18.440 Practice Midterm Two: 50 minutes, 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (20 points) Let X and Y be independent Poisson random variables with parameter 1. Compute the following. (Give a correct formula involving sums — does not need to be in closed form.)

   (a) The probability mass function for X given that $X + Y = 5$.

   (b) The conditional expectation of Y^2 given that $X = 2Y$.

   (c) The probability mass function for $X - 2Y$ given that $X > 2Y$.

   (d) The probability that $X = Y$.

2. (15 points) Solve the following:

   (a) Let X be a normal random variable with parameters (μ, σ^2) and Y an exponential random variable with parameter λ. Write down the probability density function for $X + Y$.

   (b) Compute the moment generating function and characteristic function for the uniform random variable on $[0, 5]$.

   (c) Let $X_1, \ldots, X_n$ be independent exponential random variables of parameter λ. Let Y be the second largest of the X_i. Compute the mean and variance of Y.

3. (10 points)

   (a) Suppose that the pair (X, Y) is uniformly distributed on the disc $x^2 + y^2 \leq 1$. Find f_X, f_Y.

   (b) Find also $f_{X^2+Y^2}$ and $f_{\max(x,y)}$.

   (c) Find the conditional probability density for X given $Y = y$ for $y \in [-1, 1]$.

   (d) Compute $\mathbb{E}[X^2 + Y^2]$.

4. (10 points) Suppose that X_i are independent random variables which take the values 2 and .5 each with probability 1/2. Let $X = \prod_{i=1}^{n} X_i$.

(a) Compute $\mathbb{E}X$.

(b) Estimate the $P\{X > 1000\}$ if $n = 100$.

5. (20 points) Suppose X is an exponential random variable with parameter $\lambda_1 = 1$, Y is an exponential random variable with $\lambda_2 = 2$, and Z is an exponential random variable with parameter $\lambda_3 = 3$. Assume X and Y and Z are independent and compute the following:

(a) The probability density function f_{X+Y}

(b) $\mathrm{Cov}(XY, X+Y)$

(c) $\mathbb{E}[\max\{X, Y, Z\}]$

(d) $\mathrm{Var}[\min\{X, Y, Z\}]$

(e) The correlation coefficient $\rho(\min\{X, Y, Z\}, \max\{X, Y, Z\})$.

6. (10 points) Suppose $X_1, \ldots, X_{10}$ be independent standard normal random variables. For each $i \in \{2, 3, \ldots, 9\}$ we say that i is a local maximum if $X_i > X_{i+1}$ and $X_i > X_{i-1}$. Let N be the number of local maxima. Compute

(a) The expectation of N.

(b) The variance of N.

(c) The correlation coefficient $\rho(N, X_1)$.

7. (15 points) Give the name and an explicit formula for the density or mass function of $\sum_{i=1}^{n} X_i$ when the X_i are

(a) Independent normal with parameter μ, σ^2.

(b) Independent exponential with parameter λ.

(c) Independent geometric with parameter p.

(d) Independent Poisson with parameter λ

(e) Independent Bernoulli with parameter p.