr}}) \hookrightarrow \mathcal{A} - \mathrm{mod}^{\mathrm{gr}}.$$

The functor

$$\mathcal{A} - \mathrm{mod}^{\mathrm{gr}} \to \mathcal{A} - \mathrm{mod}^{\mathrm{gr}}, \qquad M \mapsto A \otimes_{A^{s_\alpha}} M$$

preserves those subcategories so it induces an endofunctor R_α^{gr} on $\mathrm{Proj}(\mathcal{O}_\lambda^{\mathrm{gr}})$.

Exercise 7.0.3. The functor R_α^{gr} extends uniquely to all of $\mathcal{O}_\lambda^{\mathrm{gr}}$.

Question 7.0.4. What can be said about the action of $K^0(R_\alpha^{\mathrm{gr}})$ on $K^0(\mathcal{O}_\lambda^{\mathrm{gr}})$?

Recall that $s_\alpha \mapsto (K^0(R_\alpha) - 1)$ defines an action of W on $K^0(\mathcal{O}_\lambda)$. One can check that the maps $A \otimes_{A^{s_\alpha}} A \to A$ and $A \to A \otimes_{A^{s_\alpha}} A$ induces functors on the graded category

$$\mathrm{Cone}(R_\alpha^{\mathrm{gr}} \to \mathrm{Id})[1] =: {\Theta'_\alpha}^{\mathrm{gr}},$$
$$\mathrm{Cone}(\mathrm{Id}[1] \to R_\alpha^{\mathrm{gr}}) =: \Theta_\alpha^{\mathrm{gr}}.$$

Example 7.0.5. For $\mathfrak{g} = \mathfrak{sl}_2$ the second map is given by

$$k[t_1, t_2]/(t_1 - t_2) \simeq k[t] \to k[t] \otimes_{k[t^2]} k[t], \qquad t \mapsto t_1 + t_2.$$

Exercise 7.0.6. The functors $\Theta_\alpha^{\mathrm{gr}}$ and ${\Theta'_\alpha}^{\mathrm{gr}}$ also satisfy braid relations.

The generators of the braid group B is indexed by the simple reflections. Write $\tilde{s}_\alpha$ for the generator corresponding to s_α. For an arbitrary $w \in W$ with minimal decomposition $w = s_{\alpha_1} \cdots s_{\alpha_n}$ set $\tilde{w} := \tilde{s}_{\alpha_1} \cdots \tilde{s}_{\alpha_n}$.

Write $r_\alpha := K^0(R_\alpha^{\mathrm{gr}})$. Either $\tilde{s}_\alpha \to (1 - r_\alpha)$ or $\tilde{s}_\alpha \to (r_\alpha - q)$ gives a right braid group action on $K^0(\mathcal{O}_\lambda^{\mathrm{gr}})$.

The functor on $\mathcal{A}-\mathrm{mod}^{\mathrm{gr}}$ corresponding to $R_\alpha^{\mathrm{gr}} \circ R_\alpha^{\mathrm{gr}}$ comes from tensoring with the bimodule
$$A \otimes_{A^{s_\alpha}} A \otimes_{A^{s_\alpha}} A \simeq A \otimes_{A^{s_\alpha}} A \oplus A \otimes_{A^{s_\alpha}} A[1]$$
This implies that
$$r_\alpha^2 = r_\alpha + q r_\alpha$$
Rewrite this as $r_\alpha(r_\alpha - 1 - q) = 0$. Setting $\theta_\alpha := r_\alpha - 1$ it becomes
$$(\theta_\alpha + 1)(\theta_\alpha - q) = 0.$$
This is exactly the relation in the Hecke algebra
$$H_v := \mathbb{Z}[v, v^{-1}][B]/(\tilde{s}_\alpha + 1)(\tilde{s}_\alpha - v^2),$$
where $q = v^2$.

8. The Kazhdan-Lusztig conjectures

Let D denote the automorphism of the Hecke algebra H sending
$$v \mapsto v^{-1}, \qquad \tilde{s}_\alpha \mapsto \tilde{s}_\alpha^{-1}.$$
The eigenvalues of $\tilde{s}_\alpha$ are ± 1 so such an automorphism exists.

Proposition 8.0.1. *There exists a unique $\mathbb{Z}[v, v^{-1}]$ basis C_w with $D(C_w) = C_w$.*
$$C_w = v^{-\ell(w)} \sum_y P_{y,w} \tilde{y},$$
where $P_{y,w} \in \mathbb{Z}[v, v^{-1}]$ is a polynomial satisfying

(1) $P_{w,w} = 1$.
(2) $P_{y,w} = 0$ if $y \nleq w$.
(3) $P_{y,w} \in v^{-\ell(y)+\ell(w)-1}\mathbb{Z}[v^{-1}]$ for $y < w$.

Proof. Rewrite
$$0 = (\tilde{s}_\alpha + 1)(\tilde{s}_\alpha - v^2) = \tilde{s}_\alpha^2 + (1 - v^2)\tilde{s}_\alpha - v^2 \Leftrightarrow$$
$$\tilde{s}_\alpha = (v^2 - 1) + v^2 \tilde{s}_\alpha^{-1}.$$
Put $\tilde{s}'_\alpha := v^{-1}\tilde{s}_\alpha$. Then
$$D(\tilde{s}'_\alpha) = \tilde{s}'_\alpha + \text{constant}.$$
For a general element
$$\begin{aligned} D(\tilde{w}) &= \tilde{s}_{\alpha_1}^{-1} \cdots \tilde{s}_{\alpha_n}^{-1} = (\tilde{s}_{\alpha_n} \cdots \tilde{s}_{\alpha_1})^{-1} \\ &= \widetilde{(w^{-1})}^{-1} = v^{2\ell(w)}\tilde{w} + \sum_{y<w} G_y \tilde{y}. \end{aligned}$$
Putting $\tilde{w}' := v^{-\ell(w)}\tilde{w}$ this becomes
$$D(\tilde{w}') = \tilde{w}' + \sum_{y<w} G_y \tilde{y}.$$

Hence, one can rewrite the proposition as

$$C_w = \sum P'_{y,w}\tilde{y}', \qquad P'_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}].$$

The basis is constructed inductively. Set $C_1 = 1$. For a fixed w assume that we have C_y for $y < w$. Using this basis

$$\tilde{w}' - D(\tilde{w}') = \sum_{y<w} Q_y C_y.$$

Since $D^2 = \mathrm{Id}$ applying D yields

$$-(\tilde{w}' - D(\tilde{w}')) = \sum_{y<w} D(Q_y) C_y.$$

It follows that $Q_y(v^{-1}) = -Q_y(v)$ so Q_y is of the form

$$Q_y(v) = P_y(v) - P_y(v^{-1}), \qquad \text{for some } P_y \in v\mathbb{Z}[v].$$

Set

$$C_w := \tilde{w}' + \sum_{y<w} P_y(v^{-1}) C_y$$

Then

$$\begin{aligned} C_w - D(C_w) &= \tilde{w}' - D(\tilde{w}') + \sum_{y<w} (P_y(v^{-1}) - P_y(v)) C_y \\ &= \sum_{y<w} Q_y(v) C_y - \sum_{y<w} Q_y(v) C_y \\ &= 0. \end{aligned}$$

This finishes the proof. □

The C_w are called the canonical Kazhdan-Lusztig basis. The image under the map $H \to \mathbb{Z}[W], v \mapsto 1$ is denoted by $\bar{C_w}$.

Theorem 8.0.2 (Kazhdan-Lusztig conjecture). *The isomorphism* $\mathbb{Z}[W] \xrightarrow{\sim} K^0(\mathcal{O}_\lambda)$ *given by*

$$w \mapsto [\Delta_w]$$

sends $\bar{C_w}$ *to* $[L_w]$.

The conjecture yields the character formula

$$\chi_{L_w} = \sum_{y<w} P_{y,w}(1) \chi_{\Delta_y}.$$

Remark 8.0.3. Define the Kazhdan-Lusztig matrix

$$M := (P_{y,w}) \in \mathrm{Mat}_{|W|}(\mathbb{Z}[v, v^{-1}]).$$

Assuming the Kazhdan-Lusztig conjecture M^{-1} records the grading multiplicities in the Jordan-Hölder series of T'_w.

$$\begin{aligned} M &: \tilde{\Delta}'_w \mapsto \tilde{L}_v, \\ M^{-1} &: \tilde{L}_v \mapsto \tilde{\Delta}'_w. \end{aligned}$$

Let M^+ be the matrix corresponding to the automorphism $v \mapsto v^{-1}$. Combining this with BGG reciprocity, $[\Delta_v : P_y] = [L_y : \nabla_v]$ we get

$$C := ((M^+)^T)^{-1} M^{-1} : \tilde{L}_y \mapsto \tilde{P}_w$$

and C is the Cartan matrix of the graded algebra $\mathcal{A}'$.

There is also a graded version of the conjecture

Theorem 8.0.4 (Graded Kazhdan-Lusztig conjecture)**.** *The isomorphism $H \xrightarrow{\sim} K^0(\mathcal{O}_\lambda^{\mathrm{gr}})$ given by*

$$\tilde{w} \mapsto \Delta_w^{\mathrm{gr}} := \Theta_w^{\mathrm{gr}}(\tilde{L}_1)$$

sends C_w to $[L_w^{\mathrm{gr}}]$.

8.1. **Modifying the grading.** Change the grading on $\mathcal{A}$ by doubling each degree. I.e.

$$\mathcal{A}_{new,i} := \begin{cases} \mathcal{A}_{old,\frac{i}{2}} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

With this grading shift

$$K^0(\mathcal{A} - \mathrm{mod}^{\mathrm{gr}}) \simeq H_v \simeq H_q \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[v, v^{-1}].$$

If Q is a graded $\mathcal{A}$-module which is a projective generator for $\mathcal{A} - \mathrm{mod}$ then

$$\mathcal{O}_\lambda^{\mathrm{gr}} := \mathcal{A} - \mathrm{mod}^{\mathrm{gr}} \simeq \mathcal{A}' - \mathrm{mod}^{\mathrm{gr}},$$

where $\mathcal{A}' = \mathrm{End}_{\mathcal{A}}(Q)^{opp}$.

Theorem 8.1.1 (Main theorem)**.** *The module Q can be chosen in such a way that $\mathcal{A}'_i = 0$ for $i < 0$ and $\mathcal{A}'_0$ is semisimple.*

Remark 8.1.2. The theorem would be false without modifying the grading.

Example 8.1.3. For $\mathfrak{g} = \mathfrak{sl}_2$ we have the minimal projective generator $P_1 \oplus P_2 \simeq k \oplus k[t]/(t^2) = M$. In section 5.4 we showed that all Hom spaces between the indecomposable projectives are 1-dimensional except for $\mathrm{End}(P_2)$ which is 2-dimensional, we also described the algebra structure on $\mathrm{End}(P_1 \oplus P_2)$. This can be represented by the quiver with the left dot corresponding to P_1 and the right dot corresponding to P_2.

$$\begin{tikzcd} \bullet \arrow[r, bend right, "b"'] & \bullet \arrow[l, bend right, "a"'] \end{tikzcd}$$

Here ba is multiplication by t and $ab = 0$.

$$\text{Path algebra}/(ba = 0) \simeq \mathrm{End}(P_1 \oplus P_2).$$

In the old grading $\deg(ab) = 1$. Replacing P_1 by $P_1(n)$ we change the degree of a by n and b by $-n$. In the new grading $\deg(ab) = 2$ and we can arrange that $\deg(a) = 1 = \deg(b)$.

8.2. **Proof of the graded Kazhdan-Lusztig conjecture using the main theorem.** The plan is to lift the duality on $\mathcal{O}_\lambda$ to $\mathcal{O}_\lambda^{\mathrm{gr}}$ in such a way that the induced action on $K^0(\mathcal{O}_\lambda^{\mathrm{gr}})$ is the involution $D : H_v \to H_v$.

Assume that we have such a duality. The lift is unique up to shifts so for some lift $\widetilde{L_w^\vee{}'}$.

$$L_w^\vee \simeq L_w \Rightarrow \widetilde{L_w^\vee{}'} \simeq \widetilde{L_w^\vee{}'}(2n) = (\tilde{L}'_w(-2n))^\vee$$

Fix the grading

$$\tilde{L}_w := \widetilde{L_w^\vee{}'}(n) = (\tilde{L}_w(n))^\vee$$

Then $\tilde{L}_w$ satisfies $\tilde{L}_w^\vee \simeq \tilde{L}_w$ so

$$D([\tilde{L}_w]) = [\tilde{L}_w].$$

This shows that the preimage of the class $[\tilde{L}_w]$ under the isomorphism $H \tilde{\to} K^0(\mathcal{O}_\lambda^{\mathrm{gr}})$ is invariant under the involution. The next lemma shows that it also satisfies the other characterizing property of the Kazhdan-Lusztig basis, so the lemma implies that the isomorphism maps C_w to $[\tilde{L}_w]$.

Lemma 8.2.1. *The module Δ_w admits a grading lift $\tilde{\Delta}_w$ with a map $\tilde{\Delta}_w \twoheadrightarrow \tilde{L}_w$ lifting the map $\Delta_w \twoheadrightarrow L_w$.*

Proof. Since $\Delta_1 \simeq L_1$ we set $\tilde{\Delta}_1 := \tilde{L}_1$. In the non-graded version we had $\Delta_w \simeq \Theta_w(\Delta_1)$ so we define

$$\tilde{\Delta}'_w := \Theta_w^{\mathrm{gr}}(\tilde{\Delta}_1).$$

The map $\mathcal{O}_\lambda \to A-\mathrm{mod}$ sends Δ_w to k. We want the map $\mathcal{O}_\lambda^{\mathrm{gr}} \to A-\mathrm{mod}$ to send $\tilde{\Delta}_w$ to k, where k sits in degree 0. For this we need to introduce a grading shift

$$\tilde{\Delta}_w := \tilde{\Delta}'_w(-2\ell(w))$$

By definition

$$[\tilde{\Delta}_w] = t_w = v^{-\ell(w)}\tilde{w}$$
$$D(t_w) - t_w = \sum_{y<w} r_y \tilde{y}$$

The main theorem implies that the Jordan-Hölder series of $\tilde{\Delta}_w$ is of the form

$$[\tilde{\Delta}_w] = [\tilde{L}_w] + \sum_{y<w} v^{-1}\mathbb{Z}[v^{-1}][\tilde{L}_y].$$

Thus, the matrix $M \in \mathrm{Mat}_{|W|}(\mathbb{Z}[v^{-1}])$ sending $[\tilde{L}_w]$ to $[\tilde{\Delta}_w]$ is upper triangular with ones on the diagonal. The inverse matrix

$$M^{-1} = I - (I - M) + (I - M)^2 + \cdots$$

also has this form. To finish the proof we need the following lemma.

Lemma 8.2.2. *Assuming the main theorem, if $\mathcal{A}' \sim \mathcal{A}$ is a graded Morita equivalence and $\mathcal{A}'$ is positively graded then there is a bijection*

$$\{\textit{Irreducible } \mathcal{A}' \textit{ modules in degree } 0\} \quad \leftrightarrow \quad \{\tilde{L}_w[m],\ w \in W \textit{ and } m \textit{ independent of } w\}.$$

We can assume without loss of generality that m=0. Let $M = \oplus_i M_i$ be a graded module over $\mathcal{A}'$. Set $n = \min\{i \mid M_i \neq 0\}$. Then $M^+ := \oplus_{i>n} M_i$ is a submodule and we have a short exact sequence

$$0 \to M_+ \to M \to M_n \to 0$$

The Jordan-Hölder series for M only contains $\tilde{L}[i]$ with $i \geq -n$ and

$$M_n \simeq \bigoplus_i \tilde{L}_i^{d_i}[-n].$$

Applying this to $M = \tilde{\Delta}_w$ we get $\tilde{\Delta}_w \twoheadrightarrow \tilde{L}_w$. □

8.2.1. *Lifting duality to* $\mathcal{O}_\lambda^{\mathrm{gr}}$. Recall that we have the exact, self-adjoint (up to shift) functor $R_\alpha^{\mathrm{gr}}(=: \Xi_\alpha^{\mathrm{gr}})$. On $\mathcal{O}_\lambda$ we have the duality functor $\mathcal{O}_\lambda \to \mathcal{O}_\lambda^{opp}$. Notice that the opposite to the category of finite dimensional (graded) modules over a ring is naturally equivalent to the category of finite dimensional (graded) modules over the opposite ring (i.e. right modules over the original ring), the equivalence sends a module M to the dual vector space M^*. Thus the duality defines a functor

$$\mathcal{A}-\mathrm{mod}_{f.d.} = \mathcal{O}_\lambda \to \mathcal{O}_\lambda^{opp} = \mathcal{A}^{opp}-\mathrm{mod}_{f.d.}, \qquad M \mapsto M^\vee.$$

We want to upgrade it to a duality on the graded categories.

$$\mathcal{A}-\mathrm{mod}^{\mathrm{gr}} \to \mathcal{A}^{opp}-\mathrm{mod}^{\mathrm{gr}}.$$

The ordinary duality is exact so it can be written as

$$(-)^\vee \simeq \mathrm{Hom}(P, P^\vee) \otimes_{\mathcal{A}} -$$

Equipping the module $\mathrm{Hom}(P, P^\vee)$ with a grading yields a graded duality functor.

$$\mathcal{A}-\mathrm{mod}^{\mathrm{gr}} \to \mathcal{A}^{opp}-\mathrm{mod}^{\mathrm{gr}}, \qquad M \mapsto \mathrm{Hom}(P, P^\vee) \otimes_{\mathcal{A}} M.$$

To define this grading we first notice that with the old duality

$$P^\vee = \Big(\bigoplus_w R_{\alpha_1^w} \cdots R_{\alpha_n^w}(\Delta_{w_0})\Big)^\vee = \bigoplus_w R_{\alpha_1^w} \cdots R_{\alpha_n^w}(\nabla_{w_0}) =: I.$$

Hence,

$$\mathrm{Hom}(P, P^\vee) = \bigoplus_{v,w} \mathrm{Hom}(R_{\alpha_n^v} \cdots R_{\alpha_1^v} R_{\alpha_1^w} \cdots R_{\alpha_n^w}(\Delta_{w_0}), \nabla_{w_0}).$$

Recall that we have the exact sequence

$$\begin{tikzcd} \mathfrak{t} \otimes P_1 \arrow[r] \arrow[d, equal] & P_1 \arrow[r] & \nabla_{w_0} \arrow[r] \arrow[d, equal] & 0 \\ \mathfrak{t} \otimes T_{-\rho\to\lambda}(\Delta_{-\rho}) & & \tilde{T}_{-\rho\to\lambda}(\Delta_{-\rho}) & \end{tikzcd}$$

From this we get

$$\mathfrak{t} \otimes \mathrm{Hom}(-, P_1) \to \mathrm{Hom}(-, P_1) \to \mathrm{Hom}(-, \nabla_{w_0}) \to 0$$

Notice that

$$\begin{aligned}\operatorname{Hom}(R_{\alpha^v_m}\cdots R_{\alpha^v_1}R_{\alpha^w_1}\cdots R_{\alpha^w_n}(\nabla_{w_0}),P_1)&\\ &= \operatorname{Hom}_A(\tilde{T}_{\lambda\to-\rho}(R_{\alpha^v_m}\cdots R_{\alpha^v_1}R_{\alpha^w_1}\cdots R_{\alpha^w_n}(\nabla_{w_0})),\Delta_{-\rho})\\ &= \operatorname{Hom}_A(A\otimes_{A^{s_{\alpha^v_m}}}\cdots\otimes_{A^{s_{\alpha^w_n}}}k,A).\end{aligned}$$

Thus, we have the exact sequence

$$\mathfrak{t}\otimes\operatorname{Hom}_A(\bigoplus_{v,w}A\otimes_{A^{s_{\alpha^v_m}}}\cdots\otimes_{A^{s_{\alpha^w_n}}}k,A)\to\operatorname{Hom}_A(\bigoplus_{v,w}A\otimes_{A^{s_{\alpha^v_m}}}\cdots\otimes_{A^{s_{\alpha^w_n}}}k,A)\to\operatorname{Hom}(P,P^\vee)\to 0$$

The first two terms are graded and the first map is graded. This induces the desired grading on $\operatorname{Hom}(P,P^\vee)$.

Now we need to check that this duality is compatible with the involution on the Hecke algebra. Write $[\tilde{\Delta}_w]=T'_w$ and since $\Delta_{w^{-1}}*\nabla_w\simeq\Delta_1$ we have

$$D(T'_w)=T'^{-1}_{w^{-1}}=[\tilde{\nabla}_w].$$

Thus, it is enough to show that

$$\tilde{\Delta}^\vee_w\simeq\tilde{\nabla}_w.$$

The module Δ_w is the unique object in $\mathcal{O}_{\le w}$ satisfying $\operatorname{Ext}^1(\Delta_w,L_v)=0$ for $v\le w$, $\operatorname{Hom}(\Delta_w,L_v)-0$ for $v<w$ and $\operatorname{Hom}(\Delta_w,L_w)=k$. Similarly, $\tilde{\Delta}_w$ is the unique object of $\mathcal{O}^{\mathrm{gr}}_{\le w}$ with $\operatorname{Hom}_{\mathrm{gr}}(\tilde{\Delta}_w,\tilde{L}_w)=k$ placed in graded degree zero, while $\operatorname{Hom}_{\mathrm{gr}}(\tilde{\Delta}_w,\tilde{L}_v)=0$ for $v<w$ and $\operatorname{Ext}^1_{\mathrm{gr}}(\tilde{\Delta}_w,\tilde{L}_v)=0$ for $v\le w$ (here we write $\operatorname{Hom}_{\mathrm{gr}}$ for the graded Hom taking values in graded vector spaces). It follows that $\tilde{\Delta}^\vee_w$ satisfies the characterizing properties of $\tilde{\nabla}_w$: $\operatorname{Hom}_{\mathrm{gr}}(\tilde{L}_w,\tilde{\nabla}_w)=k$ placed in graded degree zero, while $\operatorname{Hom}_{\mathrm{gr}}(\tilde{L}_v,\tilde{\nabla}_w)=0$ for $v<w$ and $\operatorname{Ext}^1_{\mathrm{gr}}(\tilde{L}_v,\tilde{\nabla}_w)=0$ for $v\le w$. Hence, $\tilde{\Delta}^\vee_w\simeq\tilde{\nabla}_w$.

9. Geometry of the flag variety

The main theorem will be proved using the geometry of G/B (conceptual way: via D-modules and Riemann-Hilbert correspondence). Instead, we will use topology and work with $\operatorname{Sh}(G/B)$ (here $G/B=G/B(\mathbb{C})$). It turns out that

$$\mathcal{O}_\lambda\subset\operatorname{Perv}\subset D^b(\operatorname{Sh}(G/B)),$$

where Perv is the category of perverse sheaves. In example 3.0.10 we introduced the category of perverse sheaves compatible with a fixed cell decomposition, $\operatorname{Sh}_\Sigma(X)$. The general definition of Perv will be given later.

Definition 9.0.1 (Equivariant sheaf)**.** An equivariant sheaf for an action of a topological group H on a topological space X is a sheaf $\mathcal{F}$ on X together with an isomorphism of sheaves on $H\times X$

$$a^*(\mathcal{F})\simeq\operatorname{pr}^*(\mathcal{F}),$$

where

$$\operatorname{pr}:H\times X\to X,\ (h,x)\mapsto x,\qquad a:H\times X\to X,\ (h,x)\mapsto h(x)$$

such that the two isomorphisms of sheaves on $H \times H \times X$

$$a^*(\mathcal{F}) \simeq \mathrm{pr}^*(\mathcal{F}),$$

here $\mathrm{pr}(h_1, h_2, x) = x$ and $a(h_1, h_2, x) = h_1 h_2(x)$. The category of H-equivariant sheaves is denoted $\mathrm{Sh}_H(X)$.

Definition 9.0.2 (Wrong definition)**.** A naive equivariant complex of sheaves on X is an object in $D(\mathrm{Sh}(X))$ and an isomorphism as above satisfying the above condition. We denote this category by $D(\mathrm{Sh}_H(X))$.

Remark 9.0.3. There is a natural functor from the correct equivariant derived category to the category of naive equivariant complexes. This functor is an equivalence on the subcategory of sheaves, as well as on the category of perverse sheaves. Thus if $\mathcal{F}$ is a sheaf or a perverse sheaf, then a naive equivariant structure on $\mathcal{F}$ uniquely lifts to an actual equivariant structure. On the other hand, the category of naive equivariant complexes is not triangulated in general and does not satisfy descent, i.e. in the cases when the quotient X/H exists it does not coincide with the derived category of sheaves on the quotient.

The right definition of $D_H(\mathrm{Sh}(X))$ satisfy the following properties

(0) For an H-equivariant map $f : X \to Y$ there exist functors f_*, f^*.

$$(R)f_* : D_H(\mathrm{Sh}(X)) \to D_H(\mathrm{Sh}(X))$$

(1) If the action is free and X/H is defined then

$$D_H(\mathrm{Sh}(X)) \simeq D(X/H).$$

(2) Consider $X = E \times Y$ where E has a G action and is contractible. Let f be the projection to Y. Then

$$f^* : D_H(Y) \to D_H(X)$$

is fully faithful.

In particular, if H is contractible (e.g. H is a unipotent algebraic group over $\mathbb{C}$) and we take $E = H$. Then

$$D_H(X) \xrightarrow{m^*} D_H(H \times X) \to D(X)$$

is a full embedding.

Exercise 9.0.4. Show that its image consists of objects admitting a naively equivariant structure. Such a structure is unique if it exists.

Theorem 9.0.5. *In the notation below the first factor in the equivariance acts on the left and the second factor acts on the right.*

$$D^b(\mathcal{O}^\wedge_\lambda) \simeq D_{B\times N}(G) \simeq D_N(B\backslash G),$$

$$D^b({}^\wedge\mathcal{O}_\lambda) \simeq D_{N\times B}(G),$$

$$D^b({}^\wedge\mathcal{O}^\wedge_\lambda) \subset D_{N\times N}(G) \quad \textit{full embedding.}$$

All sending the abelian category to perverse sheaves

However, $D^b(\bar{\mathcal{O}}_\lambda) \not\simeq D_{B\times B}(G)$. The category $D_{B\times B}(G)$ is not the derived category of any abelian category but it is a DG category.

$$D_{B\times B}(G) \simeq \mathrm{DG}-\mathrm{mod}(\mathcal{A} \otimes^L_{\mathrm{Sym}(\mathfrak{t})} k).$$

$$\bar{\mathcal{O}}_\lambda \simeq \mathrm{perv}_{B\times B}(G)$$

Recall that N-orbits on $\mathcal{B} = G/B$ are indexed by W and $\mathcal{B}_w \simeq \mathbb{A}^{\ell(w)}$. Let $j_w : \mathcal{B}_w \hookrightarrow \mathcal{B}$ denote the embedding.

The category $\mathcal{C} := D_N(\mathcal{B}) \subset D(\mathcal{B})$ consists of complexes $\mathcal{F}$ such that

$$\underline{H}^i(j_w^*(\mathcal{F})) \text{ is the constant sheaf for all } w, i.$$

Recall from example 3.0.10 that $\mathcal{C}$ is generated by the exceptional collection

$$\Delta_w := j_{w!}(\underline{\mathbb{C}})[\ell(w)]$$

with dual collection

$$\nabla_w := Rj_{w*}(\underline{\mathbb{C}})[\ell(w)]$$

Definition 9.0.6 (Perverse sheaves)**.** The category of perverse sheaves $\mathcal{P}$ is the full subcategory in $\mathcal{C}$ given by

$$\mathcal{P} = \{\mathcal{F} \mid \mathrm{Hom}(\Delta_w, \mathcal{F}[i]) = 0 = \mathrm{Hom}(\mathcal{F}, \nabla_w[i]) \quad \forall i > 0, w \in W\}.$$

It is known that $\mathcal{P}$ is a highest weight category with standard objects Δ_w and costandard objects ∇_w and that $D^b(\mathcal{P}) \simeq \mathcal{C}$.

Remark 9.0.7. The Grothendieck group $K^0(\mathcal{P}) = K^0(\mathcal{C})$ is freely generated by $[\Delta_w]$.

Let $j : U \hookrightarrow X$ be an open embedding and $i : Z \hookrightarrow X$ a closed embedding with $Z \subset X\backslash U$. Then for any sheaf $\mathcal{F}$ on X there is a short exact sequence (see [Iver, 6.11])

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0. \tag{3}$$

Exercise 9.0.8. Prove the remark directly using the exact sequence.

Our goal is to construct an equivalence

$$\mathcal{L} : \mathcal{O}_\lambda \xrightarrow{\sim} \mathcal{P}.$$

Example 9.0.9. For $G = \mathrm{SL}_2$, $\mathcal{B} = \mathbb{P}^1$ and $W = \{1, s\}$. In the derived category we have

$$\underline{\mathbb{C}}_0[-2] \to \underline{\mathbb{C}} \to Rj_*(\underline{\mathbb{C}}) \to \underline{\mathbb{C}}_0[-1]$$

This comes from the short exact sequence of perverse sheaves

$$0 \to \underline{\mathbb{C}}[1] \to Rj_*(\underline{\mathbb{C}})[1] \to \underline{\mathbb{C}}_0 \to 0$$

In particular, the only non-zero stalk of $R^1 j_*(\underline{\mathbb{C}})$ is the one at 0:

$$\mathrm{stalk}(Rj_*(\underline{\mathbb{C}})) \text{ at } 0 = \varinjlim H^\bullet(\mathrm{Disc} - \{0\}) = H^\bullet(S^1).$$

The irreducibles in $\mathcal{P}$ are indexed by W. They are the minimal Goresky-MacPherson extension
$$L_w = j_{w!*}(\underline{\mathbb{C}}[\ell(w)]).$$
In particular, L_w is supported on $\bar{\mathcal{B}}_w$. They fit into the exact sequence
$$\Delta_w \to L_w \to \nabla_w$$
The Weyl group is equipped with the partial order given by
$$v \le w \quad \Leftrightarrow \quad \mathcal{B}_v \subset \bar{\mathcal{B}}_w.$$
It follows form the following easy property of $j_{!*}$ that $j_w^*(L_w) = \underline{\mathbb{C}}[\ell(w)]$.

Lemma 9.0.10. *Assume that Z is irreducible and $j : Z \hookrightarrow X$ is a locally closed embedding*

(1) If $\bar{Z}$ is closed then
$$j_{!*}(\underline{\mathbb{C}}_Z[\dim Z]) = \underline{\mathbb{C}}_{\bar{Z}}[\dim Z].$$
(2) If $f : X \to Y$ is a smooth map and $j' : T \hookrightarrow Y$ is another locally closed embedding fitting into the diagram
$$\begin{tikzcd} Z \arrow[r, hook, "j"] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r, hook, "j'"] & Y \end{tikzcd}$$
with Z open in $f^{-1}(T)$. Then
$$j_{!*}(\underline{\mathbb{C}}_Z[\dim Z]) = f^*(j'_{!*}(\underline{\mathbb{C}}_T[\dim T]))[\dim Z - \dim T].$$

Notice that for $x_w \in \mathcal{B}_w$
$$\text{stalk of } \Delta_v \text{ at } x_w = \begin{cases} 0 & v \neq w \\ \mathbb{C}(\ell(w)) & v = w \end{cases}$$
Hence, for $\mathcal{F} \in K^0(\mathcal{P})$ the Euler characteristic of the stalk at x_w computes the number of copies of $[\Delta_w]$ so
$$[\mathcal{F}] = \sum_v [\Delta_v] e_v, \qquad e_v = (-1)^{\ell(v)} \operatorname{Eul}(\text{stalk of } \mathcal{F} \text{ at } x_v),$$
Assuming $\mathcal{P} \simeq \mathcal{O}_\lambda$ this shows that
$$P_{w,y}(1) = \pm \operatorname{Eul}(\text{stalk of } j_{w!*}(\underline{\mathbb{C}}[\ell(w)]) \text{ at } x_y).$$

Our plan now is to axiomatize what $\mathcal{L}$ does to $\tilde{T}_{\lambda \to -\rho}, \Theta_w$ and R_α. We will then get a Soergel style description of the topological category, which will allow to construct $\mathcal{L}$.

Example 9.0.11. For G =SL(2) we have $\mathcal{B} = \mathbb{P}^1$, $\mathcal{B}_1 = \{0\}$ and $\mathcal{B}_s = \mathbb{P}^1 \backslash \{0\} = \mathbb{A}^1$. Let j be the open embedding $j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Applying (3) to the constant sheaf $\underline{\mathbb{C}}_{\mathbb{P}^1}$ we get
$$0 \to j_!(\underline{\mathbb{C}}_{\mathbb{A}^1}) \to \underline{\mathbb{C}}_{\mathbb{P}^1} \to \underline{\mathbb{C}}_0 \to 0$$

This gives an exact triangle in the derived category

$$\underline{\mathbb{C}}_0 \to j_!(\underline{\mathbb{C}}_{\mathbb{A}^1})[1] \to \underline{\mathbb{C}}_{\mathbb{P}^1}[1] \to \underline{\mathbb{C}}_0[1]$$

The first 3 terms form a short exact sequence of perverse sheaves.

$$0 \to L_1 \to \Delta_s \to L_s \to 0.$$

Using Example 9.0.9 we get a short exact sequence

$$0 \to \underline{\mathbb{C}}_{\mathbb{P}^1}[1] \to Rj_*(\underline{\mathbb{C}}_{\mathbb{A}^1})[1] \to \underline{\mathbb{C}}_0 \to 0$$

This can be rewritten as

$$0 \to L_s \to \nabla_s \to L_1 \to 0.$$

Sheaves supported on a point cannot have higher cohomology so

$$\mathrm{Ext}^i(L_s, L_1) = \mathrm{Ext}^{i-1}(\underline{\mathbb{C}}_{\mathbb{P}^1}, \underline{\mathbb{C}}_0) = \begin{cases} \mathbb{C} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^i(L_1, L_1) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

The long exact sequence

$$0 = \mathrm{Ext}^1(L_1, L_1) \to \mathrm{Ext}^1(L_1, \Delta_s) \to \mathrm{Ext}^1(L_1, L_s) \to \mathrm{Ext}^2(L_1, L_1)$$

implies that $\mathrm{Ext}^1(L_1, \Delta_s) = \mathbb{C}$. Let Ξ be the unique non-split extension

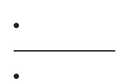
$$0 \to \Delta_s \to \Xi \to L_1 \to 0.$$

Applying the long exact sequence one more time it is easy to check that $\mathrm{Ext}^1(\Xi, L_1) = 0 = \mathrm{Ext}^1(\Xi, L_s)$. Thus, Ξ is a projective cover of L_1 in the category $Perv_N(\mathbb{P}^1)$.

The associated graded of the Jordan-Hölder filtration is then given by

$$\mathrm{gr}(\Xi) = \begin{bmatrix} L_1 \\ L_s \\ L_1 \end{bmatrix} = \begin{bmatrix} \underline{\mathbb{C}}_0 \\ \underline{\mathbb{C}}_{\mathbb{P}^1} \\ \underline{\mathbb{C}}_0 \end{bmatrix}$$

This is sometimes written as



Here the line represent the constant sheaf and the dots represent skyscraper sheaves.

Notice that $L_1 \hookrightarrow \Delta_s$ so we have another short exact sequence coming from the inclusion $L_1 \hookrightarrow \Xi$

$$0 \to L_1 \to \Xi \to \nabla_s \to 0.$$

Also one can check that $\mathrm{Ext}^1(L_1, \Xi) = 0 = \mathrm{Ext}^1(L_s, \Xi)$ so Ξ is also an injective hull of L_1 in $Perv_N(\mathbb{P}^1)$.

We will later see that Ξ is self-dual with respect to a duality interchanging the last short exact sequence with the previous one.

10. Intertwining functors

10.1. **Definition of intertwining functors.** We now define functors which will later be shown to generate a braid group action on the topological category.

Recall that $A = \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})^W_+$. We will define a functor $\mathcal{P} \to A-\mathrm{mod}$ and an action of the braid group B on $\mathcal{C}$. The G-orbits on $(G/B)^2$ are indexed by W. For a fixed $w \in W$ we pick an orbit of G on $(G/N)^2$ which projects to the G-orbit on $(G/B)^2$ corresponding to w. This orbit is denoted by X_w.

$$\begin{tikzcd} & X_w \arrow[dl, "\mathrm{pr}_1^w"'] \arrow[dr, "\mathrm{pr}_2^w"] & \\ G/N & & G/N \end{tikzcd}$$

The intertwining functors are defined as follows:

$$I_w : \mathcal{C} \to \mathcal{C}, \qquad \mathcal{F} \mapsto \mathrm{pr}^w_{2!}\mathrm{pr}_1^{w*}(\mathcal{F})[\ell(w)].$$

Note that the same definition gives a functor

$$I_w : D_H(G/N) \to D_H(G/N)$$

for any subgroup $H \subset G$.

Example 10.1.1. For $G = \mathrm{SL}_2$ we have $W = \{1, s\}$ and $G/N = \mathbb{A}^2\backslash\{0\} = V\backslash\{0\}$ for a 2-dimensional vector space V. Then one choice of X_s is given by $X_s = \{(v_1, v_2) \mid \omega(v_1, v_2) = 1\}$, where ω is a 2-form.

10.2. **Generalities on $f_!$ - derived functor of direct image with proper support.** Let $f : X \to Y$ be a map of schemes. For any sheaf $\mathcal{F}$ on X and any open subset $U \subset Y$ we define

$$\Gamma(U, f_*\mathcal{F}) = \Gamma(f^{-1}(U), \mathcal{F}),$$

$$\Gamma(U, f_!\mathcal{F}) = \{\text{sections whose support maps properly under } f\} \subset \Gamma(f^{-1}(U), \mathcal{F}).$$

It has the following properties

1. $(f \circ g)_! = f_! \circ g_!$.
2. If $f : X \hookrightarrow Y$ is an open embedding then $f_!$ is extension by zero.
3. If f is proper then $f_! = f_*$.
4. Base change: for a cartesian diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\psi"] \arrow[d, "g"'] & Z \arrow[d, "f"] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

the functors satisfy $\phi^* f_! \simeq g_! \psi^*$.

Example 10.2.1. Consider the case where $X = pt$. Then the stalk of $f_!\mathcal{F}$ at a point $y \in Y$ equals $\Gamma_c(\mathcal{F}|_{Z_y})$, where $Z_y = f^{-1}(y)$.

Remark 10.2.2. The base change formula is not true for ! replaced by $*$. Consider the following example. Let $f : Z \to Y$ and $x \in Y$. Then we have a cartesian diagram

$$\begin{tikzcd} f^{-1}(x) \simeq Y_x \arrow[r, hook, "j"] \arrow[d, "f_x"'] & Z \arrow[d, "f"] \\ \{x\} \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

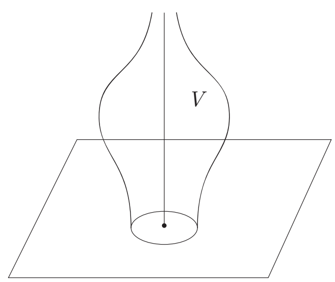
Calculating both sides

$$\begin{aligned} i^* f_* \mathcal{F} &= (f_* \mathcal{F})_x = \varinjlim_{U \ni x} \Gamma(U, f_* \mathcal{F}) \\ &= \varinjlim_{f^{-1}(U) \supset Y_x} \Gamma(f^{-1}(U), \mathcal{F}) \\ f_{x*} j^* \mathcal{F} &= f_{x*}(\mathcal{F}|_{Y_x}) = \Gamma(\mathcal{F}|_{Y_x}) \\ &= \varinjlim_{V \supset Y_x} \Gamma(V, \mathcal{F}). \end{aligned}$$

From this we see that there is a map

$$i^* f_* \mathcal{F} \to f_{x*} j^* \mathcal{F}$$

If f is not proper then there may exist open subsets $V \supset Y_x$ not containing a subset of the form $f^{-1}(U)$ for an open $U \ni x$, so the map might not be an isomorphism.



Another counterexample is the following. Let $f : U \to Y$ be an open embedding and $\phi : Z \to Y$ a closed embedding with $Z \subset Y \backslash U$. In particular, $Z \times_Y U = \varnothing$ and $\phi^* f_! = 0$ but often $\phi^* f_* \neq 0$ so base change fails.

10.3. **Verdier duality.** Let X be a smooth $\mathbb{C}$ manifold and $\mathcal{F}$ a sheaf on X. Define the Verdier dual

$$\mathbb{V}(\mathcal{F}) = \mathcal{F}^\vee := \underline{R\operatorname{Hom}}(\mathcal{F}, \underline{\mathbb{C}}[2\dim_{\mathbb{C}} X]).$$

This can be extended to not necessarily smooth manifolds. For $f : X \to Y$ the Verdier duality satisfies

$$f_! = \mathbb{V}_Y f_* \mathbb{V}_X.$$

In particular, $\mathbb{V} \circ \mathbb{V} \simeq \mathrm{Id}$.

Example 10.3.1. If $Y = pt$ and X is smooth then this amounts to Poincare duality

$$H^i(X)^* \simeq H_c^{2n-i}(X)$$

Remark 10.3.2. If X is non-smooth with a closed embedding $f : X \to Y$ into smooth Y then $f_* = f_!$ is a full embedding. Thus, we get a decomposition of $\mathbb{V}_X$ in terms of $\mathbb{V}_Y$.

For $f : X \to Y$ smooth $\mathbb{V}$ commutes with f^* up to a shift

$$\mathbb{V}_X f^* \mathbb{V}_Y(\mathcal{F}) = f^*(\mathcal{F})[2(\dim X - \dim Y)]$$

Recall the adjunction $\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}) \simeq \mathrm{Hom}(f^*\mathcal{F}, \mathcal{G})$. For f smooth $f_!$ is left adjoint to $f^!$, i.e.

$$\mathrm{Hom}(f_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}).$$

If f is a smooth map of $\mathbb{C}$ varieties then

$$f^!\mathcal{F} \simeq f^*\mathcal{F}[2(\dim X - \dim Y)].$$

In particular,

$$\mathrm{Hom}(f_!\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, f^*\mathcal{F}[2(\dim X - \dim Y)]).$$

10.4. **Properties of the intertwining functors.** We now return to the functors I_w introduced in section 10.1

Claim 10.4.1. The intertwining functors satisfy

$$I_{w_1} I_{w_2} \simeq I_{w_1 w_2} \text{ when } \ell(w_1) + \ell(w_2) = \ell(w_1 w_2).$$

Proof. It is a general fact for algebraic groups that

$$X_{w_1} \times_X X_{w_2} \simeq X_{w_1 w_2}.$$

Thus, we have a cartesian diagram

$$\begin{tikzcd} X_{w_1 w_2} \arrow[r, "\psi"] \arrow[d, "\phi"'] & X_{w_2} \arrow[d, "pr_2^{w_2}"] \\ X_{w_1} \arrow[r, "pr_1^{w_1}"'] & X \end{tikzcd}$$

By base change

$$pr_1^{w_1 *} pr_2^{w_2}{}_! \simeq \phi_! \psi^*$$

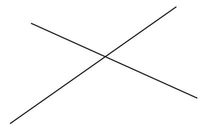
Applying this we get

$$\begin{aligned} I_{w_1} I_{w_2}(\mathcal{F}) &= pr_2^{w_1}{}_! pr_1^{w_1 *} pr_2^{w_2}{}_! pr_1^{w_2 *}(\mathcal{F})[\ell(w_1) + \ell(w_2)] \\ &\simeq pr_2^{w_1}{}_! \phi_! \psi^* pr_1^{w_2 *}(\mathcal{F})[\ell(w_1) + \ell(w_2)] \\ &\simeq (pr_2^{w_1} \circ \phi)_! (pr_1^{w_2} \circ \psi)^*(\mathcal{F})[\ell(w_1) + \ell(w_2)] \\ &\simeq pr_2^{w_1 w_2}{}_! pr_1^{w_1 w_2 *}(\mathcal{F})[\ell(w_1 w_2)] \\ &= I_{w_1 w_2}(\mathcal{F}). \end{aligned}$$

Which is what we wanted. ∎

Remark 10.4.2. The functor $I_w : D(\mathrm{Sh}(X)) \to D(\mathrm{Sh}(X))$ is not an equivalence.

To see this, consider the case $G = \mathrm{SL}_2$. Let x, y be non colinear vectors in $\mathbb{A}^2 \backslash \{0\}$ and set $\mathcal{F} := \underline{\mathbb{C}}_x$ and $\mathcal{G} := \underline{\mathbb{C}}_y$.



Then $\mathrm{Hom}^\bullet(\underline{\mathbb{C}}_x, \underline{\mathbb{C}}_y) = 0$ and

$$\begin{aligned} I_s(\underline{\mathbb{C}}_x) &= \underline{\mathbb{C}}_{\ell_x}[1], & \ell_x &= \{v \mid \langle v, x\rangle = 1\}, \\ I_s(\underline{\mathbb{C}}_y) &= \underline{\mathbb{C}}_{\ell_y}[1], & \ell_y &= \{v \mid \langle v, y\rangle = 1\}. \end{aligned}$$

Exercise 10.4.3. Show that $|\ell_x \cap \ell_y| = 1$ and $\dim \mathrm{Ext}^2(\underline{\mathbb{C}}_{\ell_x}, \underline{\mathbb{C}}_{\ell_y}) = 1$.

Let $w \in W$ and let $(G/B)_w \subset (G/B)^2$ denote the G orbit in $(G/B)^2$ corresponding to w.

$$\begin{tikzcd} & (G/B)_w \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ G/B & & G/B \end{tikzcd}$$

Define

$$\bar{I}_w : D(Sh(G/B)) \to D(Sh(G/B)), \qquad \mathcal{F} \mapsto pr_{2!} pr_1^*(\mathcal{F})[\ell(w)].$$

Note that unlike the definition of I_w there is no choice involved here. Let $\pi : X \to G/B$ be the projection. The next lemma shows that for sheaves on X which comes as a pull-back of a sheaf on G/B the definition of I_w is independent of the choice of X_w.

Lemma 10.4.4. *There is an isomorphism of functors $I_w(\pi^*\mathcal{F}) \simeq \pi^* \bar{I}_w(\mathcal{F})$.*

Proof. It is a fact that

$$X_w \simeq (G/B)_w \times_{G/B} X.$$

Hence, there is a cartesian square

$$\begin{tikzcd} X_w \arrow[r, "\phi"] \arrow[d, "pr_2^w"'] & (G/B)_w \arrow[d, "pr_2"] \\ X \arrow[r, "\pi"'] & G/B \end{tikzcd}$$

By base change $\pi^* pr_{2!} \simeq pr^w_{2\,!}\phi^*$ so

$$\begin{aligned}\pi^*\bar{I}_w(\mathcal{F}) &= \pi^* pr_{2!}pr_1^*(\mathcal{F})[-\ell(w)]\\ &\simeq pr^w_{2\,!}\phi^* pr_1^*(\mathcal{F})[-\ell(w)]\\ &\simeq pr^w_{2\,!}(pr_1\circ\phi)^*(\mathcal{F})[-\ell(w)]\\ &\simeq pr^w_{2\,!}(\pi\circ pr^w_1)^*(\mathcal{F})[-\ell(w)]\\ &\simeq pr^w_{2\,!}pr_1^{w*}\pi^*(\mathcal{F})[-\ell(w)]\\ &= I_w(\pi^*\mathcal{F}).\end{aligned}$$

Which is what we wanted. □

A proof similar to the one for I_w shows that

$$\bar{I}_{w_1}\bar{I}_{w_2} \simeq \bar{I}_{w_1w_2}, \qquad \text{when } \ell(w_1)+\ell(w_2) = \ell(w_1w_2).$$

Define another functor

$$\bar{I}'_w : D(\mathrm{Sh}(G/B)) \to D(\mathrm{Sh}(G/B)), \qquad \mathcal{F} \mapsto pr_{2*}pr_1^*(\mathcal{F})[\ell(w)].$$

Claim 10.4.5. There is an isomorphism of functors $\bar{I}_s\bar{I}'_s \simeq \bar{I}'_s\bar{I}_s \simeq \mathrm{Id}$. Hence, I_s generate a braid group action.

It turns out that $\bar{I}'_w$ and $\bar{I}_w$ are particular cases of convolution.

Definition 10.4.6 (Convolution)**.** Notice that $D_G((G/B)^2) \simeq D_B(G/B)$. Define the convolution product as

$$* : D(\mathrm{Sh}(G/B)) \times D_B(G/B) \to D(\mathrm{Sh}(G/B)), \qquad \mathcal{F} * \mathcal{G} := pr_{2*}(pr_1^*(\mathcal{F})\otimes\mathcal{G}).$$

Since G/B is compact $pr_{2!} = pr_{2*}$.

For $\mathcal{G} := j_{w!}(\underline{\mathbb{C}}[\ell(w)])$

$$\mathcal{F} * \mathcal{G} = \bar{I}_w(\mathcal{F}),$$

and for $\mathcal{G} := j_{w*}(\underline{\mathbb{C}}[\ell(w)])$

$$\mathcal{F} * \mathcal{G} = \bar{I}'_w(\mathcal{F}).$$

Consider the projections

$$\begin{tikzcd} & G/B\times G/B\times G/B \arrow[dl, "pr_{12}"'] \arrow[d, "pr_{13}"] \arrow[dr, "pr_{23}"] & \\ G/B\times G/B & G/B\times G/B & G/B\times G/B \end{tikzcd}$$

The category $D_B(\mathrm{Sh}(G/B)) \simeq D_G((G/B)^2)$ is a monoidal category with the following convolution

$$\mathcal{F} * \mathcal{G} := p_{13*}(pr_{12}^*(\mathcal{F})\otimes pr_{23}^*(\mathcal{G})).$$

Associativity follows from base change. This monoidal category acts on $D_H(G/B)$ for any subgroup $H \subset G$.

Proof of claim. Using the observation above

$$\begin{aligned}\bar{I}'_s \bar{I}_s(\mathcal{F}) &\simeq \bar{I}'_s(\mathcal{F}) * j_{s!}(\underline{\mathbb{C}}[1]) \\ &\simeq \mathcal{F} * j_{s*}(\underline{\mathbb{C}}[1]) * j_{s!}(\underline{\mathbb{C}}[1]).\end{aligned}$$

The unit for the convolution is $\underline{\mathbb{C}}_0 = \Delta_1$. Thus, we need to show that

$$j_{s*}(\underline{\mathbb{C}}[1]) * j_{s!}(\underline{\mathbb{C}}[1]) \simeq \underline{\mathbb{C}}_0,$$

where 0 is the 0-dimensional B-orbit on G/B. For this it is enough to calculate the stalk

$$\begin{aligned}&\text{Stalk at } x_s \in (G/B)_s = H^*\left({}_0\bullet \circlearrowleft^{x} \right) = 0, \\ &\text{Stalk at } 0 = H^*(\underline{\mathbb{C}}_{\mathbb{A}^1}) = \mathbb{C}.\end{aligned}$$

This finishes the proof. □

Proposition 10.4.7. *The functor*

$$I_w : D_B(G/N) \to D_B(G/N)$$

is left exact. In particular, it induces a functor

$$I_w : D^{\geq 0}(\mathcal{P}) \to D^{\geq 0}(\mathcal{P}).$$

The functor I'_w is right exact one has

$$I'_w : D^{\leq 0}(\mathcal{P}) \to D^{\leq 0}(\mathcal{P}).$$

10.5. **Convolution of (co)standard sheaves etc.** Let $\bar{\Delta}_w, \bar{\nabla}$ and $\bar{L}_w$ be the elements in $D_B(\mathrm{Sh}(B\backslash G)) = D(B\backslash G/B)$ corresponding to Δ_w, ∇_w and $L_w \in D(B\backslash G/N) = D_N(\mathrm{Sh}(B\backslash G))$. Using the isomorphism

$$(G/B)^2_{w_1} \times_{G/B} (G/B)^2_{w_2} \simeq (G/B)^2_{w_1 w_2}, \qquad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

and base change one can show that if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$

$$\begin{aligned}\Delta_{w_1} * \Delta_{w_2} &= \Delta_{w_1 w_2}, \\ \nabla_{w_1} * \nabla_{w_2} &= \nabla_{w_1 w_2}.\end{aligned}$$

We also have $\Delta_s * \nabla_s = \Delta_1$. Recall that for a simple reflection s_α we have the following extensions in $D(B\backslash G/N)$.

$$0 \to \Delta_{s_\alpha} \to \Xi_\alpha \to \Delta_1 \to 0, \tag{4}$$

$$0 \to \nabla_1 \to \Xi_\alpha \to \nabla_{s_\alpha} \to 0. \tag{5}$$

Lemma 10.5.1. *(a) The functor $\_ * \Xi_\alpha$ is exact*

$$\_ * \Xi_\alpha : \mathrm{Perv}(B\backslash G/B) \to \mathrm{Perv}(B\backslash G/N).$$

(b) The functor $\_ * \Delta_{s_\alpha}$ is left exact

$$\_ * \Delta_{s_\alpha} : D^{\geq 0}(B\backslash G/B) \to D^{\geq 0}(B\backslash G/N).$$

(c) The functor $\_ * \nabla_{s_\alpha}$ is right exact

$$\_ * \nabla_{s_\alpha} : D^{\leq 0}(B\backslash G/B) \to D^{\leq 0}(B\backslash G/N).$$

Proof. Recall that

$$\begin{aligned} D^{\leq 0}(B\backslash G/N) &= \langle \Delta_w[d] \mid d \geq 0, w \in W\rangle \\ &= \{\mathcal{F} \mid \mathrm{Ext}^{<0}(\mathcal{F}, \nabla_w) = 0, \forall w \in W\}. \\ D^{\geq 0}(B\backslash G/N) &= \langle \nabla_w[-d] \mid d \geq 0, w \in W\rangle \\ &= \{\mathcal{F} \mid \mathrm{Ext}^{<0}(\Delta_w, \mathcal{F}) = 0, \forall w \in W\}. \end{aligned}$$

To prove part (a) it suffices to check that $\bar{\Delta}_w * \Xi_\alpha \in \mathcal{P}$ and $\bar{\nabla}_w * \Xi_\alpha \in \mathcal{P}$. There are two cases

(1) $\ell(ws_\alpha) > \ell(w)$. Applying $\bar{\Delta}_w *\_$ to (4) we get

$$\bar{\Delta}_w * \Delta_{s_\alpha} = \Delta_{ws_\alpha} \to \bar{\Delta}_w * \Xi_\alpha \to \Delta_w$$

Since $\Delta_{ws_\alpha} \in \mathcal{P}$ and $\Delta_w \in \mathcal{P}$ we get that $\Delta_w * \Xi_\alpha \in \mathcal{P}$

(2) $\ell(ws_\alpha) < \ell(w)$. In this case we apply $\bar{\Delta}_w *\_$ to (5)

$$\Delta_w \to \bar{\Delta}_w * \Xi_\alpha \to \Delta_{ws_\alpha} = \bar{\Delta}_w * \nabla_{s_\alpha}$$

The same argument as before shows $\bar{\Delta}_w * \Xi_\alpha \in \mathcal{P}$

The argument for ∇_w is similar.

Part (b) follows from (4), while part (c) follows from (5). In more detail, convoluting (4) with M we get a long exact sequence

$$\cdots \to H^{i-1}(M) \to H^i(M * \Delta_s) \to H^i(M * \Xi_\alpha) \to \cdots$$

If $M \in D^{\geq 0}$ then

$$H^i(M) = 0 \qquad \text{and} \qquad H^i(M * \Xi_\alpha) = 0 \qquad \text{for } i < 0.$$

Thus, $H^i(M * \Delta_s) = 0$ for $i < 0$ so $- * \Delta_s$ is left exact. The same argument using (5) instead of (4) shows that $*\nabla_{s_\alpha}$ is right exact. □

Remark 10.5.2. Left/right exactness of $\_ * \Delta_{s_\alpha}/\_ * \nabla_{s_\alpha}$ also follows from a general theorem about direct image of perverse sheaves under affine maps.

Lemma 10.5.3. *For any $w \in W$ there exists maps*

$$\begin{aligned} L_1 = \Delta_1 &\hookrightarrow \Delta_w \\ \nabla_w &\twoheadrightarrow \Delta_1 = L_1, \end{aligned}$$

where the kernel and cokernel does not contain L_1.

Proof. The category $D(B\backslash G/B)$ is at full subcategory of $D(B\backslash G/N)$ so we can work with $\bar{\Delta}_w$ and $\bar{\nabla}_w$ instead of Δ_w and ∇_w. The proof is by induction on the length of w. For $w = e$ the statement is trivial. For $w = s_\alpha$ it can be proved in the same ways as in example 9.0.11. Let $w = s_{\alpha_1} \cdots s_{\alpha_n}$ be a reduced expression. The map $\bar{\Delta}_1 \to \bar{\Delta}_{s_\alpha}$ induces a map

$$\bar{\Delta}_1 = \bar{\Delta}_1 * \cdots * \bar{\Delta}_1 \to \bar{\Delta}_{s_{\alpha_1}} * \cdots * \bar{\Delta}_{s_{\alpha_n}} = \bar{\Delta}_w.$$

Define

$$D' := \langle L_w[d] \mid w \neq 1\rangle \subset D(B\backslash G/B).$$

Claim 10.5.4. The subcategory D' is a 2-sided ideal under convolution.

We have the short exact sequence

$$0 \to \bar{\Delta}_1 \to \bar{\Delta}_{s_\alpha} \to \bar{L}_{s_\alpha} \to 0, \tag{6}$$

i.e. $\operatorname{coker}(\bar{\Delta}_1 \to \bar{\Delta}_{s_\alpha}) = \bar{L}_{s_\alpha} \in D'$. Applying the claim inductively gives that $\operatorname{coker}(\bar{\Delta}_1 \to \bar{\Delta}_w) \in D'$.

If s is a simple reflection with $\ell(sw) > \ell(w)$ then applying $\_ * \Delta_w$ to (6) gives

$$\Delta_w = \bar{\Delta}_1 * \Delta_w \to \Delta_{sw} \to \bar{L}_s * \Delta_w$$

That D' is an ideal and $L_s \in D'$ implies that $L_s * \Delta_w \in D'$ so $\Delta_w \simeq \Delta_{sw} \bmod D'$. This proves that statement about Δ_w. The proof for ∇_w is similar.

Proof of claim. We need to show that for all $\mathcal{F} \in D(B\backslash G/B)$

$$\mathcal{F} * D' \subset D' \qquad \text{and} \qquad D' * \mathcal{F} \subset D'.$$

For $w \neq e$ we can find $s = s_\alpha$ such that $\ell(ws) < \ell(w)$. Consider the projection $\pi_\alpha : G/B \to G/P_\alpha$ with fiber $\mathbb{P}^1$. We have

$$\overline{(G/B)_w} = \pi_\alpha^{-1}(\pi_\alpha((G/B)_w)).$$

By lemma 9.0.10 this implies that

$$\bar{L}_w = \pi_\alpha^*(\mathcal{L}_w).$$

A proof of the same kind as in lemma 10.4.4 shows that

$$\mathcal{F} * \bar{L}_w = \pi_\alpha^*(\mathcal{F} * \mathcal{L}).$$

In particular, $\mathcal{F} * \bar{L}_w$ is constant on the fibers of π_α. Therefore, every perverse subsheaf and subquotient is constant on the fibers. Since L_1 is not constant on any such fiber it cannot occur in the Jordan-Hölder series of $\mathcal{F} * \bar{L}_w$. This proves that D' is a left ideal. The right ideal property is proved similarly replacing G/B by $B\backslash G$. □

□

Corollary 10.5.5. Let P_1 be the projective cover of L_1. Then

$$\dim(\operatorname{End}(P_1)) = |W| \quad \text{and} \quad \operatorname{gr}(P_1) = \bigoplus_{w \in W} \Delta_w.$$

Proof. By BGG reciprocity and the lemma $[\Delta_w, P_1] = [L_1, \nabla_w] = 1$ so $\operatorname{gr}(P_1) = \bigoplus_w \Delta_w$.

$$\begin{aligned} \dim \operatorname{End}(P_1) &= \sum_w \dim \operatorname{Hom}(P_1, \Delta_w) \\ &= \sum_w [L_1, \Delta_w] = |W|. \end{aligned}$$

Which is what we wanted. □

10.6. **Monodromy.** Our plan is to use monodromy to construct a map

$$\mathrm{Sym}(\mathfrak{t}) \to \mathrm{End}(P_1).$$

Definition 10.6.1 (Monodromy sheaf)**.** Let X be an algebraic variety with an action of a torus $T = (\mathbb{C}^*)^n$. A T-monodromy sheaf on X is a sheaf equivariant with respect to the universal cover

$$\mathbb{C}^n \xrightarrow{\exp} (\mathbb{C}^*)^n.$$

The space $\mathbb{C}^n$ is contractible so by property (2) in section 9 and the remark following it this is a full triangulated subcategory

$$D_{T_{\mathrm{mon}}}(\mathrm{Sh}(X)) \subset D(\mathrm{Sh}(X)).$$

In particular, an extension of monodromic sheaves is monodromic.

Equivariant categories are functorial in the group, i.e. given a homomorphism $G \to H$ and a space X with a G action one gets a restriction of equivariance functor $D_H(X) \to D_G(X)$. Applying this to $G = \mathbb{C}^n$, $H = (\mathbb{C}^*)^n$ we see that an equivariant sheaf is monodromic.

$$\begin{tikzcd}
D_T(\mathrm{Sh}(X)) \arrow[rr, hook] \arrow[dr, hook] & & D(\mathrm{Sh}(X)) \\
& D_{T_{\mathrm{mon}}}(\mathrm{Sh}(X)) \arrow[ur, hook] &
\end{tikzcd}$$

For $\chi_*(T) \coloneqq \mathrm{Hom}_{\text{Alg. Grp.}}(\mathbb{G}_m, T) = \pi_1(T)$ there is a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, equal] & \mathbb{C}^n \arrow[r] \arrow[d, equal] & (\mathbb{C}^*)^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \chi_*(T) \arrow[r] & \tilde{T} \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

We also denote $\chi_*(T)$ by Λ. For $\mathcal{F} \in D_{T_{\mathrm{mon}}}(\mathrm{Sh}(X))$ there is a canonical action of Λ on $\mathcal{F}$. If $\mathcal{F}$ is T-equivariant then the action is trivial, i.e. $t^*(\mathcal{F}) \simeq \mathcal{F}$ for all $t \in T$. For equivariant sheaves such an isomorphism is fixed. For monodromic sheaves it exists but is not fixed.

The torus T acts on G/N. It is known that

$$(B \times N)\text{-orbits on } G = (B \times B)\text{-orbits on } G.$$

It follows that every irreducible object $L_w = j_{w!*}(\underline{\mathbb{C}}[\ell(w)])$ in $\mathcal{P}$ lifts to $D(B\backslash G/B)$. In particular, it is monodromic. Hence, all objects in $\mathcal{P}$ (or $D^b(\mathcal{P}) = D(B\backslash G/B)$) are T-monodromic. The action of monodromy on any irreducible is unipotent. This implies that it is unipotent for all $\mathcal{F} \in \mathcal{P}$ so there is a map

$$\Lambda \to \mathrm{Aut}_{\mathrm{unip}}(\mathcal{F})$$

For unipotent automorphisms one has the logarithm map $\log : \mathrm{Aut}_{\mathrm{unip}}(\mathcal{F}) \to \mathrm{End}(\mathcal{F})$. The composition with the above map is $\mathbb{C}$-linear so it induces a map

$$\mathfrak{t} = \Lambda \otimes \mathbb{C} \to \mathrm{End}(\mathcal{F}), \qquad \forall \mathcal{F} \in \mathcal{P}.$$

This can be extended to

$$\mathrm{Sym}(\mathfrak{t}) \to \mathrm{End}(\mathrm{Id}_{\mathcal{P}}).$$

Let $\tilde{\mathcal{P}}$ be the full subcategory of $D(N\backslash G/N)$ consisting of $T\times T$-monodromic sheaves with unipotent monodromy. We just showed that $\mathcal{P}\subset\tilde{\mathcal{P}}$. Using the unipotency one can take log and extend to a map

$$\mathrm{Sym}(\mathfrak{t}\oplus\mathfrak{t})\to\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}}).$$

Lemma 10.6.2. *This map factors through* $\mathrm{Sym}(\mathfrak{t})\otimes_{\mathrm{Sym}(\mathfrak{t})^W}\mathrm{Sym}(\mathfrak{t})$.

Corollary 10.6.3. The map $\mathrm{Sym}(\mathfrak{t})\to\mathrm{End}(\mathrm{Id}_{\mathcal{P}})$ factors through $\mathbb{C}\otimes_{\mathrm{Sym}(\mathfrak{t})^W}\mathrm{Sym}(\mathfrak{t})=A$.

This will be proved in section 11.3

10.7. **Further properties of intertwining functors.**

Lemma 10.7.1. $\mathrm{Hom}(L_v,\Delta_w)=0$ *for* $v\neq e$.

Proof. The proof goes by induction in $\ell(w)$. For $w=e$ the statement is clear. For $w\neq e$ we choose w' and s_α such that $w=w's_\alpha$ and $\ell(w)=\ell(w')+1$. Consider the map $\mathcal{B}_w\to(\mathcal{P}_\alpha)_w$ which is the restriction of the projection $\pi_\alpha:\mathcal{B}=G/B\to\mathcal{P}_\alpha:=G/P_\alpha$ to the orbit corresponding to w. Since $\Delta_w=\Delta_{w'}*\bar{\Delta}_{s_\alpha}$ applying $\Delta_{w'}*\_$ to

$$0\to\bar{L}_1\to\bar{\Delta}_{s_\alpha}\to\bar{L}_{s_\alpha}\to 0$$

we get an exact triangle

$$\Delta_{w'}\to\Delta_w\to\Delta_{w'}*\bar{L}_{s_\alpha}$$

Claim 10.7.2. The sheaf $\Delta_{w'}*\bar{L}_{s_\alpha}$ is a perverse sheaf.

Proof of the Claim. For any sheaf $\mathcal{F}$ we have

$$\mathcal{F}*L_{s_\alpha}=\pi_\alpha^*\pi_{\alpha*}\mathcal{F}[1].$$

The functor $\pi_\alpha^*[1]$ sends perverse sheaves to perverse sheaves so we only need to check that $\pi_{\alpha*}\Delta_{w'}\simeq\pi_{\alpha!}\Delta_{w'}$ is a perverse sheaf. The map π_α restricts to an isomorphism $\mathcal{B}_{w'}\simeq(\mathcal{P}_\alpha)_{w'}$. It follows that

$$\pi_{\alpha!}\Delta_{w'}\simeq j'_{w'!}(\underline{\mathbb{C}}[\ell(w')]).$$

The sheaf $\underline{\mathbb{C}}_{(\mathcal{P}_\alpha)_{w'}}[\ell(w')]$ is a perverse sheaf and it is a general fact that ! and $*$ pushforward by a locally closed affine map sends perverse sheaves to perverse sheaves. This proves the claim. ☐

By the claim the above exact triangle is a short exact sequence. Suppose we have a non-zero map $L_v\to\Delta_w$. By induction $\mathrm{Hom}(L_v,\Delta_{w'})=0$ so the above short exact sequence gives the existence of a non-zero map $L_v\to\Delta_{w'}*L_{s_\alpha}$. As in the proof of claim 10.5.4 there exists an irreducible perverse sheaf L'_v on $\mathcal{P}_\alpha$ such that $L_v=\pi_\alpha^*L'_v[1]$. Hence, we need to show that $\mathrm{Hom}(\pi_\alpha^*L'_v[1],\Delta_w)=0$. By adjointness

$$\mathrm{Hom}(\pi_\alpha^*L'_v[1],\Delta_w)=\mathrm{Hom}(L'_v[1],\pi_{\alpha*}\Delta_w).$$

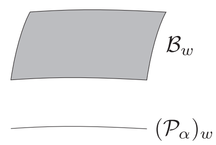
Since G/B is compact we have $\pi_{\alpha*} = \pi_{\alpha!}$. Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{B}_w \arrow[r, hook, "j_w"] \arrow[d, "\pi'_\alpha"] & \mathcal{B} \arrow[d, "\pi_\alpha"] \\
(\mathcal{P}_\alpha)_w \arrow[r, hook, "j'_w"] & \mathcal{P}_\alpha
\end{tikzcd}$$

By base change we get

$$\begin{aligned}
\pi_{\alpha!}\Delta_w = \pi_{\alpha!}j_{w!}(\underline{\mathbb{C}}[\ell(w)]) &\simeq (\pi_\alpha j_w)_!(\underline{\mathbb{C}}[\ell(w)]) \\
\simeq (j'_w\pi'_{\alpha!})_!(\underline{\mathbb{C}}[\ell(w)]) &\simeq j'_{w!}\pi'_{\alpha!}(\underline{\mathbb{C}}[\ell(w)]).
\end{aligned}$$

Notice that $\dim(\mathcal{P}_\alpha)_w = \ell(w) - 1 = \dim \mathcal{B}_w - 1$.



The cohomology with compact support is

$$H^i_c(\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$\pi'_{\alpha!}(\underline{\mathbb{C}}[\ell(w)]) = \underline{\mathbb{C}}[\ell(w)] \otimes H^\bullet_c(\mathbb{C}) = \underline{\mathbb{C}}[\ell(w) - 2].$$

I.e. $\pi_{\alpha!}\Delta_w$ is the ! extension from $(\mathcal{P}_\alpha)_w$ of $\underline{\mathbb{C}}[\ell(w) - 2]$. Let Δ'_w be the zero extension of $\underline{\mathbb{C}}[\ell(w) - 1]$ from $(\mathcal{P}_\alpha)_w$. Then $\pi_{\alpha!}\Delta_w = \Delta'_w[-1]$. Thus, we have

$$\begin{aligned}
\operatorname{Hom}(L_v, \Delta_w) &= \operatorname{Hom}(L'_v[1], \pi_{\alpha!}\Delta_w) \\
&= \operatorname{Hom}(L'_v[1], \Delta'_w[-1]) \\
&= \operatorname{Ext}^{-2}(L'_v, \Delta'_w) \\
&= 0.
\end{aligned}$$

This finishes the proof. $\square$

It is a known fact that $D^b(\mathrm{Perv}_N(\mathcal{P}_\alpha)) \xrightarrow{\sim} D_N(\mathcal{P}_\alpha)$.

Claim 10.7.3. (see e.g. [BBM]) The intertwining functor I_{w_0} sends projectives to tiltings.

Proof. Let P be projective. We know it has a filtration with standard subquotients. The formulas at the beginning of subsection 10.5 show that $I_{w_0}(\Delta_w) = \nabla_{ww_0}$. It follows that $I_{w_0}(P)$ is a perverse sheaf with a costandard filtration. To see that it is tilting it is enough to check that $\operatorname{Ext}^i(I_{w_0}(P), \nabla_w) = 0$ for $i > 0$ and any w. Since I_{w_0} is an equivalence and $\nabla_w = I_{w_0}(\Delta_{ww_0})$, this follows from $\operatorname{Ext}^i(P, \Delta_{ww_0}) = 0$. $\square$

Set $\mathcal{B} := \langle L_w \mid w \neq e \rangle$. The functor I_{w_0} is an autoequivalence of $D^b(\mathcal{P})$ preserving the subcategory of complexes with cohomology in $\mathcal{B}$. Lemma 10.7.1 states that a sheaf T is in ${}^{\perp}\mathcal{B}$ if it has a standard filtration and dually that $T \in \mathcal{B}^{\perp}$ if it has a costandard filtration. It follows that the functor

$$\mathcal{P} \to \mathcal{P}_0 = \mathcal{P}/\mathcal{B}$$

is fully faithful on tilting objects.

Our goal is to describe $\mathcal{P}_0$ and Ξ_α. These are counterparts of wall-crossing. Recall that we had a partial description as convolution with an element in $\mathrm{Perv}_B(X)$.

$$0 \to \Delta_{s_\alpha} \to \Xi_\alpha \to \Delta_e \to 0,$$
$$0 \to \nabla_e \to \Xi_\alpha \to \nabla_{s_\alpha} \to 0.$$

10.8. **Extension of** Id **by** I_s**.** In this section $G = \mathrm{SL}_2$. Our aim is to construct an extension of Id by I_s. For this we want an alternative description of Id on $\mathcal{P}$ or $\mathcal{C} = D_B(G/N)$. Set $X = G/N = \mathbb{A}^2 \backslash \{0\}$ and define

$$\begin{tikzcd} & Z := \{(v,w,t) \mid v,w \in \mathbb{A}^2, v \wedge w = 1, t \in \mathbb{A}^1\} \arrow[dl, "f"', "(v,w,t)\mapsto tv+w"] \arrow[dr, "\mathrm{pr}_{wt}", "(v,w,t)\mapsto(w,t)"'] & \\ X & & X \times \mathbb{A}^1 \arrow[d, "\mathrm{pr}_w"', "(w,t)\mapsto w"] \\ & & X \end{tikzcd}$$

Claim 10.8.1. There is an isomorphism $\mathrm{Id}_{\mathcal{C}} \simeq \mathrm{pr}_{w*}\mathrm{pr}_{wt!} f^*(\ \ )[2]$.

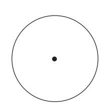
Consider the inclusion

$$i : X \to X \times \mathbb{A}^1, \qquad w \mapsto (w, 0).$$

It is a general fact that $\mathrm{pr}_{w*}\mathcal{F} \simeq i^*\mathcal{F}$ if $\mathcal{F}$ is monodromic with respect to dilations on $\mathbb{A}^1$.

Example 10.8.2. For $X = \mathrm{pt}$ all terms in the limit are the same so

$$i_0^*\mathcal{F} = \varinjlim_{\mathcal{U} \text{ nbh. of } 0} (R\Gamma(\mathcal{F}(\mathcal{U}))) = R\Gamma(\mathcal{F}(\mathcal{U})) = \mathrm{pr}_{w*}\mathcal{F}.$$



Proof of claim. Define a torus action on Z by

$$\lambda : Z \to Z, \qquad v \mapsto \lambda v,\ \ w \mapsto \lambda^{-1} w,\ \ t \mapsto \lambda^2 t, \qquad \lambda \in \mathbb{C}^*.$$

Notice that f is equivariant with respect to this action. A sheaf $\mathcal{F}$ in $\mathcal{C}$ is monodromic with respect to this action and it is also equivariant for the B-action on the second factor. Since w is conjugate to λw under B for a fixed w we get

$$\{w\} \times \mathbb{A}^1 \subset \text{an orbit of } B \times \mathbb{C}^* \quad (w,t) \mapsto (\lambda^{-1} w, \lambda^2 t),\ \ \lambda \in \mathbb{C}^*.$$

Hence, $\mathrm{pr}_{wt!} f^*$ is monodromic for dilations of t. Using the general fact we get

$$\mathrm{pr}_{w*}\mathrm{pr}_{wt!} f^*(\mathcal{F})[2] \simeq i^* \mathrm{pr}_{wt!} f^*(\mathcal{F})[2].$$

Notice that

$$Z \times_{X\times\mathbb{A}^1} X \simeq \{(v,w) \mid v \wedge w = 1\}.$$

Both f and projection to the second factor coincide with the map $\mathrm{pr} : (v,w) \mapsto w$ so using base change we get $i^*\mathrm{pr}_{wt!}f^*(\mathcal{F}) \simeq \mathrm{pr}_!\mathrm{pr}^*(\mathcal{F})$. The projection pr is a fibration with fiber $\mathbb{A}^1$ so we have $\mathrm{pr}_!\mathrm{pr}^*(\mathcal{F}) \simeq \mathcal{F} \otimes H^*_c(\mathbb{C}) \simeq \mathcal{F}[-2]$. This finishes the proof. □

We now construct the first map in the extension

$$\mathcal{F} \to I_s\mathcal{F}[1]$$

Recall that we have

$$X_s \simeq \{(v,w) \mid v,w \in \mathbb{A}^2, v \wedge w = 1\}$$

with two projections $\mathrm{pr}_1 : (v,w) \mapsto v$, $\mathrm{pr}_2 : (v,w) \mapsto w$. It can be considered it as a subspace of Z in the following way

$$X_s \simeq \{(v,w,t) \mid v,w \in \mathbb{A}^2, v \wedge w = 1, t = 1\} \overset{i'}{\hookrightarrow} Z$$

Applying $\mathrm{pr}_{w*}\mathrm{pr}_{wt!}(\ \ )[2]$ to the adjunction map $f^*\mathcal{F} \to i'_*i'^*f^*\mathcal{F}$ we obtain a map

$$\mathcal{F} \simeq \mathrm{pr}_{w*}\mathrm{pr}_{wt!}f^*\mathcal{F}[2] \to \mathrm{pr}_{w*}\mathrm{pr}_{wt!}i'_*i'^*f^*\mathcal{F}[2]$$

The map i' is a closed immersion, so it is proper and we can replace i'_* by $i'_!$ in the last expression without changing the result. Likewise, we can replace pr_{w*} by $\mathrm{pr}_{w!}$ without changing the result because we are applying it to a complex supported on $\{(w,t) \mid t = 1\}$ and pr_w is an isomorphism (hence a proper map) on this set. Using that $fi' = \mathrm{pr}_1$ and $\mathrm{pr}_w\mathrm{pr}_{wt}i' = \mathrm{pr}_2$ we get

$$\mathrm{pr}_{w*}\mathrm{pr}_{wt!}i'_*i'^*f^*\mathcal{F}[2] \simeq \mathrm{pr}_{w!}\mathrm{pr}_{wt!}i'_!i'^*f^*\mathcal{F}[2] \simeq \mathrm{pr}_{2!}\mathrm{pr}_1^*\mathcal{F}[2] = I_s(\mathcal{F})[1].$$

Thus, we have constructed the desired map.

For the second map in the extension we consider

$$\begin{tikzcd} & Z' := \{(v,w,t) \mid v,w \in \mathbb{A}^2, v \wedge w = 1, t \neq 1\} \arrow[dl, "(v,w,t)\mapsto v+tw"'] \arrow[dr, "\phi"] & \\ X & & X \times \mathbb{A}^1 \arrow[d, "\psi"] \\ & & X \end{tikzcd}$$

Let j be the inclusion $X \times \mathbb{A}^1_{t\neq 1} \hookrightarrow X \times \mathbb{A}^1$. Define the functor

$$\Xi : \mathcal{C} \to \mathcal{C}, \qquad \mathcal{F} \mapsto \mathrm{pr}_{w*}j_!j^*\mathrm{pr}_{wt!}f^*(\mathcal{F})[2]$$

Notice that the distinguished triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}$ gives a functorial triangle

$$\Xi(\mathcal{F}) \to \mathcal{F} \to I_s(\mathcal{F})[1]$$

This is the desired extension.

The formula for Ξ can be simplified as follows. Consider the cartesian diagram

$$\begin{tikzcd} Z' \arrow[r, "j'"] \arrow[d, "\mathrm{pr}'"'] \arrow[dr, "\phi"] & Z \arrow[d, "\mathrm{pr}_{wt}"] \\ X \times \mathbb{A}_{t\neq 1} \arrow[r, "j"'] & X \times \mathbb{A}^1 \end{tikzcd}$$

By base change $j^*\mathrm{pr}_{w!} \simeq \mathrm{pr}'_! j'^*$ so

$$\begin{aligned} \mathrm{pr}_{w*} j_! j^* \mathrm{pr}_{wt!} f^*(\mathcal{F})[2] &\simeq \mathrm{pr}_{w*} j_! \mathrm{pr}'_! j'^* f^*(\mathcal{F})[2] \\ &\simeq \mathrm{pr}_{w*}(j \circ \mathrm{pr}')_!(f \circ j')^*(\mathcal{F})[2] \\ &\simeq \psi_* \phi_! f^*(\mathcal{F})[2]. \end{aligned}$$

Exercise 10.8.3. Show that $\Xi(\pi^*\mathcal{F}) = \mathcal{F} * \Xi$ where $\pi : X \to \mathbb{P}^1$ is the projection.

For general G and α a simple root we consider

$$\begin{tikzcd} X & G/N_\alpha \times N_\alpha = Z \supset Z' = \{(x,t) \mid t \neq 1\} \arrow[l, "f"', "(x,t)\mapsto x \mod N"] \arrow[r, "\phi", "(x,t)\mapsto(xt \mod N, t)"'] & X \times \mathbb{A}^1 \arrow[d, "\psi"] \\ & & X \end{tikzcd}$$

We define the functor

$$\Xi_\alpha : D_B(G/N) \to D_B(G/N), \qquad \mathcal{F} \mapsto \psi_* \phi_! f^*(\mathcal{F})[2].$$

We will show later that this extends our previously defined Ξ_α functor

$$\begin{tikzcd} D(B\backslash G/N) \arrow[r, "\Xi_\alpha"] & D(B\backslash G/N) \\ D(B\backslash G/B) \arrow[u, "\mathrm{forget}"] \arrow[ur, "\Xi_\alpha"'] & \end{tikzcd}$$

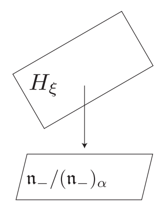
10.9. **Construction of a map $\mathcal{P}_0 \to \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}_{\mathbf{nilp}}$.** Let $\mathcal{F} \in \mathcal{P}$ be a sheaf on $\mathcal{B} = B\backslash G$ and $x_0 = \mathcal{B}_e$ the unique fixed point. Then $x_0 N_- \simeq N_- \overset{\log}{\simeq} \mathfrak{n}_-$

There is a general construction called microlocalization, which will be discussed in more detail below (see section 11.1), which associates to a perverse sheaf on a vector space V perverse sheaf $\mu(\mathcal{F})$ on the dual space V^*. For now we only need the restriction of $\mu(\mathcal{F})$ to an open part in V^*, which we describe under an additional assumption that $\mathcal{F}$ is monodromic (weakly equivariant) with respect to a contracting linear action of $\mathbb{C}^*$. In this case one can find a Zariski open dense $U \subset V^*$ such that for $\xi \in U$ we have $R^i\Gamma(V, H_\xi; \mathcal{F}) = 0$ for $i \neq 0$, while the space $R^0\Gamma(V, H_\xi; \mathcal{F})$ is identified with the fiber at ξ of a local system on U. Here $H_\xi = \{x \mid \langle \xi, x \rangle = 1\}$.

Using this one gets a local system on a Zariski open set $U \subset \mathfrak{n}_-^*$ whose fiber at $\xi \in \mathfrak{n}^*$ is $R\Gamma(\mathfrak{n}_-, H_\xi; \mathcal{F})$.

Example 10.9.1. In our case for L_v with $v \neq e$ one can use
$$U := \Big\{x = \sum_{\alpha\in\Delta^+} x_\alpha \,\Big|\, x_\alpha \neq 0 \;\; \forall \alpha \in \Delta\Big\} \quad \subset \; T^*_{x_0}\mathcal{B} \simeq (\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n},$$
where $\sum_\alpha x_\alpha$ is the image under the map
$$\mathfrak{n} \to \mathfrak{n}/[\mathfrak{n},\mathfrak{n}] = \bigoplus_{\alpha\in\Delta} \mathfrak{n}_{-\alpha}, \qquad x \mapsto \sum_{\alpha\in\Delta} x_\alpha.$$

Proof. We will show that if $\xi|_{(\mathfrak{n}_-)_{-\alpha}} \neq 0$ for all simple roots α then $R\Gamma(\mathfrak{n}_-, H_\xi; L_v) = 0$ for all $v \neq e$. If $v \neq e$ then $v = s_\alpha v'$ for some v' with $\ell(v) > \ell(v')$. In this case our sheaf L_v on $\mathfrak{n}_-$ is a pull-back under the projection $\mathfrak{n}_- \to \mathfrak{n}_-/(\mathfrak{n}_-)_{-\alpha}$.
$$L_v = \pi_\alpha^*(L'_v)$$
If $\xi|_{(\mathfrak{n}_-)_{-\alpha}} \neq 0$ then π_α maps H_ξ isomorphically to $\mathfrak{n}_-/(\mathfrak{n}_-)_{-\alpha}$ so $R\Gamma(H_\xi, L_v|_{H_\xi}) \simeq R\Gamma(\mathfrak{n}_-, L_v|_{\mathfrak{n}_-})$. Hence, $R\Gamma(\mathfrak{n}_-, H_\xi; L_v) = 0$. $\square$



10.10. **The functor Ξ_α.** Recall that $D_B(G/B)$ is a monoidal category acting on $\mathcal{C} = D(\mathcal{P})$ on the left by convolution. This action commutes with Ξ_α because the left action is defined as a composition of smooth pull-backs and proper push-forwards (because G/B is proper), these fit base change and composition isomorphisms with both $*$ and ! direct images.

So for $\mathcal{F} \in D_B(G/B)$
$$\Xi_\alpha(\mathcal{F}) = \Xi_\alpha(\mathcal{F} * \Delta_e) = \mathcal{F} * \Xi_\alpha(\Delta_e),$$
where we used the same notation for the corresponding object in $D_B(G/N)$, i.e. its pull-back. By associativity of the convolution the originally defined Ξ_α also commutes with convolution. Hence, to show that they agree on $D_B(G/B)$ it is enough to show that they agree on Δ_e. Using the exact triangle above we see that both fit into a short exact sequence
$$0 \to \Delta_{s_\alpha} \to \Xi_\alpha(\Delta_e) \to \Delta_e \to 0$$
Since $\mathrm{Ext}^1(\Delta_e, \Delta_{s_\alpha}) \simeq \mathbb{C}$ and the sequence for the original Ξ_α is non-trivial it suffices to show that the extension is non-trivial. To check this it is enough to check one of the following properties

(1) $R\Gamma(\Xi_\alpha(\Delta_e)) = 0$.
(2) The ! restriction to the closed B-orbit has rank 1.

The first one also implies that Ξ_α is exact. In particular,
$$\mathcal{F} \in \mathrm{Perv}_B(G/B) \Rightarrow \Xi_\alpha(\mathcal{F}) \in \mathcal{P}$$
so we get a functor $\Xi_\alpha : \mathcal{P} \to \mathcal{P}$.

Lemma 10.10.1. *The functor Ξ_α sends projectives to projectives.*

Proof. We need to check that for P projective
$$\mathrm{Ext}^n(\Xi_\alpha P, M) = 0 \qquad \forall M \in \mathcal{P}, n > 0.$$
For $n > 0$ the short exact sequence $0 \to I_{s_\alpha} P \to \Xi_\alpha P \to P \to 0$ produces an exact sequencce
$$0 = \mathrm{Ext}^n(P, M) \to \mathrm{Ext}^n(\Xi_\alpha P, M) \to \mathrm{Ext}^n(I_{s_\alpha} P, M) = \mathrm{Ext}^n(P, I'_{s_\alpha} M).$$
Notice that
$$\mathrm{Ext}^n(P, I'_{s_\alpha} M) \neq 0 \quad \text{only if} \quad H^n(I'_{s_\alpha} M) \neq 0.$$
Recall the direction of exactness of the functors
$$I_{s_\alpha} : D^{\geq 0} \to D^{\geq 0}, \qquad I'_{s_\alpha} : D^{\leq 0} \to D^{\leq 0}.$$
Hence, $H^n(I'_{s_\alpha} M) \neq 0$ for $n \leq 0$ only. Thus, $\mathrm{Ext}^n(\Xi_\alpha P, M) = 0$ for $n > 0$. □

Lemma 10.10.2. *The sheaf $\bigoplus_{w = s_{\alpha_1} \cdots s_{\alpha_n}} \Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(\Delta_{w_0})$ is a projective generator. Here the sum runs over all $w \in W$ and for each w we fixed a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$.*

Proof. Recall that $\Delta_{w_0} \in \mathcal{P}$ is projective so by the lemma the sheaf is projective. To find out which projective $\Xi_\alpha P$ is for a given projective P one needs to calculate $\mathrm{Hom}(\Xi_\alpha P, L)$ for L irreducible.

Claim 10.10.3. Let P be projective. Then for any L
$$\dim \mathrm{Hom}(\Xi_\alpha P, L) = \dim \mathrm{Hom}(P, \Xi_\alpha L).$$

Proof of claim. Since $\Xi_\alpha P$ is projective
$$\dim \mathrm{Hom}(\Xi_\alpha P, L) = \mathrm{Eul}(\mathrm{Hom}^\bullet(\Xi_\alpha P, L)).$$
The exact triangle $I_{s_\alpha} P \to \Xi_\alpha P \to P$ gives
$$\begin{aligned}\mathrm{Eul}(\mathrm{Hom}^\bullet(\Xi_\alpha P, L)) &= \mathrm{Eul}(\mathrm{Hom}^\bullet(I_{s_\alpha} P, L)) + \mathrm{Eul}(\mathrm{Hom}^\bullet(P, L)) \\ &= \mathrm{Eul}(\mathrm{Hom}^\bullet(P, I'_{s_\alpha} L)) + \mathrm{Eul}(\mathrm{Hom}^\bullet(P, L)).\end{aligned}$$
The corresponding exact triangle for I'_{s_α} given by $L \to \Xi_\alpha L \to I'_{s_\alpha} L$ implies
$$\begin{aligned}\mathrm{Eul}(\mathrm{Hom}^\bullet(P, I'_{s_\alpha} L)) + \mathrm{Eul}(\mathrm{Hom}^\bullet(P, L)) &= \mathrm{Eul}(\mathrm{Hom}^\bullet(P, \Xi_\alpha L)) \\ &= \dim \mathrm{Hom}(P, \Xi_\alpha L).\end{aligned}$$
This finishes the proof of the claim. □

To show that it is a projective generator it is enough to show that for a reduced expression $w^{-1} w_0 = s_{\alpha_1} \cdots s_{\alpha_n}$
$$P_w \subset \Xi_{\alpha_n} \circ \cdots \circ \Xi_{\alpha_1}(\Delta_{w_0}).$$
For this it is enough to show that
$$\dim \mathrm{Hom}(\Xi_{\alpha_n} \circ \cdots \circ \Xi_{\alpha_1}(\Delta_{w_0}), L_w) \neq 0.$$

Claim 10.10.4. Let M be a sheaf which is constant on $\mathcal{B}_{w_0}$. Then
$$\mathrm{Hom}(\Delta_{w_0}, M) = \text{stalk of restriction of } M \text{ to the open stratum } \mathcal{B}_{w_0}[-\dim \mathcal{B}].$$

Proof. Since j_{w_0} is an open embedding we have

$$\begin{aligned}\mathrm{Hom}(\Delta_{w_0}, M) &= \mathrm{Hom}(j_{w_0!}\underline{\mathbb{C}}[\ell(w_0)], M)\\ &\simeq \mathrm{Hom}(\underline{\mathbb{C}}[\ell(w_0)], j_{w_0}^* M)\\ &\simeq \mathrm{Hom}(\underline{\mathbb{C}}, j_{w_0}^* M[-\dim \mathcal{B}]).\end{aligned}$$

Since $M|_{\mathcal{B}_{w_0}}$ is constant, a homomorphism from the constant sheaf correspond to choosing an element in the stalk

$$j_{w_0}^* M[-\dim \mathcal{B}]_x = (M[-\dim \mathcal{B}]|_{\mathcal{B}_{w_0}})_x$$

Hence, $\mathrm{Hom}(\Delta_{w_0}, M)$ = stalk of restriction of M to the open stratum $\mathcal{B}_{w_0}[-\dim \mathcal{B}]$. □

The two claims imply that

$$\begin{aligned}\dim \mathrm{Hom}(\Xi_{\alpha_n} \circ \cdots \circ \Xi_{\alpha_1}(\Delta_{w_0}), L_w) &= \dim \mathrm{Hom}(\Delta_{w_0}, \Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(L_w))\\ &= \dim(\Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(L_w)[-\dim \mathcal{B}]|_{\mathcal{B}_{w_0}})_x\end{aligned}$$

It would follow that it is non-zero if we can prove that $L_{w_0} \subseteq \Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(L_w)$. When $\ell(ws_\alpha) > \ell(w)$ we have $\mathrm{supp}(\Xi_\alpha(L_w)) = \overline{\mathcal{B}_{ws_\alpha}}$ and $\Xi_\alpha(L_w)|_{\mathcal{B}_{ws_\alpha}} = \underline{\mathbb{C}}[\ell(ws_\alpha)]$. Since $\mathcal{B}_{ws_\alpha}$ is open in the support of the perverse sheaf $\Xi_\alpha(L_w)$, the latter has to contain L_{ws_α} in its Jordan-Hölder series. So $\Xi_\alpha(L_w)$ contains L_{ws_α}. Iterating this process we get that $\Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(L_w)$ contains L_{w_0} when $s_{\alpha_1} \cdots s_{\alpha_n}$ is a reduced expression for $w^{-1}w_0$. □

11. An equivalence of categories between perverse sheaves and category $\mathcal{O}$

11.1. **Microlocalization at $\mathcal{B}_e$.** We want to construct a functor

$$\mu_0 : \mathcal{P} \to \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}$$

This will play a key role in proving that $\mathcal{P} \simeq \mathcal{O}_\lambda$ for λ regular and integral.

Consider the inclusion $\mathfrak{n}_- \simeq x_0 N_- \hookrightarrow \mathcal{B}$. Pulling back along this map we get a sheaf on $\mathfrak{n}$. Consider

$$\begin{tikzcd} & \{(x,\xi) \mid (x,\xi) = 1\} = V \arrow[r, hook, "j"] \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \mathfrak{n}_- \times \mathfrak{n}_-^* \\ \mathfrak{n}_- & & \mathfrak{n}_-^* \end{tikzcd}$$

Set $\phi_0 := \mathrm{pr}_{2*} j_! \mathrm{pr}_1^*$ and take its restriction to U, where

$$U := \{\phi \mid \phi|_{(\mathfrak{n}_-)_{-\alpha}} \neq 0\} \overset{\circ}{\subseteq} \mathfrak{n}_-^*.$$

Under this composed functor L_w maps to 0 for $w \neq e$ and L_e maps to the constant sheaf.

It follows that any perverse sheaf in our category goes to a local system on U with a unipotent monodromy. The fundamental group of U is the group of cocharacters of T, for a representation of this group landing in unipotent matrices we can take log, obtaining a representation of the ring $\mathrm{Sym}(\mathfrak{t})$. Thus, we defined a functor

$$\mu_0 : \mathcal{P} \to \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}$$

taking a sheaf $\mathcal{F}$ to the generic fiber of $\phi_0(\mathcal{F})$ with the action of logarithm of monodromy.

Recall that P_1 is the projective cover of L_e. By Yoneda's lemma and uniqueness of projective cover we get

$$\mu_0(M) = \mathrm{Hom}(P_1, M) \qquad \forall M \in \mathcal{P}. \tag{7}$$

In section 10.6 we constructed a map

$$\mathrm{Sym}(\mathfrak{t}) \to \mathrm{End}(\mathrm{Id}_{\mathcal{P}}).$$

The functors are compatible with the Torus actions so for $M \in \mathcal{P}$ the action of $\mathfrak{t}$ on $\mu_0(M)$ given by the map coincides with the monodromy action of $\mathfrak{t}$ on $\mu_0(M)$.

Lemma 11.1.1. *(a) There is an isomorphism $I_w(P_1) \simeq P_1$.*
(b) The actions of $w \in W$ on $\mathfrak{t}$ and I_w on $\mathrm{End}(\mathrm{Id}_{\mathcal{P}})$ commute, i.e. we have a commutative diagram

$$\begin{tikzcd} \mathfrak{t} \arrow[r] \arrow[d, "w"] & \mathrm{End}(\mathrm{Id}_{\mathcal{P}}) \arrow[d, "I_w"] \\ \mathfrak{t} \arrow[r] & \mathrm{End}(\mathrm{Id}_{\mathcal{P}}) \end{tikzcd}$$

For $x \in \mathfrak{t}$ have $m_x^M \in \mathrm{End}(M)$

$$I_w(m_x^M) = m_{w(x)}^{I_w(M)}.$$

Proof. (a) We know that $I_w(P_1)$ is projective so it is enough to show that

$$\mathrm{Hom}(I_w(P_1), L_v) = \begin{cases} 0 & v \neq e \\ \mathbb{C} & v = e \end{cases}$$

Notice that $\mathrm{Hom}(I_w(P_1), L_v) = \mathrm{Hom}(P_1, I_w^{-1}(L_v))$. In claim 10.5.4 we proved that $\langle L_u \mid u \neq e\rangle$ is an ideal under convolution so

$$I_w^{-1}(L_v) = L_v * \nabla_{w^{-1}} \in \langle L_u \mid u \neq e\rangle.$$

Consider the short exact sequence

$$0 \to L_{s_\alpha} \to \nabla_{s_\alpha} \to L_e \to 0.$$

From this it follows that $\nabla_{s_\alpha} \simeq \nabla_e \simeq \nabla_w \mod \langle L_u \mid u \neq e\rangle$.

(b) Notice that X_w is invariant under the action of $T \simeq \{(t, w(t)) \mid t \in T\} \subset T \times T$.

$$\begin{tikzcd} & X_w \arrow[dl] \arrow[dr] & \\ T \circlearrowright X & & X \circlearrowleft T \end{tikzcd}$$

So if we make T acts on the first copy of X in the natural way, on the second one in the natural way twisted by w, and on X_w via $t \mapsto (t, w(t))$, then all the arrows in the diagram are compatible with the T action, hence pull-back and push-forward functors send monodromic sheaves to monodromic sheaves and are compatible with monodromy automorphisms.

□

Example 11.1.2. Recall that for G =SL_2 we have $X_s = \{(v, z) \mid v \wedge z = 1\}$. The action of T is given by multiplication by (t, t^{-1}) and X_s is invariant under this action.

Corollary 11.1.3. There is an isomorphism $\mu_0(I_w(M)) \simeq \mu_0(M)^w$ where ^{w} means that the $\mathfrak{t}$-action is twisted by w, i.e. $t \cdot x = w(t)x$.

Proof. By (7) and the lemma

$$\begin{aligned}\mu_0(I_w(M)) &\simeq \mathrm{Hom}(P_1, I_w(M))\\ &\simeq \mathrm{Hom}(I_w^{-1}(P_1), M)\\ &\simeq \mathrm{Hom}(P_1, M)^w\\ &\simeq \mu_0(M)^w.\end{aligned}$$

Which is what we wanted. □

Lemma 11.1.4. *There is an isomorphism* $\mu_0(\Xi_\alpha(M)) \simeq \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \mu_0(M)$.

Notice that if N does not contain L_e then $\mu_0(N) = 0$. Hence, μ_0 factors through $\mathcal{P}\big/\langle L_v \mid v \neq e\rangle$. Since this is a 2-sided ideal with respect to convolution $\mu_0 \circ \Xi_\alpha$ also factors through. The image of P_1 in $\mathcal{P}\big/\langle L_v \mid v \neq e\rangle$ is a projective generator for that category, so it suffices to construct the isomorphism for $M = P_1$. This is done later in proposition 11.3.4.

Remark 11.1.5. Both sides fit into a short exact sequence

$$0 \to \mu_0(M)^{s_\alpha} \to \mu_0(\Xi_\alpha(M)) \to \mu_0(M) \to 0.$$

For the left hand side this follows from applying the exact functor μ_0 to the exact triangle $I_{s_\alpha}(M) \to \Xi_\alpha(M) \to M$ and using $\mu_0(I_{s_\alpha}(M)) = \mu_0(M)^{s_\alpha}$.

For the right hand side notice that

$$\mathfrak{t}^* \times_{\mathfrak{t}^*/\{1,s_\alpha\}} \mathfrak{t}^* = \Delta_{\mathfrak{t}^*} \sqcup \Gamma_{s_\alpha},$$

where $\Delta_{\mathfrak{t}^*}$ is the diagonal and Γ_{s_α} is the graph of the s_α action. Moreover, the closed subvariety $\Gamma_{s_\alpha} \subset \Delta_{\mathfrak{t}^*} \sqcup \Gamma_{s_\alpha}$ is given by one equation $pr_1^*(\lambda) - pr_2^*(s_\alpha(\lambda))$ where $\lambda \in \mathfrak{t}$ is such that $\langle \lambda, \alpha \rangle \neq 0$. Thus we get an exact sequence

$$0 \to \mathcal{O}_{\Gamma_{s_\alpha}} \to \mathcal{O}_{\Delta_{\mathfrak{t}^*} \sqcup \Gamma_{s_\alpha}} \to \mathcal{O}_{\Delta_{\mathfrak{t}^*}}$$

Tensoring with $\mu_0(M)$ over $\mathcal{O}(\mathfrak{t}^*) = \mathrm{Sym}(\mathfrak{t})$ we get the desired short exact sequence.

Theorem 11.1.6. *For λ regular and integral there is an equivalence of categories* $\mathcal{P} \simeq \mathcal{O}_\lambda$.

Proof. Recall that in corollary 6.2.11 we proved that

$$\mathcal{O}_\lambda \simeq \mathcal{A}-\mathrm{mod}, \qquad \text{where } \mathcal{A} = \mathrm{End}_A\Big(\bigoplus_{w = s_{\alpha_1}\cdots s_{\alpha_n}} A \otimes_{A^{s_{\alpha_1}}} \cdots \otimes_{A^{s_{\alpha_n}}} \mathbb{C}\Big)^{\mathrm{opp}}$$

and in section 10.6 we constructed a map

$$\mathrm{Sym}(\mathfrak{t}) \to \mathrm{End}(\mathrm{Id}_{\mathcal{P}}).$$

The first step in the proof is to prove the following claim

Claim 11.1.7. For all M the map sends $\mathrm{Sym}(\mathfrak{t})^W_+$ to $0 \in \mathrm{End}(\mu_0(M))$; and for some M we get an inclusion

$$\mathrm{Sym}(\mathfrak{t})/\,\mathrm{Sym}(\mathfrak{t})^W_+ \hookrightarrow \mathrm{End}(\mu_0(M))$$

Proof of claim. To check the first part of the claim it is enough to prove the claim for a projective generator. By claim 10.10.2

$$M = \bigoplus_{w=s_{\alpha_1}\cdots s_{\alpha_n}} \Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(\Delta_{w_0})$$

is a projective generator. The lemma implies that

$$\begin{aligned}\mu_0(M) &\simeq \bigoplus_{w=s_{\alpha_1}\cdots s_{\alpha_n}} \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_{\alpha_1}}} \cdots \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_{\alpha_n}}} \mu_0(\Delta_{w_0}) \\ &\simeq \bigoplus_{w=s_{\alpha_1}\cdots s_{\alpha_n}} \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_{\alpha_1}}} \cdots \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_{\alpha_n}}} \mathbb{C}.\end{aligned}$$

If $\mathrm{Sym}(\mathfrak{t})_+^W$ acts by 0 on an A-module N then it acts by 0 on $\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} N$ so in particular it acts by 0 on $\mu_0(M)$. We know that if $w_0 = s_{\alpha_1} \cdots s_{\alpha_N}$ is a reduced expression then A acts faithfully on $A \otimes_{A^{s_{\alpha_1}}} \cdots \otimes_{A^{s_{\alpha_N}}} \mathbb{C}$, so the map $A \to \mathrm{End}(\mu_0(M))$ is injective. □

For M as above consider a map $P_1^n \to M$ such that its cokernel does not contain L_1 in its Jordan-Hölder series. Then $\mu_0(P_1^n)$ maps surjectively to $\mu_0(M)$. Since the action of A on $\mu_0(M)$ is faithful, the action of A on $\mu_0(P_1^n)$, and hence on $\mu_0(P_1)$ is also faithful. Thus, A maps injectively to $\mathrm{End}(P_1)$. In corollary 10.5.5 we proved that

$$\dim(\mathrm{End}(P_1)) = |W| = \dim(A),$$

so $A \simeq \mathrm{End}(P_1)$. Thus, we can consider μ_0 as a functor

$$\mu_0 : \mathcal{P} \to A - \mathrm{mod}\,.$$

Recall that $\mu_0 \simeq \mathrm{Hom}(P_1, -)$ is fully faithful on tilting objects. By claim 10.7.3 and corollary 11.1.3 we have a commutative diagram

$$\begin{tikzcd} \text{Projectives} \arrow[r, "\sim", "I_{w_0}"'] \arrow[d, "\mu_0"'] & \text{Tilting} \arrow[d, "\mu_0"] \\ A - \mathrm{mod} \arrow[r, "w_0"] & A - \mathrm{mod} \end{tikzcd}$$

Thus, μ_0 is also fully faithful on projectives. Since μ_0 sends the projective generator M in $\mathcal{P}$ to the projective generator $\bigoplus_{w=s_{\alpha_1}\cdots s_{\alpha_n}} A \otimes_{A^{s_{\alpha_1}}} \cdots \otimes_{A^{s_{\alpha_n}}} \mathbb{C}$ in $\mathcal{A} - \mathrm{mod} \simeq \mathcal{O}_\lambda$ it gives the equivalence of categories $\mathcal{P} \simeq \mathcal{O}_\lambda$. □

11.2. **Example for $G = \mathrm{SL}_2$.** Consider the case G=SL_2 with maps $0 \hookrightarrow \mathcal{B} = \mathbb{P}^1 \hookleftarrow \mathbb{A}^1$. We want to investigate the action of monodromy on T (previously called Ξ). Recall the short exact sequences from example 9.0.11

$$\begin{aligned}&0 \to \underline{\mathbb{C}}_0 \to T \to j_* \underline{\mathbb{C}}_{\mathbb{A}^1}[1] \to 0,\\ &0 \to j_! \underline{\mathbb{C}}_{\mathbb{A}^1}[1] \to T \to \underline{\mathbb{C}}_0 \to 0.\end{aligned}$$

In particular, $\mu_0(T)$ is 2-dimensional. We claim that when $\mathfrak{t}$ acts naturally the monodromy acts by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The plan is to embed T into something with known monodromy. For this we restrict from $\mathbb{P}^1$ to $\mathbb{P}^1\backslash\{\infty\}$. Consider the local system $\mathcal{E}$ of rank 2 on $\mathbb{C}^* = \mathbb{P}^1\backslash\{0,\infty\}$. Then we have a short exact sequence on $\mathbb{C}^*$

$$\underline{\mathbb{C}}_{\mathbb{C}^*} \to \mathcal{E} \to \underline{\mathbb{C}}_{\mathbb{C}^*}$$

The monodromy $m_{\mathcal{E}}$ on $\mathcal{E}$ is given by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, i.e. $1-m_{\mathcal{E}}$ is the composition

$$\mathcal{E} \twoheadrightarrow \underline{\mathbb{C}}_{\mathbb{C}^*} \hookrightarrow \mathcal{E}$$

Pushing forward by $j : \mathbb{C}^* \to \mathbb{P}^1\backslash\{\infty\}$ we get

$$j_*\underline{\mathbb{C}}_{\mathbb{C}^*} \to j_*\mathcal{E} \to j_*\underline{\mathbb{C}}_{\mathbb{C}^*},$$

on which $1-m_{j_*\mathcal{E}}$ acts as the composition

$$j_*\mathcal{E} \twoheadrightarrow j_*\underline{\mathbb{C}}_{\mathbb{C}^*} \hookrightarrow j_*\mathcal{E}.$$

Now we can describe the action of monodromy on T by presenting $T|_{\mathbb{P}^1\backslash\{\infty\}}$ as a quotient of $j_*(\mathcal{E})$. We claim that $T|_{\mathbb{P}^1\backslash\{\infty\}} \cong j_*(\mathcal{E})/\underline{\mathbb{C}}_{\mathbb{C}}[1]$, where the map $\underline{\mathbb{C}}_{\mathbb{C}}[1] \to j_*(\mathcal{E})$ is the composition $\underline{\mathbb{C}}_{\mathbb{C}}[1] \to j_*(\underline{\mathbb{C}}_{\mathbb{C}^*})[1] \to j_*(\mathcal{E})$.

One way to check this is to observe that T is the unique nontrivial extension

$$\underline{\mathbb{C}}_0 \to \mathrm{coker} \to j_*\underline{\mathbb{C}}_{\mathbb{C}^*}[1],$$

since $\mathrm{Ext}^1(j_*\underline{\mathbb{C}}_{\mathbb{C}^*}[1], \underline{\mathbb{C}}_0)$ is the dual space to $H^{-1}(j_*\underline{\mathbb{C}}_{\mathbb{C}^*}[1])$ which is 1-dimensional. The extension in

$$\underline{\mathbb{C}}_0 \to \mathrm{coker} \to j_*\underline{\mathbb{C}}_{\mathbb{C}^*}[1]$$

is nontrivial, since $i^*(\mathrm{coker})$ is 1-dimensional, where $i : 0 \hookrightarrow \mathbb{P}^1$; if the extension had been split it would be 3 dimensional.

We conclude that $1-m_T$ factors as

$$\begin{tikzcd} T \arrow[rr, "1-m_T"] \arrow[dr] & & T \\ & \underline{\mathbb{C}}_0 \arrow[ur] & \end{tikzcd}$$

and $\mu_0(1-m_T)$ factors as

$$\begin{tikzcd} \mu_0(T) \arrow[rr, "\mu_0(1-m_T)"] \arrow[dr] & & \mu_0(T) \\ & \mathbb{C} \arrow[ur, hook] & \end{tikzcd}$$

Since $\mu_0(\underline{\mathbb{C}}_0) = 0$ we have $\mu_0(T) \simeq \mu_0(j_*\mathcal{E}[1])$.

11.3. **Proof of corollary 10.6.3.** Recall $\mathcal{P} \subset \tilde{\mathcal{P}} \subset \mathrm{Perv}(G/N)$.

$$\begin{aligned}\mathcal{P} &= \mathrm{Perv}(B\backslash G/N) = \mathrm{Perv}_N(B\backslash G)\\ \tilde{\mathcal{P}} &= \mathrm{Perv}(B\therefore G \mathbin{.\cdot^{\cdot}} B)\\ &= N \text{ equivariant } T\times T \text{ unipotently monodromic sheaves on } G/N.\end{aligned}$$

It is known that ${}^{\wedge}\mathcal{O}^{\wedge}_{\lambda} \simeq \tilde{\mathcal{P}}$

Lemma 11.3.1. *The action of* $\mathrm{Sym}(\mathfrak{t})$ *on* $\mathrm{Id}_{\tilde{\mathcal{P}}}$ *is torsion free.*

Recall from section 6.3 that

$$\tilde{\mathcal{P}} \simeq \tilde{\mathcal{A}} - \mathrm{mod}^{f.d.}_{\mathrm{nilp}}$$

where

$$\tilde{\mathcal{A}} := \mathrm{End}(M)^{opp}; \quad M \in \mathrm{Coh}(\mathfrak{t}^* \times_{\mathfrak{t}^*//W} \mathfrak{t}^*) = \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod},$$
$$M := \bigoplus_{w = s_{\alpha_1} \cdots s_{\alpha_n}} \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_{\alpha_1}}} \cdots \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_{\alpha_n}}} \mathrm{Sym}(\mathfrak{t}).$$

The category $\tilde{\mathcal{P}}$ does not have enough projectives. E.g. $\mathbb{C}[t] - \mathrm{mod}_{\mathrm{nilp}}$ does not have enough projectives. However, it has a projective pro-object

$$\varprojlim_n \mathbb{C}[t]/t^n \in \mathrm{Proobj}(\mathbb{C}[t] - \mathrm{mod}_{\mathrm{nilp}}) \hookrightarrow \widehat{\mathbb{C}[t]} - \mathrm{mod}\,.$$

Sketch of proof. Consider

$$\begin{aligned} \mathrm{Perv}(B \ddots Bw_0B \therefore B) &= T \times T \text{ unipotently monodromic sheaves on } N\backslash Bw_0B/N \simeq T \\ &\simeq \chi_*(T) - \mathrm{mod}_{\mathrm{unip}} \\ &\simeq \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}^{f.g.}_{\mathrm{nilp}}\,. \end{aligned}$$

A functor $\mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}^{f.g.}_{\mathrm{nilp}} \to \mathrm{Perv}(B \ddots Bw_0B \therefore B)$ can be constructed as follows. Let $M \in \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}^{f.g.}_{\mathrm{nilp}}$. Since $\mathfrak{t} = k \otimes \Lambda$ exponentiating gives a unipotent representation of $\Lambda \simeq \pi_1(T)$. This gives a local system on T. We denote this functor by $M \mapsto \mathcal{E}_M$. We then formally extend the categories to include projective pro-objects, the functor extends to a uniquely defined functor between the larger categories.

Let $j : Bw_0B/N \hookrightarrow G/N$ be the inclusion. Applying the functor to the projective pro-object $\varprojlim_n \mathbb{C}[t]/t^n \in \mathrm{Proobj}(\mathbb{C}[t]{-}\mathrm{mod}_{\mathrm{nilp}})$ we get a projective pro-object in $\mathrm{Perv}(B \ddots Bw_0B \therefore B)$

$$\tilde{\Delta}_{w_0} := \varprojlim j_! \mathcal{E}_{\mathrm{Sym}(\mathfrak{t})/(\mathfrak{t})^n}$$

Since $\widehat{\mathrm{Sym}}(\mathfrak{t})$ is torsion free over $\mathrm{Sym}(\mathfrak{t})$ and the functor $M \mapsto j_! \mathcal{E}_M$ is exact, we get an object which is torsion free over $\mathrm{Sym}(\mathfrak{t})$. Define Ξ_α in the same way as in section 10.8. As in claim 10.10.2 we get a generating projective pro-object for $\tilde{\mathcal{P}}$

$$\bigoplus_{w \in W} \Xi_{\alpha_1} \circ \cdots \circ \Xi_{\alpha_n}(\tilde{\Delta}_{w_0}),$$

where for each $w \in W$ we use some reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$. Denote this generating projective pro-object by P.

The constructions for $\tilde{\mathcal{P}}$ have the same properties as the ones for $\mathcal{P}$. E.g. there is a short exact sequence

$$0 \to I_{s_\alpha}(\mathcal{F}) \to \Xi_\alpha(\mathcal{F}) \to \mathcal{F} \to 0.$$

There is also a version of μ_0

$$\mu_0 : \tilde{\mathcal{P}} \to \mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}$$

The functor μ_0 is exact and an analog of corollary 11.1.3 holds with $\mu_0(\ )^{s_\alpha}$ being $\mu_0(\ )$ with the right action twisted by s_α.

$$0 \to \mu_0(\mathcal{F})^{s_\alpha} \to \mu_0(\Xi_\alpha(\mathcal{F})) \to \mu_0(\mathcal{F}) \to 0.$$

It follows that $\mu_0(P)$ is torsion free over $\mathrm{Sym}(\mathfrak{t})$. Let $\tilde{P}_1$ denote the pro-object which is the projective cover of L_1 in $\tilde{\mathcal{P}}$. Like for $\mathcal{P}$

$$\mu_0(M) \simeq \mathrm{Hom}_{\tilde{\mathcal{P}}}(\tilde{P}_1, M).$$

It is fully faithful on projectives. This implies that a non-zero element in $\mathrm{Sym}(\mathfrak{t})$ acts injectively on a projective object so $\mathrm{End}(P)$ is torsion free over $\mathrm{Sym}(\mathfrak{t})$. We have

$$\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}}) \subset \mathrm{End}(P).$$

Thus, $\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}})$ is torsion free over $\mathrm{Sym}(\mathfrak{t})$. □

Corollary 11.3.2. The action of $\mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{t})$ on $\tilde{\mathcal{P}}$ factors through $\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t})$.

Proof. Recall the Bruhat decomposition $G = \coprod_w BwB$. Since $N\backslash BwB/N \simeq TwT$ the two monodromies on $D_{T\times T_{\mathrm{mon}}}(\mathrm{Sh}(BwB))$ differ by w. Hence, for all $t \in \mathfrak{t}$, t-w(t) acts by 0 on BwB and

$$I_t := \prod_w (t - w(t))$$

acts by zero on G. Let J denote the ideal generated by the I_t's. Notice that

$$\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\mathfrak{t}^* \times_{\mathfrak{t}^*//W} \mathfrak{t}^*)$$

We have an equality of sets (not necessarily an equality of schemes): $\mathfrak{t}^* \times_{\mathfrak{t}^*//W} \mathfrak{t}^* = \cup_w \Gamma_w$, where Γ_w is the graph of the action of w on $\mathfrak{t}^*$. Let I be the ideal in $\mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{t})$ of functions vanishing on $\cup_w \Gamma_w$. Clearly $J \subset I$ but they might not be equal. However, the two ideals coincide after localizing by all coroots $\alpha_i^\vee \in \mathfrak{t}$. Let $f \in I$. We need to show that f acts by 0. The image of f in the localization $\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}})_{(\alpha_i^\vee)}$ lies in $I_{(\alpha_i^\vee)} = J_{(\alpha_i^\vee)}$ so the image acts by 0. $\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}})$ is torsion free so there is an inclusion $\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}}) \subset \mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}})_{(\alpha_i^\vee)}$. Hence, f also acts by 0 on $\mathrm{End}(\mathrm{Id}_{\tilde{\mathcal{P}}})$. □

Corollary 11.3.3. The action of $\mathrm{Sym}(\mathfrak{t})$ on $\mathcal{P}$ factors through A.

11.3.1. *Effect of* Ξ_α *on* μ_0. We can now finish the proof of lemma 11.1.4 by proving the following proposition.

Proposition 11.3.4. *There is an isomorphism* $\mathrm{Hom}(P_1, \Xi_\alpha(P_1)) \xleftarrow{\sim} A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1)$.

Since A acts on all $M \in \mathcal{P}$ we get 2 actions of A on $\Xi_\alpha(M)$:

$$m_{\Xi_\alpha(M)} \quad \text{and} \quad \Xi_\alpha(m_M),$$

the first one is the canonical action on the object $\Xi_\alpha(M) \in \mathcal{P}$, while the second one is obtained by functoriality from the action on M.

We apply this to $M = P_1$, also $\mathrm{End}(P_1)$ acts on $\Xi_\alpha(P_1)$ by functoriality extending the action $\Xi_\alpha(m_M)$ (we will later see that in fact A maps isomorphically to $\mathrm{End}(P_1)$, so there is no actual need to extend).

Post-composing we get an action

$$A \otimes \mathrm{End}(P_1) \circlearrowright \mathrm{Hom}(P_1, \Xi_\alpha(P_1))$$

with the A action being $m_{\Xi_\alpha(M)}$. We need to show that the action factors through $A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1)$. This follows from the following lemma.

Lemma 11.3.5. *The restrictions of the two actions $\Xi_\alpha(m_M)$ and $m_{\Xi_\alpha(M)}$ to A^{s_α} coincide.*

Proof. It is enough to check this for $\tilde{\mathcal{P}}$. We need to show that the $\mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{t})$ action factors through

$$\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^{s_\alpha}} \mathrm{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\Gamma_{\mathrm{Id}} \cup \Gamma_{s_\alpha}).$$

We will use the same approach as in the proof of 11.3.2. Let I be the ideal in $\mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{t})$ of functions vanishing on $\Gamma_{\mathrm{Id}} \cup \Gamma_{s_\alpha}$. We have

$$I_{s_\alpha}(M) \to \Xi_\alpha(M) \to M.$$

On M the two actions coincide and on $I_{s_\alpha}(M)$ they differ by the s_α twist. So thinking of this as a $\mathrm{Sym}(\mathfrak{t}) \otimes \mathrm{Sym}(\mathfrak{t})$ module the product of the ideals for Γ_{Id} and Γ_{s_α} acts by 0. $\mathrm{End}(\Xi_\alpha(M))$ is torsion free so the intersection acts by 0, as in the proof of 11.3.2. □

Proof of proposition 11.3.4. The short exact sequence

$$0 \to P_1 \simeq I_{s_\alpha}(P_1) \to \Xi_\alpha(P_1) \to P_1 \to 0$$

shows that $\dim \mathrm{Hom}(P_1, \Xi_\alpha(P_1)) = 2 \dim \mathrm{End}(P_1)$. Picking an element $\phi \in \mathrm{Hom}(P_1, \Xi_\alpha(P_1))$ gives a map

$$A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1) \to \mathrm{Hom}(P_1, \Xi_\alpha(P_1)), \qquad a \mapsto a\phi.$$

Since $A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1)$ fits into the same exact sequence as $\mathrm{Hom}(P_1, \Xi_\alpha(P_1))$ it also has dimension $= 2 \dim \mathrm{End}(P_1)$. Hence, it is enough to check that for some choice of ϕ the map is onto, this would follow if we check that $\mathrm{Hom}(P_1, \Xi_\alpha(P_1))$ is a cyclic module over $A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1)$.

Since P_1 is an indecomposable projective, the ring $\mathrm{End}(P_1)$ is augmented and the augmentation ideal $\mathrm{End}(P_1)^+$ is nilpotent. So it is enough to see that

$$\frac{\mathrm{Hom}(P_1, \Xi_\alpha(P_1))}{(A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1))^+ \cdot \mathrm{Hom}(P_1, \Xi_\alpha(P_1))} \simeq \mathbb{C},$$

where $(A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1))^+$ is the augmentation ideal.

Now,

$$\frac{\mathrm{Hom}(P_1, \Xi_\alpha(P_1))}{(A \otimes_{A^{s_\alpha}} \mathrm{End}(P_1))^+ \cdot \mathrm{Hom}(P_1, \Xi_\alpha(P_1))} = \mathrm{Hom}(P_1, \Xi_\alpha(\overline{P_1})/A^+ \Xi_\alpha(\overline{P_1})),$$

where $\overline{P_1} = P_1/(\mathrm{End}(P_1)^+ \cdot P_1)$ and $A^+ \subset A$ is the augmentation ideal, where we used that $\mathrm{Hom}(P_1, \quad)$ and Ξ_α are exact. The object $\overline{P_1}$ contains L_1 with multiplicity one, so

$$\mathrm{Hom}(P_1, \Xi_\alpha(\overline{P_1})) \cong \mathrm{Hom}(P_1, \Xi_\alpha(L_1)),$$

$$\mathrm{Hom}(P_1, \Xi_\alpha(\overline{P_1})/A^+\Xi_\alpha(\overline{P_1})) \cong \mathrm{Hom}(P_1, \Xi_\alpha(L_1)/A^+\Xi_\alpha(L_1)).$$

Now, $\Xi_\alpha(L_1)$ is a sheaf on $\mathbb{P}^1$. We have the short exact sequence

$$0 \to I_s(L_1) \to \Xi_\alpha(L_1) \to L_1 \to 0.$$

We have $\Delta_1 \simeq L_1 = \underline{\mathbb{C}}_0$ so

$$I_s(L_1) \simeq I_s(\Delta_1) \simeq \Delta_s = j_!\underline{\mathbb{C}}_{\mathbb{A}^1}[1].$$

Hence, the short exact sequence becomes

$$0 \to j_!\underline{\mathbb{C}}_{\mathbb{A}^1}[1] \to \Xi_\alpha(L_1) \to \underline{\mathbb{C}}_0 \to 0.$$

This is the same short exact as in the example in section 11.2 and it is non-split, so $\Xi_\alpha(L_1)$ is isomorphic to T from example 11.2. In the example we showed that $\mu_0(T)$ is 2-dimensional with non-trivial action of $\mathfrak{n}$ – log monodromy = action of $\alpha^\vee \in \mathfrak{t}$.

In the example we showed that $\mu_0(T)_m \simeq \mathbb{C}$. Thus,

$$\mathrm{Hom}(P_1, A^+\Xi_\alpha(L_1)) = \mu_0(T)_m \simeq \mathbb{C}.$$

This finishes the proof. □

12. ℓ-adic setting

Consider G, B and $X = G/N$ over $\mathbb{F}_q$ for some $q = p^n$ with p prime. Replace $\mathcal{P}$ by the corresponding category of ℓ-adic sheaves on the base change of our variety to $\bar{\mathbb{F}}_q$:

$$\mathcal{P}_{\bar{\mathbb{Q}}_\ell} = \mathrm{Perv}_N(B\backslash G).$$

The same constructions go through. Choose an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$. For λ regular and integral

$$\mathcal{O}_\lambda \simeq \mathcal{A}_{\bar{\mathbb{Q}}_\ell} - \mathrm{mod}_{f.g.}$$

The Frobenius morphism Fr acts on $\mathcal{P}_{\bar{\mathbb{Q}}_\ell}$. There is an automorphism of $A_{\bar{\mathbb{Q}}_\ell}$ given by $\mathfrak{t} \ni x \mapsto q^{-1}x$. This induces an automorphism on $\mathcal{A}_{\bar{\mathbb{Q}}_\ell}$ we denote both automorphisms by $[q]$:

$$[q] \circlearrowright A_{\bar{\mathbb{Q}}_\ell} - \mathrm{mod}_{f.g.},$$

$$[q] \circlearrowright \mathcal{A}_{\bar{\mathbb{Q}}_\ell} - \mathrm{mod}_{f.g.}$$

Claim 12.0.1. (1) The Frobenius action on $\mathcal{P}_{\bar{\mathbb{Q}}_\ell}$ commutes with the $[q]$ action on $A-\mathrm{mod}_{f.g.}$, i.e. the following diagram is commutative

$$\begin{tikzcd} \mathcal{P}_{\bar{\mathbb{Q}}_\ell} \arrow[r] \arrow[d, "\mathrm{Fr}"] & A - \mathrm{mod}_{f.g.} \arrow[d, "{[q]}"] \\ \mathcal{P}_{\bar{\mathbb{Q}}_\ell} \arrow[r] & A - \mathrm{mod}_{f.g.} \end{tikzcd}$$

(2) The Frobenius action on $\mathcal{P}_{\bar{\mathbb{Q}}_\ell}$ commutes with the $[q]$ action on $\mathcal{A}-\mathrm{mod}_{f.g.}$, i.e. the following diagram is commutative

$$\begin{tikzcd}
\mathcal{P}_{\bar{\mathbb{Q}}_\ell} \arrow[r, "\sim", "\phi"'] \arrow[d, "\mathrm{Fr}"] & \mathcal{A}-\mathrm{mod}_{f.g.} \arrow[d, "{[q]}"] \\
\mathcal{P}_{\bar{\mathbb{Q}}_\ell} \arrow[r, "\sim", "\phi"'] & \mathcal{A}-\mathrm{mod}_{f.g.}
\end{tikzcd}$$

Here part 1 follows from relation of Fr to monodromy automorphism of a $\mathbb{G}_m$-monodromic sheaf, while part 2 follows from part 1.

Example 12.0.2. To a vector over $\bar{\mathbb{Q}}_\ell$ with an automorphism whose eigenvalues are in $\bar{\mathbb{Z}}_\ell$ one can assign a local system on $\mathbb{G}_m$. Frobenius acts on this data by raising the automorphism to power q. For example, if the vector space is one dimensional and the automorphism is given by multiplication by an nth root of 1, $(n,p)=1$, the sheaf is a direct summand in the direct image of the constant sheaf under the map of multiplication by n

$$\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$$

This map is a Galois covering with Galois group $=\mu_n$ - the group of roots of 1 of order n. Frobenius clearly acts on the Galois group by multiplication by q, which shows how it acts on such sheaves.

The general rank one local system can be obtained from this by passing to inverse limits, as is explained in the theory of ℓ-adic sheaves.

For $M \in \mathcal{O}_\lambda^{\mathrm{gr}}$ the action of $q \in \mathbb{C}^*$ gives a canonical isomorphism $[q]^*(M) \simeq M$, i.e. an equivariant structure with respect to the subgroup in $\mathbb{C}^*$ generated by q, which we will refer to as a $[q]$-equivariant structure. By the claim the $[q]$-equivariant structure on M corresponds to a Fr-equivariant structure on $\phi(M)$. Thus, a grading on $M \in \mathcal{O}_\lambda$ induces a Fr equivariant structure on $\phi(M)$

$$\mathrm{Fr}^*(\phi(M)) \simeq \phi(M).$$

Notice that shifting the grading on a graded module M by d results in multiplying the Frobenius action on $\phi(M)$ by q^d, i.e. tensoring the corresponding Weil sheaf by the Tate Frobenius module $\mathbb{Q}_\ell(d)$:

$$\phi(M(d)) = \phi(M)(d),$$

where $M(d)$ in the left hand side is the graded module given by $M(d)_i = M_{d+i}$ and (d) in the right hand side is the standard abbreviation for $\otimes\mathbb{Q}_\ell(d)$.

We will need a categorical analogue of introducing the variable v with $v^2=q$ in the coefficient ring for the Hecke algebra which was needed for construction of Kazhdan-Lusztig basis. We change the grading on $\mathcal{A}$ by doubling the degrees (see theorem 8.1.1 and discussion preceding it). We fix a square root $q^{1/2}$ of q. Again, a graded $\mathcal{A}$-module M is in particular equivariant with respect to the automorphism $[v]$, the action of $q^{-1/2}$ under the multiplicative group action corresponding to the new grading; this defines a Weil structure on the the sheaf $\phi(M)$. The shift of grading by d now corresponds to twisting by $\mathbb{Q}_\ell(\frac{d}{2})$.

To make notation on the two sides of the equivalence parallel, we denote the shift of grading on the category of graded $\mathcal{A}$ modules in the new grading on $\mathcal{A}$ by $M \mapsto M(\frac{d}{2})$, thus $M(\frac{d}{2})_i = M_{i+d}$.

For graded $\mathcal{A}$-modules M, N the $q^{i/2}$-eigenspace of the action of $[v]$ on $\mathrm{Ext}^1(M,N)$, which is identified with the $q^{i/2}$ eigenspace of Frobenius on $\mathrm{Ext}^1(\phi(M),\phi(N))$, is the $(-i)$-th graded component $\mathrm{Ext}^1_{-i}(M,N)$.

12.1. **Proof of the main theorem.**

Lemma 12.1.1. *There exists irreducible graded lifts $\tilde{L}_w$ which are self dual with respect to the duality on $\mathcal{O}^{\mathrm{gr}}_\lambda$ such that for all $w,v \in W$*

$$\mathrm{Ext}^1_i(\tilde{L}_w,\tilde{L}_v) = 0 \qquad \textit{for } i > 0.$$

The lemma implies the main theorem 8.1.1 as we now explain. Recall the ambiguity in the definition of the graded module $Q^{\mathrm{gr}} \in A-\mathrm{mod}^{\mathrm{gr}}$ and the resulting ambiguity in defining the grading on the algebra $\mathcal{A} = \mathrm{End}(Q)^{\mathrm{op}}$ (but not in defining the graded version of the category $\mathcal{O}^{\mathrm{gr}}_\lambda \cong \mathcal{A}-\mathrm{mod}^{\mathrm{gr}}$), where we used notations of theorem 8.1.1. The module Q can be taken to be a sum of pairwise nonisomorphic indecomposable modules Q_i, the graded lift Q^{gr} has the form $Q^{\mathrm{gr}} = \oplus Q_i^{\mathrm{gr}}$ where Q_i^{gr} is a graded lift of Q_i. We can modify the choice by replacing Q^{gr} by $(Q^{\mathrm{gr}})' = \oplus Q_i^{\mathrm{gr}}(\frac{d_i}{2})$ for some integers d_i. The choice of the graded lifts $\tilde{L}_w$ fixes that choice; notice that $\tilde{L}_w$ will correspond to irreducible modules concentrated in graded degree zero. Theorem 8.1.1 then follows from

Lemma 12.1.2. *Let B be a finite dimensional graded algebra; assume also that B is based (i.e. all irreducible modules are one dimensional). Suppose that for every irreducible representations L_1, L_2 concentrated in graded degree zero the natural grading on $\mathrm{Ext}^1(L_1,L_2)$ satisfies $\mathrm{Ext}^1_n(L_1,L_2) = 0$ for $n \leq 0$. Then B satisfies $B_n = 0$ for $n < 0$ and B_0 is semisimple.*

Proof. Let J be the Jacobson radical of B, so that we have a short exact sequence $0 \to J \to B \to \oplus L \to 0$, where $L = \bigoplus\limits_i L_i$. Then we get $\mathrm{Ext}^1(L,L) \cong (J/J^2)^*$. It is a standard fact that a subspace surjecting to J/J^2 generates B as a ring over the ring spanned by the central idempotents. Since the idempotents have degree zero and generators surjecting to J/J^2 have positive degree, the lemma is proved. □

The lemma implies that the grading on $\mathcal{A}$ was positive so it yields a proof of the main theorem 8.1.1. Thus, proving the lemma finishes the proof of the Kazhdan-Lusztig conjecture.

For a sheaf $\mathcal{F}$ let $\mathcal{F}(n)$ denote the sheaf with Frobenius action multiplied by q^n. The following theorem is a part of the generalization of Weil conjectures by Beilinson-Bernstein-Deligne and Gabber.

Theorem 12.1.3 (Beilinson-Bernstein-Deligne, Asterisque 100)**.** *Let X be an algebraic variety over $\mathbb{F}_q$ and let Z_1, Z_2 be locally closed smooth and irreducible of dimension d_1, d_2 with inclusions $j_1 : Z_1 \hookrightarrow X$, $j_2 : Z_2 \hookrightarrow X$. If $q^{n/2}$ is an eigenvalue of Fr acting on*

$$\mathrm{Ext}^1\left(j_{1!*}\underline{\bar{\mathbb{Q}}_\ell}[d_1]\left(\tfrac{d_1}{2}\right), j_{2!*}\underline{\bar{\mathbb{Q}}_\ell}[d_2]\left(\tfrac{d_2}{2}\right)\right)$$

then $n > 0$.

Let $\tilde{L}_w$ be irreducible graded lifts corresponding to $j_{w!*}\left(\underline{\bar{\mathbb{Q}}_\ell}[\ell(w)]\left(\frac{\ell(w)}{2}\right)\right)$ under ϕ.

Corollary 12.1.4. The grading on $\mathrm{Ext}^1(\tilde{L}_w, \tilde{L}_v)$ is such that components of degree ≥ 0 vanish.

Proof. To show that $\mathrm{Ext}^1_i(\tilde{L}_w, \tilde{L}_v) = 0$ for $i > 0$ is equivalent to showing that the action of Fr on $\mathrm{Ext}^1(j_{w!*}\left(\underline{\bar{\mathbb{Q}}_\ell}[\ell(w)]\left(\frac{\ell(w)}{2}\right)\right), j_{v!*}\left(\underline{\bar{\mathbb{Q}}_\ell}[\ell(v)]\left(\frac{\ell(v)}{2}\right)\right))$ has no q^i eigenvalues for $i < 0$. This is exactly the statement of the theorem. $\square$

To finish the proof of lemma 12.1.1 we need to show that the graded lifts $\tilde{L}_w$ are self dual. We will give two different proofs of this.

Proof 1. In this proof we use duality on $\mathrm{Perv}_N(\mathcal{B})$ in the ℓ-adic setting. In char 0 the Verdier duality was defined as

$$\mathcal{F} \mapsto R\underline{\mathrm{Hom}}(F, \underline{\mathbb{C}}[2\dim X]).$$

In the ℓ-adic setting it is defined as

$$\mathcal{F} \mapsto R\underline{\mathrm{Hom}}(F, \underline{\bar{\mathbb{Q}}_\ell}[2\dim X](\dim X)).$$

It sends perverse sheaves to perverse sheaves.

<u>Facts</u>:

(1) If j is a locally closed embedding of an irreducible smooth subvariety then $j_{!*}\left(\underline{\bar{\mathbb{Q}}_\ell}[d]\left(\frac{d}{2}\right)\right)$ is self-dual.
(2) μ_0 commutes with duality.

Exercise 12.1.5. The duality on $\mathcal{O}_\lambda^{\mathrm{gr}}$ can be characterized as the only exact contravariant functor which sends irreducibles to themselves (up to grading shift) such that

$$\mu_0(M^\vee) \simeq \mu_0(M)^*.$$

Hence the exercise shows that our duality on $\mathcal{O}_\lambda^{\mathrm{gr}}$ is compatible with Verdier duality on ℓ-adic sheaves equivariant under Fr (sheaves with a fixed Fr-equivariant structure are called Weil sheaves). I.e. for $M \in \mathcal{O}_\lambda^{\mathrm{gr}}$

$$\phi(M^\vee) \simeq \text{Verdier dual}(\phi(M)).$$

Using fact 1 we get that $\tilde{L}_w = \phi^{-1}\left(j_{w!*}\left(\underline{\bar{\mathbb{Q}}_\ell}[\ell(w)]\left(\frac{\ell(w)}{2}\right)\right)\right)$ is self dual with respect to the duality on $\mathcal{O}_\lambda^{\mathrm{gr}}$. $\square$

Proof 2. This proof is based on the following

Claim 12.1.6. A positive grading is unique up to replacing $\tilde{L}_w$ by $\tilde{L}_w(d)$, where the same integer d is used for all $w \in W$.

Self-duality follows from the Claim. We know that a lift $\tilde{L}_w$ is unique up to a shift by some d_w.

Let $\tilde{L}_w$ be graded irreducibles with grading on $\mathrm{Ext}^1(\tilde{L}_w, \tilde{L}_y)$ positive for all $y \in W$. Duality sends irreducibles to irreducibles and

$$\mathrm{Ext}^i(M^\vee, N^\vee) = \mathrm{Ext}^i(N, M)$$

Thus, $\tilde{L}_w^\vee$ also satisfy the condition on positivity of grading on Ext^1. By the claim there exists an integer d independent of w such that

$$\tilde{L}_w^\vee = \tilde{L}_w(d).$$

Duality can be defined for graded modules in the grading with $\deg(x) = 1$ for $x \in \mathfrak{t}$. We are working with the grading where $\deg(x) = 2$ so d has to be even. Then replacing $\tilde{L}_w$ by $\tilde{L}_w(\frac{d}{2})$ we get the self-dual choice. □

The rest of the section is devoted to proving the claim.

Proof of claim. (i) Non-vanishing for $\mathrm{Ext}^1_1(\tilde{L}_w, \tilde{L}_{ws_\alpha})$: Let $w, s_\alpha \in W$. Assume that $\ell(ws_\alpha) > \ell(w)$. Recall the short exact sequence

$$0 \to L_0 \to \Delta_{s_\alpha} \to L_{s_\alpha} \to 0.$$

There exists a corresponding short exact sequence for the graded lifts.

$$0 \to \tilde{L}_0\left(\tfrac{1}{2}\right) \to \tilde{\Delta}_{s_\alpha} \to \tilde{L}_{s_\alpha} \to 0.$$

Using convolution we get

$$0 \to \tilde{\Delta}_w\left(\tfrac{1}{2}\right) \to \tilde{\Delta}_{ws_\alpha} \to \tilde{\Delta}_w \star \tilde{L}_{s_\alpha} \to 0.$$

Let E be the quotient of $\tilde{\Delta}_{ws_\alpha}$ by the maximal subobject not containing $\tilde{L}_w(d)$, $\tilde{L}_{ws_\alpha}(d)$ in its Jordan-Hölder series ($d \in \mathbb{Z}$). We claim that E fits into the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\Delta}_w\left(\frac{1}{2}\right) \arrow[r] \arrow[d] & \tilde{\Delta}_{ws_\alpha} \arrow[r] \arrow[d] & \Delta_w \star \tilde{L}_{s_\alpha} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{L}_w\left(\frac{1}{2}\right) \arrow[r] & E \arrow[r] & \tilde{L}_{ws_\alpha} \arrow[r] & 0
\end{tikzcd}$$

Recall that there exist a graded lift $\tilde{\Delta}_w \twoheadrightarrow \tilde{L}_w$ with kernel only containing $\tilde{L}_v\left(\frac{d}{2}\right)$ with $v < ws_\alpha$ and $d > 0$. If the displayed diagram does not exist, then we have an irreducible subobject $\tilde{L}_v\left(\frac{d}{2}\right)$ in $\tilde{\Delta}_w \star \tilde{L}_{s_\alpha}$ such that the induced extension class in $\mathrm{Ext}^1(\tilde{L}_v\left(\frac{d}{2}\right), \tilde{L}_w\left(\frac{1}{2}\right))$ is nonzero. This contradicts positivity of grading on Ext^1 between the fixed graded lifts of irreducibles.

Thus we got an extension of $\tilde{L}_{ws_\alpha}$ by $\tilde{L}_w\left(\frac{1}{2}\right)$. We need to show that it is non-trivial. It is enough to prove this in the non-graded setting. Assume that $N \simeq L_w \oplus L_{ws_\alpha}$. Then $\Delta_{ws_\alpha}\big/M \simeq L_w \oplus L_{ws_\alpha}$ so $\ker \simeq M \oplus L_w$. Thus, L_w is a submodule in Δ_{ws_α}. By proposition 4.1.1 this is only possible for $w = e$, so in all other cases the sequence is non-trivial. The case $w = e$ has been already been proved.

If $\ell(ws_\alpha) < \ell(w)$ one can apply duality to show that $\mathrm{Ext}^1_1(\tilde{L}_w, \tilde{L}_{ws_\alpha}) \neq 0$ also in this case.

(ii) Uniqueness: Let $\tilde{L}'_w$ be another set of irreducibles with a positive grading. The lift of one irreducible is unique up to a shift so for each $w \in W$ there is a $d_w \in \mathbb{Z}$ such that $\tilde{L}'_w \simeq \tilde{L}_w(d_w)$. We need to prove that d_w is independent of w. We just proved that for all $w, s_\alpha \in W$

$$\mathrm{Ext}^1_1(\tilde{L}_w, \tilde{L}_{ws_\alpha}) \neq 0$$

Now,

$$\mathrm{Ext}^1_{i+d_w-d_{ws_\alpha}}(\tilde{L}_w(d_w), \tilde{L}_{ws_\alpha}(d_{ws_\alpha})) = \mathrm{Ext}^1_i(\tilde{L}_w, \tilde{L}_{ws_\alpha})$$

so for $\tilde{L}'_w$ to have a positive grading we must have

$$d_{ws_\alpha} \leq d_w \quad \forall w, s_\alpha \in W.$$

This implies that $d_w = d_v$ for all $w, v \in W$. □

Remark 12.1.7. Kazhdan-Lusztig polynomials are interpreted as graded Euler characteristic of $\mathrm{Ext}^\bullet(\tilde{L}_w, \tilde{\nabla}_y)^*$ = stalk of $L_w = j_{w!*}\underline{\bar{\mathbb{Q}}_\ell}[l(w)]\left(\frac{l(w)}{2}\right)$ at a point of $\mathcal{B}_y$. By the Koszul property of $\mathcal{A}$ it is also true that the coefficients of $P_{w,v}$ are $\pm \dim H^i(L_w|_{X_v})$, cohomology spaces of the stalks.

12.2. **What happens if λ is singular?** Recall that for λ regular we have

$$H = \mathbb{Z}[q, q^{-1}][B]/(\tilde{s}_\alpha + 1)(\tilde{s}_\alpha - q) \simeq K^0(\mathcal{O}^{\mathrm{gr}}_{\bar{\lambda}}),$$

where q is a formal variable. Over $\mathbb{F}_q$ we have

$$H/(q - p^n) \xrightarrow{\sim} \mathbb{Z}\Big[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)\Big]$$

We have a Bruhat decomposition

$$G(\mathbb{F}_q) = \coprod_w B(\mathbb{F}_q) w B(\mathbb{F}_q).$$

The map is given by $\tilde{s}_\alpha \mapsto \delta_{Bs_\alpha B}$. Here δ_Z denotes the constant sheaf supported on Z. Also

$$K^0(\mathrm{Perv}_N(B\backslash G), \mathrm{Fr}) \to \underline{\bar{\mathbb{Q}}_\ell}\left[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/N(\mathbb{F}_q)\right],$$
$$\mathcal{F} \mapsto \Big(x \mapsto \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}, H^i(\mathcal{F}_x))\Big).$$

Question 12.2.1. What happens if λ is singular?

Assume that λ, μ are positive and λ is regular but μ is not necessarily regular. Recall the translation functor $T_{\lambda\to\mu}$ which sends $\Delta_{w\cdot\lambda}$ to $\Delta_{w\cdot\mu}$ and

$$T_{\lambda\to\mu}(L_w) = \begin{cases} \text{Irreducible} & \text{if } w \text{ has minimal length} \\ 0 & \text{otherwise} \end{cases}$$

Write $W_\mu := \mathrm{Stab}_W(\mu)$ this is the Weyl group of the Levi corresponding to μ. The sign representation $W_\mu \to \mathbb{C}^*$, $s_\alpha \mapsto -1$ deforms to a representation sign of $H_q(W_\mu)$.

Proposition 12.2.2. *For μ positive but not necessarily regular we have*

$$K^0(\mathcal{O}^{\mathrm{gr}}_\mu) \simeq H_q \otimes_{H_q(W_\mu)} \textit{sign}.$$

The translation functor gives a map

$$\begin{tikzcd} K^0(\mathcal{O}^{\mathrm{gr}}_\lambda) \arrow[r] \arrow[d, no head, "\wr"] & K^0(\mathcal{O}^{\mathrm{gr}}_\mu) \arrow[d, no head, "\wr"] \\ H_q \arrow[r] & H_q \otimes_{H_q(W_\mu)} \mathrm{sign} \end{tikzcd}$$

the kernel is $\langle \tilde{s}_\alpha + 1 \mid s_\alpha \in W_\mu \rangle$. The latter space is identified with the subspace in $K^0(\mathrm{Perv}_N(G/B))$ spanned by the classes of sheaves pulled back under the projection $G/B \to G/P_\alpha$.

12.2.1. *An interpretation of $H_q \otimes_{H_q(W_\mu)}$ sign in terms of $G(\mathbb{F}_q)$.* For λ regular

$$H_q = \mathbb{C}\big(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)\big/N(\mathbb{F}_q)\big).$$

Given a subset Σ of simple roots consider functions

$$\begin{tikzcd} \Psi_\Sigma : N(\mathbb{F}_q) \arrow[r] \arrow[d] & \mathbb{C}^* \\ \prod_{\alpha \text{ simple}} \mathbb{F}_q = N/[N,N](\mathbb{F}_q) & \end{tikzcd}$$

such that Ψ_Σ is non-trivial on the summand corresponding to $\alpha \Leftrightarrow \alpha \in \Sigma$.

Example 12.2.3. For G =SL$_2$ one such map is given by

$$\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mapsto e^{2\pi i \frac{a}{p}}.$$

Define

$$\mathbb{C}(X/(N,\Psi)) \coloneqq \{f : X \to \mathbb{C}^* \mid f(nx) = \Psi(n) f(x) \;\; \forall n \in N, x \in X\}.$$

Proposition 12.2.4. *For $\Sigma = \{\alpha \mid s_\alpha \in W_\mu\}$ there exist an isomorphism*

$$H_q \otimes_{H_q(W_\mu)} \mathrm{sign} \simeq \mathbb{C}\big(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/N(\mathbb{F}_q), \Psi_\Sigma\big)$$

Example 12.2.5. If μ is regular then $\Sigma = \varnothing$ so Ψ is trivial. If $\mu = -\rho$ then Σ is the set of all simple roots. The functions in this case are called B-invariant Whittaker functions.

Let w_0^μ denote the longest element in W_μ and set $N^{w_0^\mu} \coloneqq w_0^\mu(N)$. There are maps

$$H_q = \mathbb{C}\big(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/N^{w_0^\mu}(\mathbb{F}_q)\big) \underset{\mathrm{Av}_{N,\Psi_\Sigma}}{\overset{\mathrm{Av}_{N^{w_0^\mu}}}{\rightleftarrows}} \mathbb{C}\big(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/N(\mathbb{F}_q), \Psi_\Sigma\big) = H_q \otimes_{H_q(W_\mu)} \mathrm{sign}$$

These maps are called averaging maps. They are defined in the following way

$$\mathrm{Av}_H(f) = \tfrac{1}{|H|} \sum_{h \in H} f^h,$$
$$\mathrm{Av}_{H,\Psi}(f) = \tfrac{1}{|H|} \sum_{h \in H} f^h \Psi(h),$$

where f^h is f composed with right multiplication by h.

Lemma 12.2.6. *The composition* $\mathrm{Av}_{N^{w_0^\mu}} \circ \mathrm{Av}_{N,\Psi_\Sigma} : H_q \to H_q$ *is the right multiplication by* $\xi = \sum_{w \in W_\mu} (-q)^{-\ell(w)} T_w$.

Proof. Notice that the averaging maps commute with left multiplication. Therefore the composition must be given by right multiplication by some element, call it ξ'. It remains to check that $\xi = \xi'$. The element ξ is characterized by:

(1) It lies in $H_q(W_\mu)$.
(2) Coefficient at T_1 is 1.
(3) $(1 + T_{s_\alpha})\cdot$ this element $= 0$ for $s_\alpha \in W_\mu$.

So we need to check that ξ' also satisfies these properties. Property (1) follows from the equality $\Psi_\Sigma(0) = 1$, property (2) is obvious. To check (3) consider the projection $G/B \to G/P_\alpha$ for a simple root $\alpha \in \Sigma$. The element $1 + T_{s_\alpha}$ corresponds to $\delta_{B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)}$, so we only need to check that the averaging map kills $\delta_{B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)}$. This follows from the equality $\sum_{x \in \mathbb{F}_q} \Psi_\Sigma(u_\alpha(x) = 0$, where u_α is an isomorphism between the additive group and the corresponding root subgroup. □

12.3. **Category $\mathcal{O}_{\bar{\mu}}$ for nonregular μ.** We work with varieties over a field k of characteristic p so $x \mapsto x^{p^n}$ is a homomorphism of algebraic groups. Consider the Artin-Schreier map

$$AS_n : \mathbb{G}_a \to \mathbb{G}_a, \qquad x \mapsto x^{p^n} - x.$$

This is a homomorphism with a discrete kernel identified with the additive group of F_{p^n}, hence it is a Galois cover with that Galois group.

The additive group of F_{p^n} acts on $\mathrm{AS}_{n,*}(\underline{\bar{\mathbb{Q}}_\ell})$. For a character χ of $(\mathbb{F}_{p^n}, +)$ we can consider the summand in $\mathrm{AS}_*(\underline{\bar{\mathbb{Q}}_\ell})$ where $\mathbb{F}_{p^n}$ acts by χ. This summand is called the character sheaf and it is denoted by $\mathcal{F}_\chi$. The function

$$x \mapsto \mathrm{Tr}(\mathrm{Fr}, (\mathcal{F}_\chi)_x)$$

is the character χ. Pulling back along the addition map $+ : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$.

$$(+)^*(\mathcal{F}_\chi) = \mathcal{F}_\chi \boxtimes \mathcal{F}_\chi = \mathrm{pr}_1^*(\mathcal{F}_\chi) \otimes \mathrm{pr}_2^*(\mathcal{F}_\chi).$$

Let X a variety with an action of $\mathbb{G}_a$

$$\begin{tikzcd} & \mathbb{G}_a \times X \arrow[dl, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[dr, "a"] & \\ \mathbb{G}_a & X & X \end{tikzcd}$$

Definition 12.3.1. A $(\mathbb{G}_a, \mathcal{F}_\chi)$-equivariant sheaf on X is a pair $(\mathcal{F}, \phi)$ where $\mathcal{F} \in D(\mathrm{Sh}(X))$ and ϕ is an isomorphism

$$a^*(\mathcal{F}) \overset{\phi}{\simeq} \mathcal{F}_\chi \boxtimes \mathcal{F} \simeq \mathrm{pr}_1^*(\mathcal{F}_\chi) \otimes \mathrm{pr}_2^*(\mathcal{F})$$

such that on $\mathbb{G}_a \times \mathbb{G}_a \times X$ with action $\mathrm{ac} : \mathbb{G}_a \times \mathbb{G}_a \times X \to X$, $(g, h, x) \mapsto ghx$. The two isomorphisms

$$\mathcal{F}_\chi \boxtimes \mathcal{F}_\chi \boxtimes \mathcal{F} \simeq \mathrm{ac}^*(\mathcal{F})$$

arising from this data are required to coincide.

Remark 12.3.2. (i) Compare with the naive definition of equivariance in the derived category.

(ii) The same works if $\mathbb{G}_a$ is replaced by an algebraic group H with a sheaf $\mathcal{F}_X \neq 0$ such that

$$m^*(\mathcal{F}_X) \simeq \mathcal{F}_X \boxtimes \mathcal{F}_X,$$

(where $m : H \times H \to H$ is the multiplication) satisfying the associativity constraint (equality of two isomorphisms between sheaves on $H \times H \times H$).

In particular, if $\psi : H \to \mathbb{G}_a$ is a homomorphism of algebraic groups, we can take $\mathcal{F}_X$ to be the pullback of the Artin Schreier sheaf. We will denote the resulting category of twisted equivariant sheaves by $D(\mathrm{Sh}(X/N, \psi))$. We will also use the notation $\mathrm{Perv}(X/N, \psi)$ etc.

(iii) When H is connected and unipotent we get a full triangulated subcategory in $D(\mathrm{Sh}(X))$.

Let $\mathcal{O}^{\bar{\mu}}_{\bar{\mathbb{Q}}_\ell}$ denote $\mathcal{O}_{\bar{\mu}}$ considered over the field $\bar{\mathbb{Q}}_\ell$. Using the maps $H \times X \overset{a}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X$ define functors

$$\mathrm{Av}_H(\mathcal{F}) := a_* \mathrm{pr}_2^*(\mathcal{F}),$$
$$\mathrm{Av}_{H,\mathcal{F}_X}(\mathcal{F}) := a_*(\mathrm{pr}_2^*(\mathcal{F}_X) \otimes \mathcal{F}).$$

Theorem 12.3.3 (Variation of Milicic-Soergel)**.** *There is an equivalence of categories*

$$\mathcal{O}^{\bar{\mu}}_{\bar{\mathbb{Q}}_\ell} \simeq \mathrm{Perv}(B\backslash G/(N, \Psi_\Sigma)), \qquad \Sigma = \{\alpha \mid s_\alpha \in W_\mu\},$$

such that for λ regular the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Perv}(B\backslash G/N^{w_0^\mu}) \arrow[r, no head, "\sim"] \arrow[d, shift right, "\mathrm{Av}_{N,\Psi}"'] & \mathcal{O}^{\bar{\lambda}}_{\bar{\mathbb{Q}}_\ell} \arrow[d, shift right] \\
\mathrm{Perv}(B\backslash G/N, \Psi_\Sigma) \arrow[r, no head, "\sim"] \arrow[u, shift right, "\mathrm{Av}_{N^{w_0^\mu}}"'] & \mathcal{O}^{\bar{\mu}}_{\bar{\mathbb{Q}}_\ell} \arrow[u, shift right]
\end{tikzcd}$$

where the maps on the right are translation functors and Ψ_Σ is given by:

$$N \xrightarrow{\sum_{\alpha\in\Sigma} \alpha} \mathbb{G}_a$$

Exercise 12.3.4. The kernels of the downward arrows in the two columns agree.

In particular, the Theorem provides a description of the endo-functor of wall-crossing functor of $\mathrm{Perv}(B\backslash G/N)$ corresponding to the translation functor $T_{\mu\to\lambda}T_{\lambda\to\mu}$, namely it is the composition of two arrows in the left column. Recall that we have already used something we called topological wall-crossing functors, especially in the case when $W_\mu = \{1, s_\alpha\}$. To compare the two constructions consider the diagram

$$\begin{tikzcd}[column sep=small] & G/N^\alpha \times N_{-\alpha} \arrow[dl, "f"', "(x,t)\mapsto\bar{x}"] \arrow[dr, "(x,t)\mapsto(\overline{xt},t)", "h"'] \arrow[ddr, "(x,t)\mapsto\overline{xt}"', "g"] & \\ G/N & & G/N \times N_{-\alpha} \arrow[d] \\ & & G/N \end{tikzcd} \tag{8}$$

where $N_{-\alpha} \simeq \mathbb{G}_a$ is the part corresponding to $-\alpha$ and $N^\alpha \subset N$ is the radical of the parabolic of type α. Here $\bar{x}$ is the image of x under the projection $G/N^\alpha \twoheadrightarrow G/N$. In the ℓ-adic setting the composition of the two arrows in the left column of the diagram from the Theorem can be rewritten as

$$g_*(\mathrm{pr}_2^*(\mathrm{AS}) \otimes f^*(\mathcal{F}))$$

12.4. **Vanishing cycles.** A general reference for vanishing cycles is section 8.6 in [KS].
Let $f : X \to \mathbb{A}^1$ be some map. Set $Z := f^{-1}(\{0\})$.

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 \end{tikzcd}$$

From f we construct two functors on the derived categories of constructible sheaves

$$\psi_f, \phi_f : D_{\mathrm{cons}}(X) \to D_{\mathrm{cons}}(Z)$$

The functors ψ_f and ϕ_f are called nearby cycles and vanishing cycles. Both commute with duality, satisfy proper base change isomorphism and send perverse sheaves to perverse sheaves. Moreover, $\psi_f(\mathcal{F})$ only depends on the restriction $\mathcal{F}|_U$, $U = X \setminus Z$, i.e. it is actually a functor $\psi_f(\mathcal{F}) : D_{\mathrm{cons}}(U) \to D_{\mathrm{cons}}(Z)$. There exists an exact triangle

$$i^*\mathcal{F}[-1] \to \psi_f(\mathcal{F}) \to \phi_f(\mathcal{F}) \to i^*\mathcal{F}.$$

12.4.1. *Construction over $\mathbb{C}$ in classical topology.* Using the exponential map

$$\mathbb{C} \overset{\exp}{\to} \mathbb{C}\backslash\{0\} \subset \mathbb{C}$$

define a fibre product $\tilde{X} := X \times_{\mathbb{C}} \mathbb{C} \overset{\pi}{\to} X$. The nearby cycles are defined as

$$\psi_f(\mathcal{F}) = i^*\pi_*\pi^*(\mathcal{F})[-1]$$

The second functor ϕ_f is called vanishing cycles, it can be computed from the exact triangle knowing ψ, i^* (the exact triangle does not quite provide a definition for ϕ because of nonfunctoriality of cone, though it defines $\phi_f(\mathcal{F})$ up to an isomorphism for each $\mathcal{F}$).

A more intuitive (though requiring more work to make formal sense of) description of ψ is as follows. For a sufficiently small open neighborhood U of Z in X a general theorem guarantees existence of a retract map $c : U \to Z$. Choose such a neighborhood U; then locally for small enough positive ε the complex $c_*(\mathcal{F}|_{f^{-1}(\varepsilon)})[-1]$ is canonically identified with $\psi_f(\mathcal{F})$.

12.4.2. *Another construction of $R_\alpha : \mathcal{P} \to \mathcal{P}$.* Let us make an additional assumption that $\mathbb{G}_m$ acts on X contracting X to Z so that $f(tx) = tf(x)$. Denote this contraction map by p. Let $i : Z \hookrightarrow X$ and $j : \{x \mid f(x) \neq 1\} \hookrightarrow X$ be the inclusions. $\mathcal{F}$ is monodromic with respect to the action. Thus,

$$\begin{aligned} i^*(\mathcal{F}) &= p_*(\mathcal{F}) \\ \psi(\mathcal{F}) &= p_*(\mathcal{F}|_{f=1}) \\ \phi(\mathcal{F}) &= p_*(j_! j^*(\mathcal{F})). \end{aligned}$$

Under the same conditions in the ℓ-adic setting

$$\phi_f(\mathcal{F}) = \pi_*(\mathcal{F} \otimes f^*(\mathcal{F}_X)).$$

We will use it in such a situation; namely, we let $X = G/N \times N_\alpha^-$, the function ϖ is the projection to $N_\alpha^- \cong \mathbb{A}^1$ and the sheaf is $h_* f^*(\mathcal{F})$ (notations of diagram (8)). Thus, we see that the topological wall-crossing functor can be rewritten in terms of vanishing cycles:

$$R_\alpha(\mathcal{F}) = \phi_\varpi h_* f^*(\mathcal{F}).$$

12.5. **Generalities about microlocalization.** Recall that for λ regular integral we used microlocalization at $\mathcal{B}_e$ to construct a map $\mu_0 : \mathrm{Perv}_N(B\backslash G) \to \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod}$

$$\begin{tikzcd} \mathcal{O}_{\bar{\lambda}} \arrow[r, dash, "\sim"] \arrow[d] & \mathrm{Perv}_N(B\backslash G) \arrow[d, "\mu_0"] \\ \mathcal{O}_{-\rho} \simeq A - \mathrm{mod} & \mathrm{Sym}(\mathfrak{t}) - \mathrm{mod} \end{tikzcd}$$

We now describe microlocalization for a general smooth algebraic variety X over $\mathbb{C}$ and Z a locally closed, irreducible and smooth subvariety. Denote the conormal bundle to Z by $T_Z^*(X)$. Notice that $T_Z^*(X) \subset T^*(X)|_Z$ is Lagrangian in $T^*(X)$. One can define a functor

$$\mu_Z : D_{\mathrm{cons}}(X) \to D_{\mathrm{cons}}(T_Z^*(X))$$

sending perverse sheaves to perverse sheaves such that for generic $(z,\xi) \in T_Z^*(X)$

$$\mu_Z(\mathcal{F})_{z,\xi} = \phi_f(\mathcal{F})|_z,$$

where f is a function on a neighborhood of z such that $f|_z = 0$ and $df|_z = \xi$. If (z,ξ) is generic the right hand side is independent of f. Let $N_Z(X)$ denote the normal bundle to Z. This vector bundle is dual to $T_Z^*(X)$ so there is a Fourier transform

$$D_{\mathrm{cons}}(N_Z(X)) \xrightarrow{\mathrm{Fou}} D_{\mathrm{cons}}(T_Z^*(X))$$

Inside $D_{\mathrm{cons}}(N_Z(X))$ we have the specialization $\mathrm{Sp}(\mathcal{F})$. To describe $\mathrm{Sp}(\mathcal{F})$ we need a degeneration of X to $N_Z(X)$. Assume that X is affine so $X = \mathrm{Spec}(\mathcal{O}(X))$. Let I be the ideal of Z

$$I \hookrightarrow \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(Z)$$

Then

$$N_Z(X) = \mathrm{Spec}(\oplus_n I^n/I^{n+1}).$$

E.g. if Z =pt then $I/I^2 = T^*_Z(X)$. $I^n/I^{n+1} \simeq \mathrm{Sym}^n(I/I^2)$ so

$$T_Z(X) = \mathrm{Spec}(I/I^2)$$

Note that $X \times (\mathbb{A}^1\backslash\{0\}) = \mathrm{Spec}(\mathcal{O}_X[t,t^{-1}])$. Consider the following maps from the graded ring $\mathcal{O}_{\mathrm{Rees}} = \oplus_n I^n$

$$\begin{tikzcd} & \mathcal{O}_{\mathrm{Rees}} \arrow[dl] \arrow[dr, hook] & \\ \oplus_n I^n/I^{n+1} & & \mathcal{O}_X[t,t^{-1}] \end{tikzcd}$$

Set $\tilde{X} := \mathrm{Spec}(\mathcal{O}_{\mathrm{Rees}})$. Thus, there are inclusions

$$\begin{tikzcd} N_Z(X) \arrow[r, hook] & \tilde{X} \arrow[d, "f"] & X \times (\mathbb{A}^1\backslash\{0\}) \arrow[l, hook'] \\ \{0\} \arrow[r, hook] & \mathbb{A}^1 & \mathbb{A}^1\backslash\{0\} \arrow[l, hook'] \end{tikzcd}$$

Define the specialization

$$\mathrm{Sp}(\mathcal{F}) := \psi_f(\mathcal{F} \boxtimes \underline{\mathbb{C}}_{\mathbb{A}^1\backslash\{0\}}[1])$$

This also works for ℓ-adic sheaves.

12.6. **Fourier transform.** For V the total space of a vector bundle over Z and V^* the total space of the dual vector bundle there is a Fourier transform

$$\mathrm{Fou}: D_{\mathrm{cons}}(V) \to D_{\mathrm{cons}}(V^*)$$

(1) In the ℓ-adic setting

$$\begin{tikzcd} & \langle\ ,\ \rangle : V \times_Z V^* \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[r] & \mathbb{A}^1 \\ V & & V^* \end{tikzcd}$$

The Fourier transform is given by

$$\mathrm{Fou}: \mathcal{F} \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(\mathcal{F}) \otimes \langle\ ,\ \rangle^*(\mathcal{F}_\Psi))[r],$$

where $\mathcal{F}_\Psi$ is the Artin-Schreier sheaf.

(2) For dilation monodromic (also called conical) sheaves this is equivalent to

$$\phi_{\langle\ ,\ \rangle}\mathrm{pr}_1^*(\mathcal{F})[r]$$

vanishing cycles.

Claim 12.6.1. This is supported on $\{0\} \times_Z V^*$ so we got a functor

$$\mathrm{Perv}(V) \to \mathrm{Perv}(V^*)$$

Exercise 12.6.2. Check that the functor

$$\begin{tikzcd} \mu_0 : \mathrm{Perv}_N(\mathcal{B}) \arrow[r] \arrow[dr] & \text{local systems on } U \subset (\mathfrak{g}/\mathfrak{b})^* \\ & \mathrm{Sym}(\mathfrak{t}) \arrow[u, hook] \end{tikzcd}$$

coincides with the restriction $\mu_Z|_U$ for $Z = \mathcal{B}_e$.

Given $\mathcal{F}$ there are only finitely many Z (up to replacing Z by a dense open subset) such that $\mathrm{supp}(\mu_Z(\mathcal{F}))$ is dense in $T^*_Z(X)$. Indeed, there exists a stratification $X = \coprod_i X_i$ such that $\mu_Z(X) \neq 0$ iff Z is dense in the closure of a stratum. One defines Lagrangian cycle

$$\begin{aligned} SS(\mathcal{F}) &:= \sum_i m_{X_i} T^*_{X_i}(X) \quad \subset T^*(X), \\ m_{X_i} &= \dim H^{-n}(\mu_{X_i}(\mathcal{F}))|_{(z,\xi)} \\ &= (-1)^n \mathrm{Eul}\, \mu_{X_i}(\mathcal{F})|_{(z,\xi)} \qquad \text{(microlocal stalks)} \end{aligned}$$

where $n = \dim X$ and $(z,\xi) \in T^*_{X_i}(X)$ is generic.

On an open dense subvariety of $\mathrm{SS}(\mathcal{F})$ we have local systems $\mu_{X_i}(X)|_{U_i}$, $U_i \subset T^*_{X_i}(X)$. So $\mathrm{Perv}(X)$ and $D_{\mathrm{cons}}(X)$ are related to $T^*(X)$. Suppose G is connected and acts on X. Then we have a map

$$m_G : T^*(X) \to \mathfrak{g}^*, \qquad \xi \mapsto f_\xi,$$

where f_ξ is defined as

$$f_\xi(x) = \langle a(x), \xi \rangle, \qquad x \in \mathfrak{g},\ a(x) \in \mathrm{Vect}(X) \text{ is the action vector field.}$$

For $\mathcal{F} \in D^G_{\mathrm{cons}}$ we have $\mathrm{SS}(\mathcal{F}) \subset m^{-1}(0)$. If X_i are G-invariant then $\mu_{X_i}(\mathcal{F})$ is also G-equivariant. E.g. when the action is free $Y = X/G$ and $T^*(Y) = m^{-1}(0)/G$.

If G is a torus and $\mathcal{F}$ is monodromic then $\mathrm{SS}(\mathcal{F}) \subset m^{-1}(0)$ and $\mu_X(\mathcal{F})$ is monodromic.

13. Pass to geometry of coherent sheaves

We want a description of category $\mathcal{O}$ in terms of coherent sheaves on the cotangent bundle of $\mathcal{B}$. Recall that

$$\begin{aligned} \mathcal{O}_{\bar\lambda} &= \mathfrak{g} - \mathrm{mod}_{\bar\lambda} \text{- generalized central character } \bar\lambda,\ \mathfrak{t} \text{ acts diagonally.} \\ &\simeq \mathcal{A} - \mathrm{mod}\,. \\ {}^\wedge\mathcal{O}^\wedge_{\bar\lambda} &= \mathfrak{g} - \mathrm{mod}\text{- generalized central character } \bar\lambda \\ &\simeq \tilde{\mathcal{A}} - \mathrm{mod}_{\mathrm{nilp}}, \end{aligned}$$

where $\tilde{\mathcal{A}} = \mathrm{End}(M)$. We define the Springer variety $\tilde{\mathcal{N}}$ and the Grothendieck variety $\tilde{\mathfrak{g}}$ as follows

$$\begin{aligned} \tilde{\mathcal{N}} &:= T^*\mathcal{B} = \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g}^* \mid x \in \mathfrak{b}^\perp\}, \\ \tilde{\mathfrak{g}} &:= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g}^* \mid x \in \mathrm{rad}(\mathfrak{b})^\perp\} = \frac{T^*(G/N)}{T}. \end{aligned}$$

Theorem 13.0.1. *The categories* $\mathcal{O}_{\bar\lambda}$ *and* $\tilde{\mathcal{A}} - \mathrm{mod}$ *can be realized as full subcategories*

$$\mathcal{O}_{\bar\lambda} \subset \mathrm{Coh}^G(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) = \mathrm{Coh}^{G\times T}\big(m_G^{-1}(T^*(B\backslash G) \times T^*(G/N)\big),$$
$$\tilde{\mathcal{A}} - \mathrm{mod} \subset \mathrm{Coh}^G(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) = \mathrm{Coh}^{G\times T\times T}\big(m_G^{-1}(T^*(G/N)^2)\big).$$

Remark 13.0.2. The moment equation is imposed only for G-equivariance for $G{\times}T$ restricted to $T \times T$. This is the quasi-classical counterpart of monodromicity.

The theorem is best explained in the setting of D-modules. We worked with $\mathrm{Perv}_N(B\backslash G) = \mathrm{Perv}_G(B\backslash G \times G/N)$.

13.0.1. *Compatibilities.* Use the canonical isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ and $\mathfrak{t}^* \simeq \mathfrak{t}$. For any Borel $\mathfrak{b}$ and Cartan $\mathfrak{t}$ we have a canonical isomorphism $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \simeq \mathfrak{t}$. Thus, there is a map

$$\tilde{\mathfrak{g}} \to \mathfrak{t}, \quad (\mathfrak{b}, x) \mapsto x \bmod [\mathfrak{b},\mathfrak{b}].$$

Recall that $\mathfrak{g}//G \simeq \mathfrak{t}//W$ so there is a projection $\mathfrak{g} \to \mathfrak{t}//W$. The projection has a section κ called the Konstant slice. These maps fit into a commutative diagram together with the natural projections

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathfrak{t} \arrow[r] & \mathfrak{t}//W \arrow[u, bend right, dashed, "\kappa"'] \end{tikzcd}$$

In particular, there is a map

$$\tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t}$$

An element $x \in \mathfrak{g}$ is regular if $\dim \mathfrak{z}_{\mathfrak{g}}(x) = \mathrm{rank}(\mathfrak{g})$. When restricted to the regular part the above map is an isomorphism

$$\tilde{\mathfrak{g}}^{\mathrm{reg}} \xrightarrow{\sim} \mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{t}//W} \mathfrak{t}.$$

Let e, h, f be a $\mathfrak{sl}_2$ triple with e, f regular. Then

$$\mathrm{Im}(\kappa) = f + \mathfrak{z}(e)$$

One can show that

$$\mathfrak{t} \times_{\mathfrak{t}//W} \mathfrak{t} \simeq (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \times_{\mathfrak{g}} \mathfrak{t}//W \quad \subset \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}},$$

where the last fibre product is using κ. With this identification $\mathrm{Spec}(A) = \{0\} \times_{\mathfrak{t}//W} \mathfrak{t} \subset \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$.

$$\begin{tikzcd} \mathrm{Coh}(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \arrow[r] \arrow[d] & \mathrm{Coh}(\mathfrak{t} \times_{\mathfrak{t}//W} \mathfrak{t}) \arrow[d] \\ \mathrm{Coh}(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) \arrow[r] & \mathrm{Coh}(\{0\} \times_{\mathfrak{t}//W} \mathfrak{t}) \end{tikzcd}$$

For a conjugacy class of a parabolic P with decomposition $P = N_P L$, where L is the Levi

$$G/P \times \mathfrak{g} = \{P\text{-parabolics in the given conjugacy class}\}$$

Consider the subset

$$\tilde{\mathfrak{g}}_P := \{(\mathfrak{p}, x) \in G/P \times \mathfrak{g} \mid x \in \mathfrak{p}\} = \frac{T^*(G/N_P)}{L}.$$

There are projections

$$\tilde{\mathfrak{g}} \xrightarrow{\pi_P} \tilde{\mathfrak{g}}_P \to \mathfrak{g}$$

The first map is generically $|W_L| \to 1$ and the second is generically $(W : W_L) \to 1$. Viewing $\mathcal{O}_\lambda$ as a subcategory of $\mathrm{Coh}(\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$ the wall-crossing functors $T_{\mu\to\lambda}T_{\lambda\to\mu}$ corresponds to the functors $\pi_P^*\pi_{P*}$. The functors $\pi_P^*\pi_{P*}$ are called coherent wall-crossing. Consider the inclusions

$$\delta : \tilde{\mathcal{N}} \hookrightarrow \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}, \qquad \delta' : \tilde{\mathfrak{g}} \hookrightarrow \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}.$$

These give a correspondence with the Vermas

$$\delta_*(\mathcal{O}) \leftrightarrow \Delta_e, \qquad \delta'_*(\mathcal{O}) \leftrightarrow \tilde{\Delta}_e.$$

14. Generalization to the affine setting

Recall the Kazhdan-Lusztig story. We defined a braid group action

$$B \circlearrowright D^b(\mathcal{O}_{\bar{\lambda}})$$

We also got an action

$$H_q = \mathbb{Z}[B]/(T_{s_\alpha} + 1)(T_{s_\alpha} - q) \circlearrowright K^0(\mathcal{O}^{\mathrm{gr}}_{\bar{\lambda}}) \simeq H_q.$$

Hence, $\mathcal{O}_{\bar{\lambda}}$ categorifies H_q as a H_q-module with a canonical basis given by the irreducibles. We used a toplogical realization. We also have a coherent realization.

Remark 14.0.1. Let $\mathfrak{g}^{r.s.}$ denote the set of regular semisimple elements in $\mathfrak{g}$. The braid group can be realized as a fundamental group

$$B = \pi_1(\mathfrak{t}^{\mathrm{reg}}_{\mathbb{C}}/W) = \pi_1(\mathfrak{g}^{r.s.}/G).$$

E.g. for $\mathfrak{g} = \mathfrak{sl}_n$ the braid group is given by

$$\pi_1\Big(\Big\{(z_1, \dots z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j, \sum_i z_i = 0\Big\}/S_n\Big)$$

There are several different ways to generalize this picture.

- One can replace the regular representation of H_q by another.

  Remark 14.0.2. For $\mathfrak{g} = \mathfrak{sl}_n$ a categorification of an irreducible module is given by finite dimensional representations of a finite W-algebra with regular integral central character.

- One can replace W by W_{aff}. Then $\mathfrak{g} - \mathrm{mod}$ is replaced by one of the following
  (i) $\mathfrak{g}_k - \mathrm{mod}$ with $\mathrm{char}(k) = p$.
  (ii) $\hat{\mathfrak{g}}_{\mathbb{C}} - \mathrm{mod}$ where $\hat{\mathfrak{g}}_{\mathbb{C}}$ is an affine Lie algebra.
  (iii) Modules over the quantum group U_q, where q is a root of 1.

There is a notion of a categorification of a $\mathfrak{g}$-module when $\mathfrak{g}$ is a Kac-Moody Lie algebra (Chuang-Rouquier). Examples are constructed using, say, quiver varieties.

Unlike in that setting, we work with an action of a group on the (derived) category. The group B can be realized as π_1 of something.

14.1. **Affine versions of B, W and H.** For any Dynkin graph one can define a group by generators $\tilde{s}_\alpha$ and relations

$$\underbrace{\tilde{s}_\alpha \tilde{s}_\beta \cdots \tilde{s}_{\alpha \text{ or } \beta}}_{m_{\alpha\beta}} = \underbrace{\tilde{s}_\beta \tilde{s}_\alpha \cdots \tilde{s}_{\beta \text{ or } \alpha}}_{m_{\alpha\beta}}$$

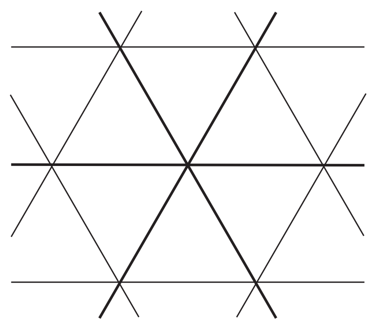
An algebraic group gives an affine Dynkin diagram. The affine braid group B_{aff} is defined as the group corresponding to that diagram.

Imposing the relation $\tilde{s}_\alpha^2 = 1$ one get the affine Weyl group.

$$B_{\text{aff}}/\langle \tilde{s}_\alpha^2 \mid \alpha \text{ simple root}\rangle = W_{\text{aff}} = W \rtimes R,$$

where R is the coroot lattice. The group W_{aff} acts on $\mathfrak{t}_\mathbb{C}$ by affine linear transformations. As a subgroup of $\operatorname{Aut}(\mathfrak{t}_\mathbb{C})$ the affine Weyl group is generated by reflections of

$$H_{\alpha,n} := \{v \mid (\alpha, v) = n\} \qquad \alpha \text{ root}, n \in \mathbb{Z}$$



If G is simply connected then $\mathfrak{t}_\mathbb{C} \to T_\mathbb{C} \simeq \mathfrak{t}_\mathbb{C}/R$ and the affine braid group can also be realized as a fundamental group

$$B_{\text{aff}} = \pi_1\left(\frac{\mathfrak{t}_\mathbb{C} \setminus \bigcup_{\alpha,n} H_{\alpha,n}}{W_{\text{aff}}}\right) = \pi_1\big(T^{\text{reg}}/W\big) = \pi_1\big(G^{r.s.}//G\big).$$

For G not necessarily simply connected define

$$B'_{\text{aff}} := \pi_1\left(\frac{\mathfrak{t}_\mathbb{C} \setminus \bigcup_{\alpha,n} H_{\alpha,n}}{W_{\text{aff}}}\right),$$

here π_1 is as an orbifold. One can also write B'_{aff} as

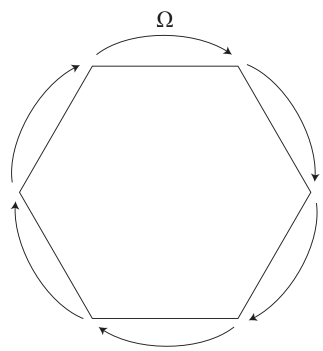
$$B'_{\text{aff}} = \Omega \rtimes B_{\text{aff}}, \qquad \Omega = \pi_1(G)$$

so $B_{\text{aff}} \subset B'_{\text{aff}}$ with equality iff G is simply connected. One can also define an extended affine Weyl group

$$W'_{\text{aff}} := W \rtimes \Lambda,$$

where Λ is the coweights of G. $W_{\text{aff}} \subset W'_{\text{aff}}$ with equality iff G is simply connected.

A connected component of $\mathfrak{t}_{\mathbb{C}} \backslash \bigcup_{\alpha,n} H_{\alpha,n}$ is an alcove. W_{aff} acts simply transitively on the set of alcoves. The action of W'_{aff} is transitive and the stabilizer of an alcove is Ω . Thus Ω acts on the fundamental alcove permuting its codimension one faces, which are identified with vertices of the affine Dynkin diagram. For example, for $G = PGL_n$ the group Ω is cyclic and the action of a generator on the affine Dynkin graph looks like this:



Analogously to the non-affine case we define

$$H_{\text{aff}} \coloneqq \mathbb{Z}[q, q^{-1}][B_{\text{aff}}] \Big/ (\tilde{s}_\alpha + 1)(\tilde{s}_\alpha - q).$$

For G not simply connected set

$$H'_{\text{aff}} \coloneqq \mathbb{Z}[q, q^{-1}][B'_{\text{aff}}] \Big/ (\tilde{s}_\alpha + 1)(\tilde{s}_\alpha - q).$$

Recall that

$$H_q \otimes_{q \to p^n} \mathbb{C} = \mathbb{C}\big(B(\mathbb{F}_q) \backslash G(\mathbb{F}_q) / B(\mathbb{F}_q)\big)$$

Replacing the Borel with a subgroup I defined later one can get a similar description for H'_{aff}

$$H'_{\text{aff}} \otimes_{q \to p^n} \mathbb{C} = \mathbb{C}\big(I \backslash G(F) / I\big),$$

where $F = \mathbb{F}_q((t))$ or another local non-archimedean field with residue field $\mathbb{F}_{q^n}$. The inclusions

$$F \coloneqq \mathbb{F}_{q^n}((t)) \supset \mathbb{F}_{q^n}[[t]] =: O, \qquad O \to \mathbb{F}_{q^n}$$

induce maps of the algebraic groups

$$\begin{tikzcd} G(F) & G(O) \arrow[l, hook'] \arrow[d] \\ & G(\mathbb{F}_{q^n}) \end{tikzcd}$$

In fact, $G(O)$ is a maximal compact subgroup of $G(F)$. Define $I \subset G(O)$ as the preimage of $B(\mathbb{F}_{q^n})$. We can assume WLOG that $T(O) \subset I$. One can show that

$$\begin{tikzcd} I\backslash G(F)/I \arrow[r, leftrightarrow] & W'_{\text{aff}} \arrow[d, equal] \\ & \mathrm{Norm}(T(F))\big/T(O) \arrow[ul, "\sim"'] \end{tikzcd}$$

Notice that

$$T(F) \simeq (F^*)^r, \qquad T(F)\big/T(O) \simeq \mathbb{Z}^r \simeq \Lambda \text{ coweights of } T$$

Since $F^*/O^* \xrightarrow{\sim} \mathbb{Z}$ as vector spaces we get

$$\Lambda = T(F)\big/T(O) \to \mathrm{Norm}(T(F))\big/T(O) \simeq W'_{\text{aff}} \to \mathrm{Norm}(T(F))\big/T(F) = W.$$

A lattice $L \subset F^n$ is a rank n O submodule.

Example 14.1.1. For G =GL(n)

$$G/B(\mathbb{F}_q) = \{(\mathbb{F}_q \supset V_{n-1} \supset \cdots \supset V_1 \supset \{0\}) \mid \dim V_i = i\}$$
$$G(F)/I = \{L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \mid L_i \text{ lattice}, L_n = t^{-1}L_0\}.$$

Example 14.1.2. The $G(O)$-orbits of the set of sublattices $L \subset O^n$ are in bijection with

$$\{(d_1 \geq d_2 \geq \cdots \geq d_n \geq 0)\} \subset \Lambda^*.$$

The T_w are the standard basis in H'_{aff}. The isomorphism we claimed existed is given by

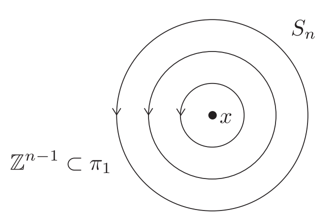
$$H'_{\text{aff}} \otimes_{q \to p^n} \mathbb{C} \xrightarrow{\sim} \mathbb{C}\big(I\backslash G(F)/I\big), \qquad T_w \mapsto \delta_{IwI}.$$

W_{aff} can be defined either as a group generated by reflections (Coxeter presentation) or as $W \rtimes R$. In the later W_{aff} is generated by $\{s_\alpha \mid \alpha \text{ is a finite simple root}\}$ and R. This upgrades to a presentation of B_{aff} with generators $\{\tilde{s}_\alpha \mid \alpha \text{ finite simple root}\}$ and $\{t_\lambda \mid \lambda \in \Lambda\}$ and relations

(i) $t_\lambda t_\mu = t_{\lambda+\mu}$
(ii) $\underbrace{\tilde{s}_\alpha \tilde{s}_\beta \cdots \tilde{s}_{\alpha \text{ or } \beta}}_{m_{\alpha\beta}} = \underbrace{\tilde{s}_\beta \tilde{s}_\alpha \cdots \tilde{s}_{\beta \text{ or } \alpha}}_{m_{\alpha\beta}}$
(iii) If $\langle \alpha^\vee, \lambda \rangle = -1$ then $\tilde{s}_\alpha t_\lambda \tilde{s}_\alpha = t_{s_\alpha(\lambda)}$.

Example 14.1.3. For G =SL(n).

$$B_{\text{aff}} = \pi_1\Big((z_1, \ldots, z_n) \in \mathbb{C}^n \mid \prod z_i = 1, z_i \neq z_j \text{ for } i \neq j\Big)$$



14.2. **Affine braid group actions.** Several of the braid group actions we have encountered so far can be extended to actions of the affine braid group.

Claim 14.2.1. Set $Z := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ and $Z' := \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$.

(1) The action of B on $D^b(\mathcal{O}_{\bar{\lambda}})$ (resp. $D^b(\tilde{\mathcal{A}} - \mathrm{mod})$) extends naturally to an action on

$$D^b(\mathrm{Coh}^G(Z')) \qquad (\text{resp. } D^b(\mathrm{Coh}^G(Z))),$$

where the action of $\tilde{w}$, $w \in W$, can be described as follows. Recall

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r] & \mathfrak{g} \times_{\mathfrak{t}//W} \mathfrak{t} \\ W \circlearrowright \tilde{\mathfrak{g}}^{\mathrm{reg}} \arrow[u, hook] \arrow[r, "\sim"] & \mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{t}//W} \mathfrak{t} \circlearrowleft W \arrow[u, hook] \end{tikzcd}$$

Let $\Gamma_w \subset \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ be the closure of the graph of the w action on $\tilde{\mathfrak{g}}^{\mathrm{reg}}$. Define

$$\Gamma_w^Z := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \Gamma_w \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \subset Z \times Z,$$
$$\Gamma_w^{Z'} := (Z' \times Z') \cap \Gamma_w^Z.$$

Consider the projections

$$\begin{tikzcd} & \Gamma_w^Z \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ Z & & Z \end{tikzcd}$$

The action of $\tilde{w}$ is defined as

$$\tilde{w} : \mathcal{F} \mapsto \mathrm{pr}_{2*}\mathrm{pr}_1^*(\mathcal{F}).$$

(2) The actions on $D^b(\mathrm{Coh}^G(Z))$ and $D^b(\mathrm{Coh}^G(Z'))$ extends to an action of B_{aff}.

$$\Lambda = \{\text{weights of } G\} = \{\text{weights of } T\} = \mathrm{Pic}^G(G/B) \subset \mathrm{Pic}(G/B).$$

Denote the line bundle on G/B corresponding to the weight $\lambda \in \Lambda$ by $\mathcal{O}(\lambda)$. The action of t_λ is given by

$$t_\lambda : \mathcal{F} \to \mathcal{F} \otimes \mathrm{pr}_2^*(\mathcal{O}(\lambda)) \qquad \lambda \in \Lambda.$$

This also give an action on $D^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(Z))$ and $D^b(\mathrm{Coh}^{G\times\mathbb{C}^*}(Z'))$ so that

$$K^0(D\,\mathrm{Coh}^{G\times\mathbb{C}^*}(Z)) \simeq H_{\mathrm{aff}}(G) \simeq K^0(D\,\mathrm{Coh}^{G\times\mathbb{C}^*}(Z))$$

with the action of B_{aff} by right multiplication (see [Kazhdan-Lusztig, Ginzburg]) FiXme Fatal: find the precise ref?

(3) There is an action of B'_{aff} on

$$D^b(\mathrm{Coh}^K(\tilde{\mathcal{N}})) \qquad \text{and} \qquad D^b(\mathrm{Coh}^K(\tilde{\mathfrak{g}})),$$

where K is a subgroup of $G \times \mathbb{C}^*$. Let $\pi : \tilde{\mathfrak{g}} \to G/B$ be the projection. For any subgroup $H \subset G$ the action of t_λ on $D^b(\mathrm{Coh}^H(\tilde{\mathfrak{g}}))$ is given by

$$t_\lambda : \mathcal{F} \mapsto \mathcal{F} \otimes \pi^*(\mathcal{O}(\lambda)).$$

There exist a section

$$W \subset W'_{\text{aff}} \twoheadleftarrow B'_{\text{aff}} \quad (w \mapsto \tilde{w}: W'_{\text{aff}} \dashrightarrow B'_{\text{aff}})$$

Let Γ_w be the closure of the graph of w acting on $\tilde{\mathfrak{g}}^{\text{reg}}$ and let $\text{pr}^w_{1,2} : \Gamma_w \to \tilde{\mathfrak{g}}$ be the projections. The action of $\tilde{w}$, $w \in W$, on $D^b(\text{Coh}^H(\tilde{\mathfrak{g}}))$ is given by

$$\tilde{w} : \mathcal{F} \mapsto \text{pr}^w_{2*}\text{pr}^{w*}_1(\mathcal{F}).$$

For the action of $\tilde{w}$ on $D^b(\text{Coh}(\tilde{\mathcal{N}}))$ replace Γ_w by

$$\Gamma'_w := \Gamma_w \cap (\tilde{\mathcal{N}} \times \tilde{\mathcal{N}})$$

and the projections $\text{pr}^w_{1,2}{}' : \Gamma'_w \to \tilde{\mathcal{N}}$

$$\tilde{w} : \mathcal{F} \to \text{pr}^w_{2*}{}'\text{pr}^{w\prime *}_1(\mathcal{F}).$$

The induced action on $K^0(\text{Coh}^{G\times\mathbb{C}^*}(\tilde{\mathfrak{g}}))$ is the H_{aff} module

$$M_{\text{asph}} = H_{\text{aff}} \otimes_H \text{sign},$$

where H is the finite Hecke algebra. This module is called the anti-spherical module.

$$M_{sph} = H_{\text{aff}} \otimes_H \text{trivial representation}.$$

This is called the spherical module. For $F = \mathbb{F}_{p^n}((t))$ and $q \rightsquigarrow p^n$

$$H_{\text{aff}} \simeq \mathbb{C}\big(I\backslash{}^LG(F)/I\big) \circlearrowright \mathbb{C}\big(I\backslash{}^LG(F)/{}^LG(O)\big) \simeq M_{\text{sph}}$$

Example 14.2.2. For $\mathfrak{g} = \mathfrak{sl}_2$ $W = \{1, s\}$. The graph has two components

$$\Gamma'_s = (\mathbb{P}^1 \times \mathbb{P}^1) \cup \Delta \quad \subset (T^*\mathbb{P}^1)^2,$$

where Δ is the diagonal. Thus, there is a short exact sequence with functions vanishing on one component and functions vanishing on the other component

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta_{\mathbb{P}^1}) \to \mathcal{O}_{\Gamma'_s} \to \mathcal{O}_\Delta \to 0.$$

Here $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta_{\mathbb{P}^1})$ is the sheaf of functions on $\mathbb{P}^1 \times \mathbb{P}^1$ which vanish on the divisor $\Delta_{\mathbb{P}^1}$.

This can also be described as a a spherical reflection. Let $\text{pr}_2 : T^*\mathbb{P}^1 \times T^*\mathbb{P}^1 \to T^*\mathbb{P}^1$ be the projection on the second factor. Define

$$S(\mathcal{F}) := \text{pr}_{2*}(\text{pr}^*_1(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta_{\mathbb{P}^1}))[1]$$

Using the short exact sequence above we get an exact triangle

$$S(\mathcal{F})[-1] \to \tilde{s}(\mathcal{F}) \to \mathcal{F} \to S(\mathcal{F}).$$

Each fiber of $S(\mathcal{F})$ is isomorphic to $H^*(\mathcal{F} \otimes_{\mathcal{O}(T^*\mathbb{P}^1)} \mathcal{O}(-1))[1]$.

Exercise 14.2.3. Show that $S(\mathcal{F}) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{Hom}^\bullet(\mathcal{F}, \mathcal{O}_{\mathbb{P}^1}(-1))^*$.

14.3. **Categorical reflection.** For an abelian group with a $\mathbb{Z}$-valued bilinear pairing there is a reflection for each $v \in X$ given by

$$x \mapsto x - 2\frac{(v,x)}{(v,v)}v.$$

If $(v,v) = 2$ then this is

$$x \mapsto x - (v,x)v.$$

One can define a categorical analog of this. Let $\mathcal{C}$ be a triangulated category over k satisfying that

$$\mathrm{Hom}^\bullet(X,Y) \text{ is finite dimensional for all } X,Y \in \mathbb{C}.$$

An element $V \in \mathcal{C}$ is spherical of dimension d for $d > 0$ if

$$\mathrm{Ext}^i(V,V) = \begin{cases} k & i = 0,d \\ 0 & i \neq 0,d \end{cases}$$

For such a V a categorical analog of reflection is given by

$$X \mapsto \mathrm{Cone}(X, \mathrm{Hom}^\bullet(X,V)^* \otimes V)[-1]$$

Note that Cone is not functorial so we need some additional assumptions to make it a functor. If d is even then taking K^0 we get to the above reflection with

$$([X],[Y]) = \mathrm{Eul}(\mathrm{Ext}(X,Y))$$

Example 14.3.1. The object $\mathcal{O}_{\mathbb{P}^1} \in D^b(\mathrm{Coh}(T^*\mathbb{P}^1))$ (and hence $\mathcal{O}_{\mathbb{P}^1}(i)[j]$ for any i and j) is spherical with $d = 2$. Indeed, for X smooth

$$\mathrm{Ext}_{\mathrm{Coh}(T^*X)}(\mathcal{O}_X, \mathcal{O}_X) = \bigoplus H^i(\Omega^j(X)).$$

If X is smooth projective this is equal to $H^*(X)$. For $\mathbb{CP}^1$

$$H^i(\mathbb{CP}^1) = \begin{cases} \mathbb{C} & i = 0,2 \\ 0 & i \neq 0,2 \end{cases}$$

The spherical reflection corresponding to $\mathcal{O}_{\mathbb{P}^1}[-1]$ is the $\tilde{s}$ from the previous example.

Let $\Sigma \subset \mathfrak{g}$ be a transversal slice to the G-orbit of $e \in \mathcal{N}$, i.e. an affine linear space with $T_e\Sigma \oplus T_eG(e) = \mathfrak{g}$. Set

$$\tilde{\Sigma} := \tilde{\mathcal{N}} \times_{\mathfrak{g}} \Sigma.$$

The B'_{aff}-action on $D^b(\mathrm{Coh}(\tilde{\mathcal{N}}))$, $D^b(\mathrm{Coh}(\tilde{\mathfrak{g}}))$ induces an action on $D^b(\mathrm{Coh}(\tilde{\Sigma}))$ using the same formula as before on

$$\begin{tikzcd} & \Gamma'_w(\Sigma) := \Gamma'_w \cap \tilde{\Sigma} \times \tilde{\Sigma} \arrow[dl] \arrow[dr] & \\ \tilde{\Sigma} & & \tilde{\Sigma} \end{tikzcd}$$

14.3.1. *Subregular slice.* There exists a unique orbit on $\mathcal{N}$ of codimension 2. Let e be in this orbit. Then

$$\Sigma \cap \mathcal{N} \simeq \mathbb{A}^2/\Gamma,$$

where Γ is a finite subgroup in $\mathrm{SL}(2)$.

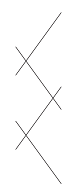
Example 14.3.2. For $\mathfrak{g} = \mathfrak{sl}_n$ a possible choice of e and Σ is

$$e = \left(\begin{array}{cc|c|cc} 0 & 0 & & & \\ \hline 0 & 0 & 1 & & \\ \hline & & 0 & 1 & \\ & & & \ddots & \ddots \end{array}\right), \qquad \Sigma = \{XY = Z^n\} \subset \mathbb{A}^3.$$

Here e has two Jordan blocks of size 1 and $n-1$. Consider the projection

$$\tilde{\Sigma} \to \Sigma \cap \mathcal{N} = \mathbb{A}^2/\Gamma \ni e.$$

The fiber over e is a reducible variety with components $\mathbb{CP}^1$ intersecting in the following way.



Exercise 14.3.3. Show that $D^b(\mathrm{Coh}(\tilde{\Sigma})) \simeq D^b(\mathrm{Coh}^\Gamma(\mathbb{A}^2))$.

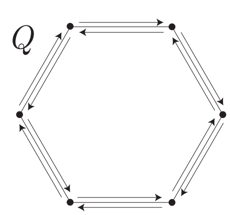
In general if G is simply laced then the components of the fiber over e are in bijection with the vertices of the Dynkin diagram (see [Slo]).

Exercise 14.3.4. Show that $\mathcal{O}_{\ell_i}$, where ℓ_i is a component of the fiber over e, is a spherical object with $d = 2$ and that $s_\alpha \in B_{\mathrm{aff}}$ acts by spherical reflection for $\mathcal{O}_{\mathbb{P}^1}(-1)[1]$.

We have $\mathrm{Coh}^\Gamma(\mathbb{A}^2) = R-\mathrm{mod}$ where

$$\begin{aligned} R &= \text{ preprojective algebra} \\ &= \text{ path algebra of the quiver Q with} \\ &\qquad \sum_{w-v} \pm e_{w\to v} e_{v \to w} = 0 \qquad \forall \text{ vertex } v \end{aligned}$$

The sign rule is the following. Fixing an orientation, the sign is plus if $v \to w$ agrees with the orientation. Otherwise, it is minus.



For that e and G simply laced $K^0(\mathrm{Coh}(\tilde{\Sigma}))$ is the reflection representation of W'_{aff} and $K^0(\mathrm{Coh}^{\mathbb{C}^*}(\tilde{\Sigma}))$ is the reflection representation of H'_{aff}.

It turns out that the result of exercise 14.3.3 is true is greater generality.

Exercise 14.3.5. For G simply laced B'_{aff} acts on a slice to a subregular nilpotent $e \in \mathcal{N}$ with $\dim G(e) = \dim \mathcal{N} - 2$. Show that

$$D^b(\tilde{\Sigma}) \simeq D^b(\mathrm{Coh}^{\Gamma}(\mathbb{A}^2))$$

and

$$\mathrm{Coh}^{\Gamma}(\mathbb{A}^2) = R - \mathrm{mod},$$

where for some quiver Q

$$R = \text{preprojective algebra} \;=\; \text{path algebra of } Q \text{ with relations}$$

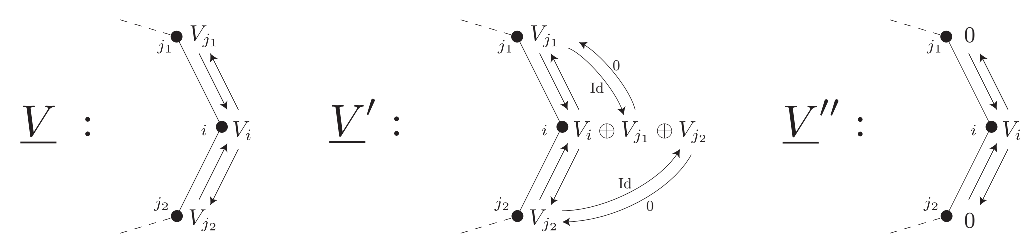
$$\sum_{w-v} \pm e_{w\to k} e_{k\to v} = 0 \qquad \forall \text{ vertex } v$$

For the first equivalence see [KV].

Exercise 14.3.6. (1) Let ρ be an irreducible representation of Γ. Let k_0 denote the skyscraper at 0. Show that $\rho \otimes k_0 \in \mathrm{Coh}^{\Gamma}(\mathbb{A}^2)$ is a spherical object. The spherical reflection gives an action of B_{aff} on $D^b(\mathrm{Coh}^{\Gamma}(\mathbb{A}^2))$.

(2) Write down the action on the preprojective algebra side. Let $\underline{V}$ be a representation. Define representations $\underline{V}'$ and $\underline{V}''$ as follows

$$V'_j := \begin{cases} V_j & i \neq j \\ V_i \oplus \bigoplus_{k-i} V_k & i = j \end{cases}, \qquad V''_j := \begin{cases} 0 & j \neq i \\ V_i & j = i \end{cases}$$



Consider the three term complex

$$S_i(\underline{V}) := \quad \underline{V}'' \to \underline{V}' \to \underline{V}''$$

given by 0 away from i and for i

$$V_i \xrightarrow{0_{V_i} + \sum_{i-j} \pm e_{ij}} V_i \oplus \bigoplus_{i-j} V_j \xrightarrow{\mathrm{Id}_{V_i} + \sum_{i-j} -e_{ji}} V_i$$

in the first sum the sign is - if $i \to j$ and + if $j \to i$. Extend this to complexes. This defines an autoequivalence S_i of the derived category of modules over the preprojective algebra. We get an action of the affine braid group on this category, where the i-th generator (in the Coxeter presentation) acts by S_i. Cf. [BGP] by Bernstein, Gelfand and Ponomarev.

14.4. **Where does W_{aff} and H_{aff} arise in representation theory?** One answer: representations in characteristic $p >$ the Coxeter number. In characteristic 0 we have $Z(\mathcal{U}(\mathfrak{g})) \simeq \mathrm{Sym}(\mathfrak{t})^W$ but in characteristic p we only have an inclusion $Z(\mathcal{U}(\mathfrak{g}_k)) \subset \mathrm{Sym}(\mathfrak{t}_k)^W$. $\mathrm{Sym}(\mathfrak{t}_k)^W$ is called the Harish-Chandra center and is denoted by Z_{HCh}. So for $\lambda \in \mathfrak{t}_k^*$ we can look at representations with corresponding generalized character of Z_{HCh}. For G simply laced

$$\text{Integral characters of } Z_{\mathrm{HCh}} = \frac{\Lambda/p\Lambda}{W} = \Lambda/(W \ltimes p\Lambda) = \Lambda/W'_{\mathrm{aff}}.$$

As in characteristic 0 we define walls

$$\Lambda \supset H_{\alpha,n} := \{\lambda \mid \langle\lambda,\alpha\rangle = np\} \qquad \alpha \text{ coroot }, n \in \mathbb{Z}.$$

One can define translation functors between categories of modules with different generalized integral central characters. They share many of the properties we have seen for translation functors in characteristic zero, in particular we have translation functors to and from the wall, so we can define wall crossing functors acting on the category $\mathfrak{g}-\mathrm{mod}_0$ of $\mathfrak{g}$-modules with generalized central character of the trivial representation (or another integral regular central character). The wall crossing functors give an action of B_{aff} on $D^b(\mathfrak{g}-\mathrm{mod}_0)$. Let $\mathrm{Coh}_{\mathcal{B}_0}(\tilde{\mathfrak{g}})$ be the modules set theoretically supported on the zero section. B_{aff} also act on $D^b(\mathrm{Coh}_{\mathcal{B}_0}(\tilde{\mathfrak{g}}))$.

Theorem 14.4.1. *For k of characteristic $p >$ Coxeter number*

$$D^b(\mathfrak{g}-\mathrm{mod}_0) \simeq D^b(\mathrm{Coh}_{\mathcal{B}_0}(\tilde{\mathfrak{g}}))$$

and the B_{aff}-actions are compatible under the equivalence.

15. Canonical basis

Recall that for λ regular integral $\mathcal{O}_{\bar{\lambda}} \simeq \mathcal{A}-\mathrm{mod} \subset \mathrm{Coh}^G(\tilde{\mathcal{N}} \times \tilde{\mathfrak{g}})$. On $D^b(\mathcal{A}-\mathrm{mod})$ we defined a B-action where w acts by the functor I_w. The I_w are right exact functors, i.e. they send

$$D^{\le 0}(\mathcal{A}-\mathrm{mod}) \to D^{\le 0}(\mathcal{A}-\mathrm{mod})$$

and there is an exact triangle $\mathrm{Id} \to \Xi_\alpha \to I_{s_\alpha}$.

The B-action on $D^b(\mathrm{Coh}^G(\tilde{\mathcal{N}} \times \tilde{\mathfrak{g}})) \supset D^b(\mathcal{A}-\mathrm{mod})$ extends to a B_{aff}-action. This comes from an action of B_{aff} on $D^b(\mathrm{Coh}(\tilde{\mathfrak{g}}))$. We are interested in a canonical basis in $K^0(\mathrm{Coh}(\mathcal{B})) \simeq H^*(\mathcal{B})$ and the corresponding category of representations

$$K^0(\mathrm{Coh}(\mathcal{B})) \simeq K^0(\mathrm{Coh}_{\mathcal{B}}(\tilde{\mathfrak{g}})) \simeq K^0(\mathrm{Coh}_{\mathcal{B}}(\tilde{\mathcal{N}})) \xrightarrow{q\text{-deformation}} K^0(\mathrm{Coh}_{\mathcal{B}}^{\mathbb{C}^*}(\tilde{\mathfrak{g}}))$$

The $\mathbb{C}^*$ action on the right hand side is by dilating the fibers. This can be enriched to $K^0(\mathrm{Coh}_{\mathcal{B}}^{\mathbb{C}^*\times T}(\tilde{\mathfrak{g}}))$.

Theorem 15.0.1. *(1) There are equivalences of categories $D^b(\mathrm{Coh}(\tilde{\mathfrak{g}})) \simeq D^b(A-\mathrm{mod}_{f.g.})$ and $D^b(\mathrm{Coh}(\tilde{\mathcal{N}})) \simeq D^b(A'-\mathrm{mod}_{f.g.})$ for some algebras A and A' satisfying*

(i) The element $\tilde{s}_\alpha \in B_{aff}$ acts by right exact functors on the right hand side, i.e.

$$\tilde{s}_\alpha : D^{\le 0}(A-\mathrm{mod}_{f.g.}) \to D^{\le 0}(A-\mathrm{mod}_{f.g.})$$

(ii) The following diagram is commutative

$$\begin{tikzcd} D^b(A-\mathrm{mod}) \arrow[r] \arrow[dr, "exact"'] & D^b(\tilde{\mathfrak{g}}-\mathrm{mod}) \arrow[d, "R\Gamma"] \\ & D^b(\mathrm{Coh}(\mathfrak{g})) \end{tikzcd}$$

Here Γ stands for global sections thought of as a module over $\mathcal{O}(\mathfrak{g})$, i.e. a coherent sheaf on $\mathfrak{g}$ (thus $R\Gamma$ here is synonymous to $R\pi_*$ where π is the projection $\tilde{\mathfrak{g}} \to \mathfrak{g}$).

(iii) This characterize A uniquely up to Morita equivalence.

(2) The classes of irreducible objects in $A-\mathrm{mod}^{\mathbb{C}^*}$ form a canonical basis in

$$\begin{aligned} K^0(A-\mathrm{mod}^{\mathbb{C}^*}_{\mathit{fin.\ length}}) &\simeq K^0(\mathrm{Coh}^{\mathbb{C}^*}_{\mathcal{B}}(\tilde{\mathfrak{g}})) \\ &\simeq H^*(\mathcal{B})[v,v^{-1}] \end{aligned}$$

Lusztig's definition of canonical basis is based on an involution i on $K^0(\mathrm{Coh}^{\mathbb{C}^*}(\tilde{\mathfrak{g}})), K^0(\mathrm{Coh}^{\mathbb{C}^*}(\tilde{\mathcal{N}}))$.

$$i([\mathcal{F}]) \coloneqq \kappa(\tilde{w}_0(S(\mathcal{F}))),$$

where κ is the Chevalley involution - an involution of G such that

$$\forall g \in G \ \ \exists x \in G: \quad \kappa(g) = xgx^{-1}$$

and S is the Grothendieck Serre duality $\mathcal{F} \mapsto R\,\mathrm{Hom}(\mathcal{F},\mathcal{O})$.

The involution is identity on non q-deformed K^0. For $G = \mathrm{SL}_n$ it is $g \mapsto ({}^t g)^{-1}$. There is a pairing on $K^0(D^b(\mathrm{Coh}^{\mathbb{C}^*}_{\mathcal{B}}(\tilde{\mathcal{N}})))$ given by

$$\langle [\mathcal{E}],[\mathcal{F}]\rangle \coloneqq \sum_i v^i \sum_j (-1)^j \dim \mathrm{Ext}^j_i(\mathcal{E},\mathcal{F}) \quad \in \mathbb{Z}[v,v^{-1}].$$

Here Ext^j_i denotes the i'th graded component of Ext^j.

Definition 15.0.2 (Lusztig)**.** A basis (C_s) in $K^0(\mathrm{Coh}^{\mathbb{C}^*}_{\mathcal{B}}(\tilde{\mathcal{N}}))$ is canonical if $i(C_s) = C_s$ and $\langle C_a, C_b\rangle \in \delta_{ab} + v\mathbb{Z}[v]$.

Exercise 15.0.3. If a canonical basis exists it is unique up to replacing C_s by $-C_s$.

15.0.1. *The canonical basis in this representation of H_{aff}.* Over a field of characteristic $p >$ Coxeter number

$$A-\mathrm{mod}_{\substack{\text{finite length, 0 generalized} \\ \text{central character}}} = \mathfrak{g}-\mathrm{mod}_{\substack{\text{generalized central character} \\ \text{of the trivial representation}}}$$

There is a closely related setting in characteristic 0, namely quantum groups at a root of 1. The analogues of Kazhdan-Lusztig conjectures here are known as a result of work of many people on "Lusztig program", the proofs use affine Kac-Moody group (loop group for the Langlangs dual group ${}^L G$) and the corresponding Lie algebra.

A similar thing works if $\tilde{\mathcal{N}}$ is replaced by a resolution of the slice. Let $e \in \mathcal{N}$ and Σ_e slice to the orbit $G(e)$

$$\mathbb{C}^* \circlearrowright \quad X := \tilde{\mathcal{N}} \times_{\mathfrak{g}} \Sigma_e.$$

Notice that X contains the Springer fiber $\mathcal{B}_e$. $K^0(D^b(\mathrm{Coh}^{\mathbb{C}^*}_{\mathcal{B}_e}(X)))$ is called the standard module for H_{aff}. It was defined by Kazhdan, Lusztig and Ginzburg in the 80's.

To get a group which is most directly related to representations in positive characteristic one can pass to the non-graded version:

$$\begin{array}{c} K^0(D^b(\mathrm{Coh}^{\mathbb{C}^*}_{\mathcal{B}_e}(X))) \\ \downarrow \\ K^0(\mathrm{Coh}(\mathcal{B}_e)) \simeq K^0(\mathrm{Coh}_{\mathcal{B}_e}(X)) \\ \downarrow \\ H_*(\mathcal{B}_e) \end{array}$$

Proposition 15.0.4. *Set $A_e := A' \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{O}(\Sigma_e)$. Then*

(1) $D^b(\mathrm{Coh}(X)) \simeq D^b(A_e - \mathrm{mod})$.
(2) $A_e - \mathrm{mod} = \mathfrak{g} - \mathrm{mod}$ *in characteristic $p > h$ where the center of the enveloping algebra acts through a certain quotient isomorphic to $\mathcal{O}(\Sigma_e)$.*

Another related category of representations is that of the finite W-algebra W_e which is a quantization of Σ_e. In particular, the category of $W_e - \mathrm{mod}$ over k of characteristic $p > h$ where a part of the center acts by a fixed integral regular character is also equivalent to $A_e - \mathrm{mod}$.

Proposition 15.0.5. *In type A we have*

(1) *$H_*(\mathcal{B}_e) \supset H_{top}(\mathcal{B}_e)$ is an irreducible representation of S_n.*
(2) *$K^0(W_e - \mathrm{mod}_{f.d.}/\mathbb{C}) \simeq H_{top}(\mathcal{B}_e)$.*

There are many examples of representation categories $\mathcal{A}$ with an action of B or B_{aff} on $D^b(\mathcal{A})$. Recall that

$$B_{\mathrm{aff}} = \pi_1\left(\frac{\mathfrak{t}^* \setminus \text{ hyperplanes defined over } \mathbb{R}}{\text{Symmetry}}\right)$$

A natural generalization of this to other situations when the symmetry group does not act transitively on the set of components in the real locus of the complement (alcoves) seems to be the following:

One should consider an abelian category assigned to each alcove and a derived equivalence attached to a homotopy class of the path connecting two alcoves, producing a functor from the subgroupoid in the Poincare groupoid to the 2-category of categories.

Notice that a presentation for this subgroupoid similar to the standard presentation for the braid group appears in [Sal].

An attempt to axiomatize further properties of this situation can be found in [ABM]. The main axiom (inspired by the notion of Bridgeland stability condition) prescribes exactness properties of the functor corresponding to the loop which goes around a hyperplane in the positive directions; such a loop generalizes a standard generator of the (affine) braid group.

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