18.336 Mid-term Solutions

Problem 1 (30 points): Velocity

(a) Plugging in $e^{i(\theta m - \phi n)}$, and employing the usual trig identites, we find:

$$\sin(\phi/2) = \pm a\lambda \sin(\theta/2)$$

This gives a group velocity $v_g = d\phi/d\theta/\lambda$ of:

$$\frac{v_g}{a} = \frac{\cos(\theta/2)}{\sqrt{1-(a\lambda)^2 \sin^2(\theta/2)}}.$$

This is actually exactly the same as the group velocity for the leap-frog method from pset 3. It clearly is 1 for $\theta = 0$ and, for $|a\lambda| < 1$, decreases to 0 for $\beta\Delta x = \theta = \pi$. If we've forgotten what the plot looks like, with a little more work we can show that it decreases monotonically. If we take its derivative, the numerator of $dv_g/d\theta$ looks like (recall that $(u/v^{1/2})' = (2u'v - uv')/2v^{3/2}$ by the product rule):

$$\begin{aligned} &-\sin(\theta/2)\cdot[1-(a\lambda)^2\sin^2(\theta/2)] \\ &\quad+\cos(\theta/2)\cdot(a\lambda)^2\sin(\theta/2)\cos(\theta/2) \\ &\qquad= -\sin(\theta/2)\cdot[1-(a\lambda)^2] \le 0 \end{aligned}$$

so the curve must be monotonically decreasing, with zero slope at $\theta = 0$ and negative slope elsewhere. Furthermore, we can show that it must be concave downward, by taking the numerator of $d^2 v_g/d\theta^2$, obtaining:

$$\begin{aligned} &-\cos(\theta/2)\cdot[1-(a\lambda)^2\sin^2(\theta/2)] \\ &\quad-3\sin(\theta/2)\cdot(a\lambda)^2\sin(\theta/2)\cos(\theta/2) \\ &= -\cos(\theta/2)\cdot[1+2(a\lambda)^2\sin^2(\theta/2)] \le 0, \end{aligned}$$

with equality (zero curvature) only at $\theta = \pi$. Putting this together, we conclude that the plot must look something like Fig. 1 (which is $a\lambda = 0.9$).

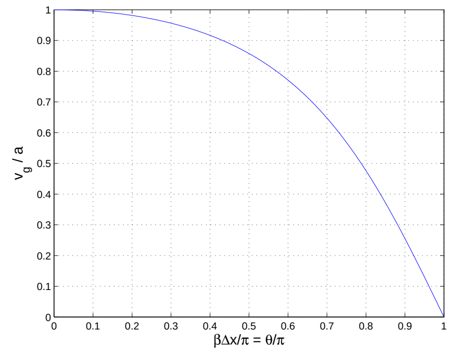

Figure 1: Group velocity v_g/a vs. θ/π for leap-frog wave equation, for $|a\lambda| = 0.9$.

(b) The pulse must be travelling to the **left**. The reason is that we saw from part (a) that faster spatial frequencies (larger θ) travel more slowly than smaller θ. At the start, the pulse has the same oscillation frequency everywhere in its envelope, but eventually the trailing edge of the pulse will have faster oscillations and the leading edge will have slower oscillations (this is called a "chirped pulse" in the signal-processing community). Since the figure showed a pulse with slower oscillations at the left, and there are no boundaries to change the pulse direction, we conclude that we are looking at the left-going pulse.

Problem 2 (30 points): Stability

(a) The simplest thing is to consider a flat solution ($u_x = 0$), which gives $u_t = \sigma u$ and thus has solutions that grow as $e^{\sigma t}$. More generally, we can look at a wave $e^{i(\beta x - \omega t)}$, and plugging this in we see that $-i\omega = -ia\beta + \sigma$, or $\beta = \omega/a - i\sigma/a$, which leads to solutions that grow as $e^{\sigma x/a}$. This is exponentially growing towards the right if a is positive and to the left if a is negative, and is therefore exponentially *growing in the direction of propagation* regardless of the sign of a—this argument can be generalized to $\sigma(x)$ not constant via the coordinate-stretching approach where we get $\exp[\int^x \sigma(x')dx'/a]$ growth. Or, for constant σ, $\omega = a\beta + i\sigma$ so every Fourier component grows in time as $e^{\sigma t}$. Another way to show that all constant-σ solutions are exponentially growing is to write $u(x,t) = f(x - at, t)$, which yields the equation $f_t - af_x = -af_x + \sigma f$, and we see that $f_t = \sigma f$ so that $f(x,t) = f(x,0)e^{\sigma t}$.

(b) Plugging in the usual $u^n_m = g^n e^{i\theta m}$, we find:

$$(g-1) = -ia\lambda g \sin\theta + \sigma\Delta t,$$

or

$$|g| = \frac{1+\sigma\Delta t}{\sqrt{1+(a\lambda)^2\sin^2\theta}} \leq 1+\sigma\Delta t,$$

which is a sufficient condition for stability as we showed in class. So, it is unconditionally stable.

(c) Cal is over-simplifying the definitions. Saying it is "stable" *doesn't* mean that "it doesn't blow up." Rather, it means that it doesn't blow up *in a finite time* as $\Delta t \to 0$. And indeed, the exact solution doesn't blow up in a finite time either (it is *well-posed*, which is a condition for Lax's theorem too). So, there is no contradiction: the solution converges to the exact solution, which blows up as $t \to \infty$.

Problem 3 (30 points): Accuracy

The collocation solution will be $\tilde{c}_k = \tilde{d}_k/k^4$, compared to the exact solution $c_k = d_k/k^4$. As we showed in class, the L_2 error in u is exactly equal to the L_2 error in the Fourier coefficients, which is the sum of two terms: a discretization error in the $\tilde{c}_k$ for $|k| \leq M$, and a truncation error for the missing Fourier coefficients with $|k| > M$.

$$\sqrt{\sum_{|k|\leq M} |\tilde{c}_k - c_k|^2 + \sum_{|k|>M} |c_k|^2}.$$

The first sum is of terms $|\tilde{c}_k - c_k|^2 = |\tilde{d}_k - d_k|^2/k^8 = O(\frac{1}{M^{2\ell}})\frac{1}{k^8}$. A sum of $1/k^8$ is a convergent series and is just sum number for large M, so we get $O(\frac{1}{M^{2\ell}})$. The second sum is of terms $|c_k|^2 = |d_k|^2/k^8 = O(\frac{1}{|k|^{2\ell+8}})$; when summed, this gives $O(\frac{1}{M^{2\ell+7}})$ by the usual series bound. Obviously, the first term dominates, and so the overall error is $O(\frac{1}{M^{\ell}})$.

Extra credit (5 points): Poisson

(a) A singular matrix means that the solution is not unique, and indeed that is the case here: we can add any constant to ϕ and it still solves the same equation.

(b) To fix this, we just need to specify a specific solution ϕ that we want out of all the infinite possibilities. The simplest way is to just set the value of ϕ at a point, e.g. require $\phi(0) = 0$. By periodicity, this means that $\phi(2\pi) = 0$ too, and thus we have just changed periodic boundary conditions into Dirichlet boundary conditions (at least in 1d)!

Note that the problem does *not* have anything to do with whether $\int \rho dx = 0$...the solution ϕ is still not unique regardless. For our spectral solution in class, we set the zeroth Fourier coefficient of ϕ to zero, which is equivalent to requiring $\int \phi dx = 0$, another possible choice (albeit a slightly harder choice to enforce in a finite-difference scheme). If we take our formulation from (b) above and plug in a ρ that does *not* have zero integral, we will still get a solution, but it will be one for which ϕ has a discontinuous slope (since $\int \rho dx = \phi'(2\pi) - \phi'(0)$ by integrating the differential equation).