When stationary modes aren't stationary: Coupling of modes and adiabatic processes in dynamic eigenproblems

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1 Introduction

Many physical and mathematical problems involve the study of *harmonic modes*, solutions which oscillate sinusoidally in time. For example, the vibrations of a drum or a piano string (acoustic waves), the propagation of light (an electromagnetic wave) in a medium or down an optical fiber, and the allowed energies of an electron bound to a nucleus (a quantum probability wave), are all described by harmonic-mode solutions of the corresponding wave equation. An important question, with very general solutions, is what happens to these harmonic modes if we allow them to *weakly couple* to one another. For example, if we bring a piano string next to a tuning fork, vibrations in one will excite vibrations in the other—they are *resonantly coupled* if they have the same vibrational frequency. Even more interesting things can happen if you *change* the vibrational modes with time—for example, if you start a piano string vibrating and then change its tension, or change how far away the tuning fork is, or change the shape of an optical fiber (e.g. by bending it) as light propagates down it, or shake an atom with an external field. In this case, there is a general *adiabatic theorem* that tells you what happens if you change the system *slowly* enough.

Put another way, strictly harmonic modes arise in *linear*, *time-invariant* systems. If we can extend this analysis to *almost* linear time-invariant systems, we will have greatly expanded the reach of our understanding.

2 Two coupled pendula

We'll center our discussion on a simple physical example. Suppose that we have two swinging pendula (denoted by $k = 1,2$) of lengths L_k and at angles θ_k with vertical. A single rigid pendulum swinging under gravity is described by the second-order ODE $\frac{d^2\theta_k}{dt^2} = \ddot{\theta}_k = -\frac{g}{L_k}\sin\theta_k \approx -\frac{g}{L_k}\theta_k$, approximated for small θ_k. This is just a harmonic oscillator with angular frequency $\omega = \sqrt{g/L_k}$. Now, however, suppose that we *couple* the two pendula: for example, when one swings, suppose it exerts a force on the other proportional to $\theta_1 - \theta_2$. We then obtain equations of the form:

$$\begin{aligned} \ddot{\theta}_1 &= -\omega_1^2\theta_1 + c\theta_2 \\ \ddot{\theta}_2 &= -\omega_2^2\theta_2 + c\theta_1, \end{aligned}$$

where $\omega_k^2 = \sqrt{g/L_k} + c$ and c is the proportionality constant of the coupling. For later convenience, we will set $c = \kappa\bar{\omega}^2$, where $\bar{\omega}^2 = \frac{\omega_1^2+\omega_2^2}{2}$ and we will assume $\kappa \ll 1$.

We'll be using this as a model system to study two important concepts: what are the *stationary* or *harmonic* modes of the system, and how do these states evolve as the system *changes*, for example if we change the length of one of the pendula *as it is swinging*.

In a *linear*, *time-invariant* set of differential equations like this one, you can always look for *harmonic solutions*, or *stationary modes*, or *eigenmodes*, of the form $\theta_k = u_k e^{i\omega t}$ (the "physical" solution is just the real part) where u_k is a constant and ω is the eigenfrequency. To find them, we just plug $\theta_k = u_k e^{i\omega t}$ into the differential equations, and obtain a linear eigenproblem where ω^2 is the eigenvalue:

$$\omega^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 & -\kappa\bar{\omega}^2 \\ -\kappa\bar{\omega}^2 & \omega_2^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

This gives a quadratic equation $\lambda^2 - 2\bar{\omega}^2\lambda + (\omega_1^2\omega_2^2 - \kappa^2\bar{\omega}^4) = 0$ for the eigenvalue $\lambda = \omega^2$. Obviously, if $\kappa = 0$ the solutions are $\omega = \omega_1$ and $\omega = \omega_2$, the frequencies of the individual pendula. More generally, we get:

$$\lambda = \bar{\omega}^2 \pm \sqrt{\left(\frac{\omega_1^2 - \omega_2^2}{2}\right)^2 + \kappa^2\bar{\omega}^4}$$

2.1 Resonant coupling

Let's start by taking a simple case: suppose $\omega_1 = \omega_2 = \bar{\omega}$, i.e. the pendula have the *same length*. In this case the eigenvalues are just $\omega^2 = \lambda = \bar{\omega}^2 \cdot (1 \pm \kappa)$ and thus $\omega = \bar{\omega}\sqrt{1 \pm \kappa} \approx \bar{\omega} \cdot (1 \pm \kappa/2)$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}$: when the pendula are swinging *together* they have a *lower* frequency [there is no resistance to their swinging since $c(\theta_1 - \theta_2) = 0$], and when they are swinging *oppositely* they have a *higher* frequency (the resistance κ to their separation increases the "spring constant"). Now, what if we start *just one* of the pendula swinging, with zero initial velocity and initial amplitude $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? This initial condition is satisfied by a *superposition* of the two eigenvectors:

$$\begin{aligned} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\bar{\omega}(1+\kappa/2)t} + \begin{pmatrix} 1 \\ +1 \end{pmatrix} e^{i\bar{\omega}(1-\kappa/2)t} \\ &= e^{i\bar{\omega}(1+\kappa/2)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ +1 \end{pmatrix} e^{-i\bar{\omega}\kappa t}\right]. \end{aligned}$$

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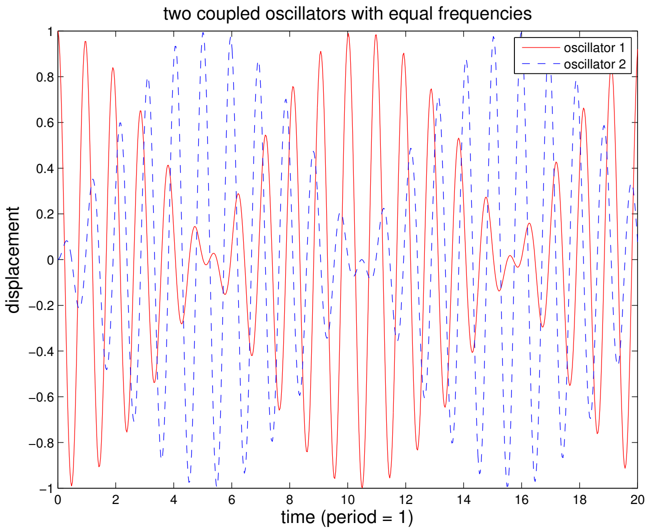

Figure 1: Plot of $\theta_k(t)$ for two coupled pendula of equal length, where one is started swinging: the energy periodically exchanges between them.

At $t = 0$ this gives $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, but at $t = \pi/(\bar{\omega}\kappa)$ this gives $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and then back to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 2\pi/(\bar{\omega}\kappa)$! That is, the energy in the system seems to *oscillate periodically* back and forth between the two pendula, repeating every $1/\kappa$ periods $2\pi/\bar{\omega}$ of the isolated pendula! This is precisely what we see in fig. 1, where we have used a period $2\pi/\bar{\omega} = 1$ and $\kappa = 1/10$. This behavior is typical of *resonant coupling* of two oscillators.

2.2 Anti-crossings and time evolution

On the other hand, suppose the frequencies ω_1 and ω_2 are very different, with $|\omega_1^2 - \omega_2^2| \gg \kappa^2\bar{\omega}^2$. In this case, the two oscillators are out of resonance from one another, and the coupling shouldn't have much effect: the pendula should just swing separately. If we solve the eigenequation, this is precisely what we find:

$$\lambda \approx \omega_{1,2}^2 \pm \frac{\kappa^2\bar{\omega}^4}{\omega_1^2 - \omega_2^2} \approx \omega_{1,2}^2 \pm (\text{small}).$$

That is, the eigensolutions are almost those of the isolated pendula. In fig. 2, we plot the two eigenfrequencies $\omega = \sqrt{\lambda}$ as a function of ω_1 (e.g. by changing L_1) while keeping ω_2 fixed. For $\kappa = 0$, we just get two straight lines (blue dashed) corresponding to the two pendula swinging separately. When $\kappa \neq 0$, however, the two (solid red) lines *couple* at the point where the eigenvalues *cross*, leading to what is called an *avoided crossing* or an *anti-crossing*.

Suppose we start with $L_1 \ll L_2$ so that the frequencies are very different. If we start pendulum 1 swinging, pendulum 2 should barely move. Now, suppose that we start increase L_1 *as the pendula are swinging*. If we are able to change the system slowly enough, a remarkable thing happens when L_1 goes through L_2: all of the energy transfers to pendulum 2, so that pendulum 1 (almost) stops swinging! Precisely this

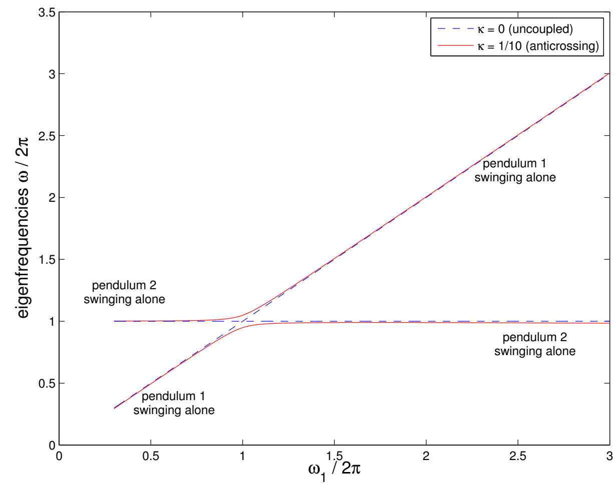

Figure 2: Eigenvalues of the coupled pendula, as a function of ω_1 for fixed $\omega_2 = 2\pi$, for $\kappa = 0$ (dashed blue) and $\kappa = 1/10$ (solid red). For $\kappa \neq 0$, the two pendula couple where their eigenvalues cross, leading to an avoided crossing or *anti-crossing*.

behavior is shown in figure 3(top), a numerical simulation of the ODE with Matlab. What is happening is that the system *follows the eigenvectors* as they change continuously: it starts out in the ω_1 eigenvalue (where pendulum 1 is swinging alone) and then follows it *around the anti-crossing* into the ω_2 eigenvalue (where the pendulum 2 is swinging alone). Figure 3(top) shows the corresponding eigenvalues ω as a function of time—the system adiabatically follows the upper eigenvalue curve. This is quite a general result, and is known as the *adiabatic theorem.*

3 Generalization and proof

Let's cast the problem in a more general form. It turns out that a second-order ODE is inconvenient, but we can always convert each second-order ODE into two first-order ODEs.

3.1 Real-symmetric first-order formulation

It will be convenient to write things in the form:

$$\dot{\vec{x}} = iA\vec{x} \tag{1}$$

where A is an $N \times N$ matrix. If A is a *constant*, the time-harmonic modes $\vec{x} = \vec{u}e^{i\omega t}$ satisfy the eigenproblem

$$\omega\vec{u} = A\vec{u}.$$

If we are looking at systems without gain or dissipation, then ω must be real: the solution oscillates, without exponential growth or decay. This gives us a hint that A may have a *special form*: we can usually write the problem in a form where A is *real-symmetric*: A is equal to its transpose.[1] Real-symmetric

[1]More generally, A is typically *Hermitian*: A is equal to the complex conjugate of its transpose. Here, we simplify life by sticking with real A.

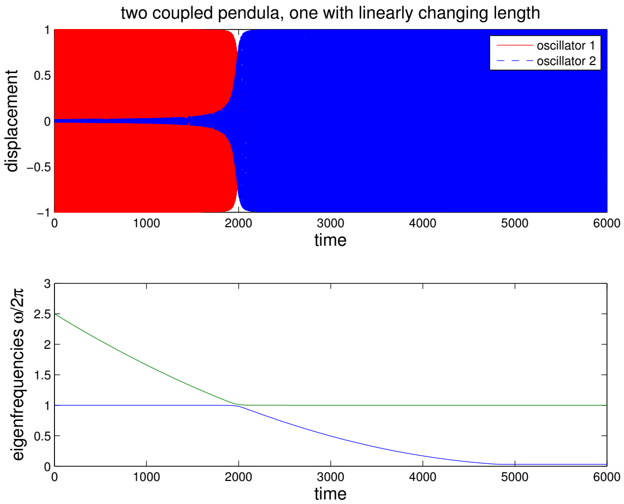

Figure 3: *Top:* pendula amplitudes when the length L_1 of one pendulum is varied from about $1.6L_2$ to about $0.18L_2$. As L_1 passes through L_2 and the frequencies are equal, almost all of the energy of oscillation adiabatically transfers to the second pendulum. *Bottom:* The corresponding frequency eigenvalues $\omega/2\pi = 1/\text{period}$ of the coupled-oscillator system, showing the slight avoided crossing from the weak coupling ($\kappa = 1/80$). The oscillator adiabatically "follows" the upper eigenvalue curve.

matrices have three very nice properties: there are N linearly independent eigenvectors $\vec{u}_n$ with eigenvalues ω_n (the matrix is never *defective*); the eigenvalues are purely *real*, and the eigenvectors can be chosen real and *orthogonal*: $\vec{u}_n \cdot \vec{u}_m = 0$ for $n \neq m$, and for convenience we will choose $\vec{u}_n \cdot \vec{u}_n = 1$.

For example, it is a simple exercise to show that our two coupled harmonic-oscillator equations above can be written in this real-symmetric form, by introducing two auxiliary variables α_k and writing:

$$\begin{aligned}
\dot{\theta}_1 &= i\tilde{\omega}_1\alpha_1 - i\tilde{\kappa}\alpha_2 \\
\dot{\theta}_2 &= i\tilde{\omega}_2\alpha_2 - i\tilde{\kappa}\alpha_1 \\
\dot{\alpha}_1 &= i\tilde{\omega}_1\theta_1 - i\tilde{\kappa}\theta_2 \\
\dot{\alpha}_2 &= i\tilde{\omega}_2\theta_2 - i\tilde{\kappa}\theta_1
\end{aligned}$$

where $\tilde{\omega}_k^2 + \tilde{\kappa}^2 = \omega_k$ and $\tilde{\kappa}(\tilde{\omega}_1 + \tilde{\omega}_2) = c = \kappa\bar{\omega}^2$.

3.2 Coupled-mode equations

Now, we want to consider a case where $A(t)$ is *not* constant, but is some *slowly varying* function of time. The trick is that, since it is *almost* constant, we can *almost* have harmonic modes. So, we define the "instantaneous" harmonic modes $\vec{u}_n(t)$ to satisfy the eigenproblem at time t:

$$A(t)\vec{u}_n(t) = \lambda_n(t)\vec{u}_n(t).$$

At *every* time we therefore have a complete set of eigenvectors that are continuously changing, and we use this set as a *basis* for our solution vector $\vec{x}(t)$:

$$\vec{x}(t) = \sum_n c_n(t)\vec{u}_n(t)e^{i\int^t \lambda_n(t')dt'}. \qquad (2)$$

Why did we choose this particular form? Consider what happens if A is a constant. In this case, the eigenvectors are constants and $\vec{u}_n e^{i\lambda_n t}$ is an exact solution of the equations, and thus the coefficients $c_n(t)$ are *constants*.

If A is changing slowly, then $c_n(t)$ will *almost* be constant, and we can exploit this to understand the system. In fact, the **adiabatic theorem** tells us that, in the limit as we change A more and more slowly, the c_n ***exactly* approach constants**.

Now, how do we solve for $c_n(t)$? The $c_n(0)$ are given by our initial conditions $\vec{x}(0)$, and to get the equation for c_n at other times we just substitute eq. (2) into eq. (1):

$$\begin{aligned}
\dot{\vec{x}} &= \sum_n \left[\dot{c}_n\vec{u}_n + c_n\dot{\vec{u}}_n + i\lambda_n c_n\vec{u}_n\right] e^{i\int^t \lambda_n dt'} \\
&= iA(t)\vec{x} = \sum_n i\lambda_n c_n\vec{u}_n e^{i\int^t \lambda_n dt'},
\end{aligned}$$

where on the first line we have used the product rule and the fact that $\frac{d}{dt}\int^t \lambda_n dt' = \lambda_n(t)$, and on the second line we have used the eigen-equation for $\vec{u}_n$. Now, however, a lovely thing has happened: the $i\lambda_n$ terms on the two lines *cancel*. In what's left over, we can take the dot product of both sides with $\vec{u}_m$ for some m to pick out the $\dot{c}_m$ term (recalling the orthogonality above):

$$\dot{c}_m = -\sum_n c_n\vec{u}_m\cdot\dot{\vec{u}}_n e^{i\int^t(\lambda_n-\lambda_m)}. \qquad (3)$$

Thus, we have arrived a set of ordinary differential equations for the c_n: the **coupled-mode equations**, which tell us how one "eigenmode" n couples to other "eigenmodes" m as the system evolves. These equations are *much* nicer than our original equations, however, because we can evaluate them *approximately* in the case where A is slowly varying, in which case $\dot{\vec{u}}_n$ is *small* and c_n is *nearly constant.*

Before we continue, let's make one simplification. It turns out that the $n = m$ term in eq. (3) is zero. The reason is simple: $\vec{u}_m\cdot\dot{\vec{u}}_m = \frac{1}{2}\frac{d}{dt}(\vec{u}_m\cdot\vec{u}_m)$ by the product rule, but $\vec{u}_m\cdot\vec{u}_m = 1$ is a constant by our choice of normalization.[2] So, we chan change eq. (3) to use $\sum_{n\neq m}$.

There is another simplification that we could make: it turns out that we could write $\dot{\vec{u}}_n$ in terms of $\vec{u}_n$ and dA/dt, but that's not necessary for our analysis so we skip it here.

3.3 Adiabatic theorem

Suppose that we start out with some initial condition $c_n(0)$ and consider $\Delta c_n(t) = c_n(t) - c_n(0)$. We would like to show that, as A changes more and more slowly, $\Delta c_n \to 0$. To quantify how slowly A changes, let's write A as a function $A(t/T)$ for some timescale T — the larger T is, the more slowly A changes. Furthermore, we'll change variables from t to $\tau = t/T$, so that $\frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}$. Eq. (3) now becomes:

$$\frac{d(\Delta c_m)}{d\tau} = -\sum_{n\neq m}[c_n(0) + \Delta c_n(\tau)]\vec{u}_m\cdot\frac{d\vec{u}_n}{d\tau}e^{iT\int^\tau(\lambda_n-\lambda_m)d\tau'}. \qquad (4)$$

Notice that T now only appears in the exponent.

[2] More generally, if we had complex-Hermitian A, $\vec{u}$ would not be real and our dot product would be of the form $\vec{u}_m^*\cdot\vec{u}_m = 1$. In this case, $\vec{u}_m^*\cdot\dot{\vec{u}}_m$ is purely imaginary, and this imaginary part gives us something called "Berry's phase" [1,2].

Now, let's *assume* that Δc_n is small for all n, and expand the solution in powers of Δc_n. We'll calculate Δc_m to lowest order, and show *a posteriori* that it indeed goes to zero for large T, thus justifying our power expansion. (That is, if the lowest-order term goes to zero for large T, the higher-order terms will go to zero even faster.)

To lowest order in Δc_n, we just solve eq. (4) where $\Delta c_n = 0$ on the right-hand side. In this case, the right-hand side is completly known, and the zeroth order solution $\Delta c_m^{(0)}(\tau_0)$ at some time τ_0 is just an integral:

$$\Delta c_m^{(0)}(\tau_0) = -\sum_{n \neq m} c_n(0) \int_0^{\tau_0} \vec{u}_m \cdot \frac{d\vec{u}_n}{d\tau} e^{iT \int^{\tau} (\lambda_n - \lambda_m) d\tau'} d\tau.$$

If we wanted, we could then plug this solution back into eq. (4) and integrate again to get the first-order correction, and repeat ad nauseam. The key thing is to show that $\Delta c_m^{(0)}$ is small, so that the series expansion converges, and in particular to show that $\lim_{T\to\infty} \Delta c_m^{(0)} = 0$.

Let's look at each one of the integrals that we have to do in the above $\sum_{n\neq m}$:

$$F(T) = \int_0^{\tau_0} \vec{u}_m \cdot \frac{d\vec{u}_n}{d\tau} e^{iT \int^{\tau} (\lambda_n - \lambda_m) d\tau'} d\tau$$

This may look like a mess, but it really has remarkably simple properties, that we can reveal just by a change of variables. The key thing is the observation that we made back with the coupled pendula: *the eigenvalues (almost) never cross*, because if they "tried" to there is almost always an anti-crossing at that point. This means that $\lambda_n - \lambda_m$ is *always the same sign* and nonzero, and hence the function $y(\tau) = \int_0^{\tau} (\lambda_n - \lambda_m) d\tau'$ is *monotonically increasing or decreasing.* This lets us do a change of variables to $\tau(y)$. In this change of variables, the above integral takes on the form:

$$F(T) = \int_0^{y_0} f(y) e^{iTy} dy, \tag{5}$$

where $f(y) = \vec{u}_m \cdot \frac{d\vec{u}_n}{d\tau} / \frac{dy}{d\tau}$ is just some function of y depending only on how A (and thus u_n) is changing.

But eq. (5) is just a Fourier transform of $f(y)$ (restricted to the interval $[0, y_0]$), where T takes the role of the "frequency!" Or, if we restrict ourselves to $T = 2\pi\ell/y_0$ for integers ℓ, it is the ℓ-th coefficient in a Fourier series expansion of $f(y)$. Either way, we know that the Fourier coefficients have to go to zero eventually for large frequencies—the Fourier transform/series converges whenever $\int |f(y)|^2$ is finite. This means that $\lim_{T\to 0} F(T) = 0$, and hence $\Delta c_m^{(0)} \to 0$ for large T (slow transitions). Q.E.D.

3.4 Smoothness and adiabaticity

Because we reduced the problem to a simple Fourier transform, we can say a lot more about the problem because we know a lot about the properties of Fourier transforms and Fourier series. In particular, we can say *how fast* $\Delta c_n \to 0$ as $T \to \infty$!

The basic fact is that the rate at which a Fourier transform or a Fourier series goes to zero for high frequencies depends on the *smoothness* of the function being transformed. You probably learned in 18.03 that the Fourier series coefficients of a square-wave, which is discontinuous, go as $\sim \frac{1}{\ell}$ for the ℓ-th term. And for a triangle wave, which is continuous with discontinuous slope, the coefficients go as $\sim \frac{1}{\ell^2}$. The same holds true in general: if k derivatives of $f(y)$ are continuous, the Fourier transform $F(T)$ goes asymptotically as $\frac{1}{T^{k+1}}$. And if $f(y)$ is *infinitely* differentiable, $F(T)$ generally decreases *exponentially* with some power of T.

So, to approach the adiabatic limit, we want not only to change A as *slowly* as possible, but we also want to change it as *smoothly* as possible.

4 Further reading

The adiabatic theorem has been most commonly derived in the context of Schrodinger's equation in quantum mechanics [2, 3], where it has been extensively studied (including cases where eigenvalues cross, or where there are a continuum of eigenvalues) [4–8], but coupled-mode equations and adiabatic theorems of the same form appear in many fields, e.g. in electromagnetism [9, 10].

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