# 18.600: Lecture 5

# Problems with all outcomes equally likely, including a famous hat problem

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# Outline

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# Equal likelihood

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- What is $P(A)$ for a general set $A \subset S$?
- Answer: $|A|/|S|$, where $|A|$ is the number of elements in $A$.

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Hat problem

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- $1 - \prod_{i=0}^{22} \frac{365-i}{365}$

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# Recall the inclusion-exclusion identity

- $$P(\cup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \dots$$
  $$+ (-1)^{(r+1)} \sum_{i_1<i_2<\dots<i_r} P(E_{i_1}E_{i_2}\dots E_{i_r})$$
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- The notation $\sum_{i_1<i_2<i_r}$ means a sum over all of the $\binom{n}{r}$ subsets of size $r$ of the set $\{1, 2, \dots, n\}$.

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- $1 - P(\cup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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