18.600: Lecture 36

Risk Neutral Probability and Black-Scholes

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Outline

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Black-Scholes

Call quotes and risk neutral probability

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- Will spend time giving financial *interpretations* of the math.
- Can interpret this lecture as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, risk neutral probability, etc.)

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- In particular, the risk neutral expectation of tomorrow's (interest discounted) stock price is today's stock price.
- Implies **fundamental theorem of asset pricing**, which says discounted price $\frac{X(n)}{A(n)}$ (where A is a risk-free asset) is a martingale with respected to **risk neutral probability**.

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- **General Black-Scholes conclusion:** If g is any function then the price of a contract that pays $g(X)$ at time T is

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

  where N is normal with mean μ and variance $T\sigma^2$.

Black-Scholes example: European call option

- A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

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- Write this as

$$e^{-rT}E_{RN}[\max\{0, e^N - K\}] = e^{-rT}E_{RN}[(e^N - K)1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx.$$

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- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function Φ.
- Price of European call is $\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$ where $d_1 = \frac{\ln(\frac{X_0}{K})+(r+\frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(\frac{X_0}{K})+(r-\frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$.

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Determining risk neutral probability from call quotes

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- Differentiating under the integral, we find that

$$e^{rT} C'(K) = \int f(x)(-1_{x>K})dx = -P_{RN}\{X > K\} = F_X(K) - 1,$$

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- We can look up $C(K)$ for a given stock symbol (say GOOG) and expiration time T at cboe.com and work out approximately what F_X and hence f_X must be.
- Try this out for near term option (so e^{rT} is essentially one).

Perspective: implied volatility

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- "Implied volatility" is the value of σ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
- If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices. In practice, when the implied volatility is viewed as a function of strike price (sometimes called the "volatility smile"), it is not constant.

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- **Fixes:** variable volatility, random interest rates, Lévy jumps....