18.440 Midterm 1, Fall 2009: 50 minutes, 100 points

1. **Carefully and clearly show your work on each problem (without writing anything that is technically not true).**

2. **No calculators, books, or notes may be used.**

1. (20 points) Evaluate the following explicitly:

(a) $\sum_{i=0}^{10} 9^i \frac{10!}{i!(10-i)!} = (1+9)^{10} = 10^{10}$ *by the Binomial theorem.*

(b) $\sum_{i=5}^{9} 2^{-9} \frac{9!}{i!(9-i)!} = \frac{1}{2}(\frac{1}{2}+\frac{1}{2})^9 = \frac{1}{2}$ *by the Binomial theorem and* $\binom{9}{i} = \binom{9}{9-i}$.

2. (20 points) Six people, labeled $\{1,2,3,4,5,6\}$, each own one hat. They throw their hats into a box, and each person removes and holds onto one of the six hats (with all of the 6! hat orderings being equally likely). Let M be the number of people who get their own hat. A pair of people is called a *swapped pair* if each one has the other person's hat. For example, if person 1 has person 4's hat and person 4 has person 1's hat, then 1 and 4 constitute a single swapped pair. (There can be at most three swapped pairs.) Let S be the number of swapped pairs. Compute the following:

(a) $\mathbb{E}[M]$ *Let* X_i *be* 1 *if* i*th person gets own hat,* 0 *otherwise. Then*

$$\mathbb{E}[M] = \mathbb{E}[\sum_{i=1}^{6} X_i] = \sum_{i=1}^{6} \mathbb{E}X_i = 6\frac{1}{6} = 1$$

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(b) $\mathbb{E}[S]$ *Let* $X_{i,j}$ *be* 1 *if* i *and* j *are a swapped pair, zero otherwise. Then*

$$\mathbb{E}[S] = \mathbb{E}[\sum_{1 \le i < j \le 6} X_i] = \binom{6}{2} \mathbb{E}X_{1,2} = \binom{6}{2} \frac{1}{6}\frac{1}{5} = \frac{1}{2}.$$

(c)

$$\mathrm{Var}[M] = \sum_{i=1}^{6}\sum_{j=1}^{6} \mathrm{Cov}[X_i, X_j] = 6\mathrm{Cov}[X_1, X_1] + 30\mathrm{Cov}[X_1, X_2] = 6\frac{5}{36} + 30\frac{1}{180} = 1.$$

The first equality is a basic property of variance. The second equality comes from expanding the sum and collapsing symmetrically equivalent terms (e.g., $\mathrm{Cov}(X_1, X_1) = \mathrm{Cov}(X_2, X_2)$). We have $\mathrm{Cov}(X_1, X_1) = \mathrm{Var}(X_1) = \frac{5}{36}$. Also, $\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2] = \frac{1}{30} - \frac{1}{36} = \frac{1}{180}$.

3. (20 points) Let D_1 and D_2 be the outcomes (in $\{1, 2, 3, 4, 5, 6\}$) of two independent fair die rolls. Let Y_i be the random variable which is equal to 1 if $D_1 = i$ and 0 otherwise. Compute the following:

(a) $\mathbb{E}[D_1^2 D_2^2] = E[D_1^2]\mathbb{E}[D_2^2] = \frac{(91)^2}{36} = \frac{8281}{6}$ *by independence and* $\mathbb{E}D_i^2 = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$.

(b) $\mathrm{Var}[D_1 - D_2] = \mathrm{Var}[D_1] + \mathrm{Var}[D_2] - 2\mathrm{Cov}[D_1, D_2] = 2\mathrm{Var}[D_1] = \frac{35}{6}$.

(c) $\mathrm{Cov}(Y_1 + Y_2 + Y_3, Y_5 + Y_6) = 6\mathrm{Cov}[Y_1, Y_5] = -\frac{1}{6}$ *by bilinearity of covariance.*

(d) $\mathrm{Var}[\sum_{i=1}^{6} Y_i]$. *The sum is constant, so the variance is zero.*

4. (20 points) Let X_1, X_2, and X_3 be independent Poissonian random variables with parameters $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, respectively. Compute the probabilities of the following events:

(a) The largest of X_1, X_2, and X_3 is at least 1. *One minus the probability they are all zero is* $1 - e^{-\lambda_1}e^{-\lambda_2}e^{-\lambda_3} = 1 - e^{-6}$.

(b) The largest of X_1, X_2, and X_3 is exactly 1. *The probability that each* X_i *is* 1 *or* 0 *is*

$$\prod_{i=1}^{3}(e^{-\lambda_i} + \lambda_i e^{-\lambda_i}) = e^{-\lambda_1\lambda_2\lambda_3}\prod_{i=1}^{3}(1 + \lambda_i) = 24e^{-6}.$$

Subtracting the probability that the X_i *are all zero yields* $23e^{-6}$.

5. (20 points) There are ten children: five attend school A, three attend school B, and two attend school C. Suppose that a pair of two children is chosen uniformly at random from the set of all possible pairs of children. Let a be the number of students in the random pair that attend school A and let b be the number in the pair that attend school B. (So both a and b take values in the set $\{0, 1, 2\}$.)

(a) Compute $\mathbb{E}[ab]$. *Product will be non-zero (and equal to* 1*) only if* $a = b = 1$. *The expectation is the probability of this:* $\frac{15}{\binom{10}{2}} = \frac{1}{3}$.

(b) Given that the two children in this pair attend the same school, what is the conditional probability that they both attend school A?

$$\frac{\binom{5}{2}}{\binom{5}{2} + \binom{3}{2} + \binom{2}{2}} = \frac{10}{14} = \frac{5}{7}.$$