18.440 Midterm 2, Spring 2014: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

1. (20 points) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 3/4 and tails with probability 1/4. Let X_i be 1 if the ith toss comes up heads and 0 otherwise.

(a) Compute $E[X_1]$ and $\mathrm{Var}[X_1]$.

(b) Compute $\mathrm{Var}[X_1 + 2X_2 + 3X_3 + 4X_4]$.

(c) Let Y be the number of heads in the first 4800 tosses of the biased coin, i.e.,

$$Y = \sum_{i=1}^{4800} X_i.$$

Use a normal random variable to approximate the probability that $Y \geq 3690$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer.

2. (10 points) Suppose that a fair six-sided die is rolled just once. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the number that comes up. Let Y be 1 if the number on the die is in $\{1, 2, 3\}$ and 0 otherwise.

(a) What is the conditional expectation of X given that $Y = 0$?

(b) What is the conditional variance of Y given that $X = 2$?

3. (20 points) Let X be a uniform random variable on the set $\{-2, -1, 0, 1, 2\}$. That is, X takes each of these values with probability $1/5$. Let Y be an independent random variable with the same law as X, and write $Z = X + Y$.

(a) What is the moment generating function $M_X(t)$?

(b) What is the moment generating function $M_Z(t)$?

4. (20 points) Two soccer teams, the Lions and the Tigers, begin an infinite soccer games starting at time zero. Suppose that the times at which the Lions score a goal form a Poisson point process with rate $\lambda_L = 2/\text{hour}$. Suppose that the times at which the Tigers score a goal form a Poisson point process with rate $\lambda_T = 3/\text{hour}$.

(a) Write down the probability density function for the amount of time until the first goal by the Lions.

(c) Write down the probability density function for the amount of time until the first goal by *either* team is scored.

(c) Compute the probability that the Tigers score no goals at all during the first two hours.

(d) Compute the probability that the Lions score exactly three goals during the first hour.

5. (20 points) Let X and Y be independent uniform random variables on $[0, 1]$. Write $Z = X + Y$. Write $W = \max\{X, Y\}$.

(a) Compute and draw a graph of the probability density function f_Z.

(b) Compute and draw a graph of the cumulative distribution function F_W.

(c) Compute the variances $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Var}(Z)$.

(d) Compute the covariance $\mathrm{Cov}(Y, Z)$ and the correlation coefficient $\rho(Y, Z)$.

6. (10 points) Let X and Y be independent exponential random variables, each with with parameter $\lambda = 5$.

(a) Let f be the joint probability density function for the pair (X, Y). Write an explicit formula for f.

(b) Compute $E[X^2 Y]$.