18.600: Lecture 32

Markov Chains

Scott Sheffield

MIT

Outline

Outline

Markov chains

Examples

Ergodicity and stationarity

Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.

Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.
- Interpret X_n as state of the system at time n.

Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.
- Interpret X_n as state of the system at time n.
- Sequence is called a **Markov chain** if we have a fixed collection of numbers P_{ij} (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state i, there is probability P_{ij} that system will next be in state j.

Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.
- Interpret X_n as state of the system at time n.
- Sequence is called a **Markov chain** if we have a fixed collection of numbers P_{ij} (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state i, there is probability P_{ij} that system will next be in state j.
- Precisely,
  $P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}$.

Markov chains

- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.
- Interpret X_n as state of the system at time n.
- Sequence is called a **Markov chain** if we have a fixed collection of numbers P_{ij} (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state i, there is probability P_{ij} that system will next be in state j.
- Precisely, $P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}$.
- Kind of an "almost memoryless" property. Probability distribution for next state depends only on the current state (and not on the rest of the state history).

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny.

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny.
- If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny.
- If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- In this climate, sun tends to last longer than rain.

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny.
- If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- In this climate, sun tends to last longer than rain.
- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny.
- If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- In this climate, sun tends to last longer than rain.
- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?
- Given that it is sunny today, how many days to I expect to have to wait to see a rainy day?

Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny.
- If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- In this climate, sun tends to last longer than rain.
- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?
- Given that it is sunny today, how many days to I expect to have to wait to see a rainy day?
- Over the long haul, what fraction of days are sunny?

Matrix representation

- To describe a Markov chain, we need to define P_{ij} for any $i, j \in \{0, 1, \ldots, M\}$.

Matrix representation

- To describe a Markov chain, we need to define P_{ij} for any $i, j \in \{0, 1, \ldots, M\}$.
- It is convenient to represent the collection of transition probabilities P_{ij} as a matrix:

$$A = \begin{pmatrix} P_{00} & P_{01} & \ldots & P_{0M} \\ P_{10} & P_{11} & \ldots & P_{1M} \\ . & & & \\ . & & & \\ . & & & \\ P_{M0} & P_{M1} & \ldots & P_{MM} \end{pmatrix}$$

Matrix representation

- To describe a Markov chain, we need to define P_{ij} for any $i, j \in \{0, 1, \ldots, M\}$.
- It is convenient to represent the collection of transition probabilities P_{ij} as a matrix:

$$A = \begin{pmatrix} P_{00} & P_{01} & \ldots & P_{0M} \\ P_{10} & P_{11} & \ldots & P_{1M} \\ . & & & \\ . & & & \\ . & & & \\ P_{M0} & P_{M1} & \ldots & P_{MM} \end{pmatrix}$$

- For this to make sense, we require $P_{ij} \geq 0$ for all i, j and $\sum_{j=0}^{M} P_{ij} = 1$ for each i. That is, the rows sum to one.

Transitions via matrices

- Suppose that p_i is the probability that system is in state i at time zero.

Transitions via matrices

- Suppose that p_i is the probability that system is in state i at time zero.
- What does the following product represent?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} & \dots & P_{0M} \\ P_{10} & P_{11} & \dots & P_{1M} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{M0} & P_{M1} & \dots & P_{MM} \end{pmatrix}$$

Transitions via matrices

- Suppose that p_i is the probability that system is in state i at time zero.
- What does the following product represent?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} & \dots & P_{0M} \\ P_{10} & P_{11} & \dots & P_{1M} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{M0} & P_{M1} & \dots & P_{MM} \end{pmatrix}$$

- Answer: the probability distribution at time one.

Transitions via matrices

- Suppose that p_i is the probability that system is in state i at time zero.
- What does the following product represent?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} & \dots & P_{0M} \\ P_{10} & P_{11} & \dots & P_{1M} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{M0} & P_{M1} & \dots & P_{MM} \end{pmatrix}$$

- Answer: the probability distribution at time one.
- How about the following product?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} A^n$$

Transitions via matrices

- Suppose that p_i is the probability that system is in state i at time zero.
- What does the following product represent?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} & \dots & P_{0M} \\ P_{10} & P_{11} & \dots & P_{1M} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{M0} & P_{M1} & \dots & P_{MM} \end{pmatrix}$$

- Answer: the probability distribution at time one.
- How about the following product?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} A^n$$

- Answer: the probability distribution at time n.

Powers of transition matrix

- We write $P_{ij}^{(n)}$ for the probability to go from state i to state j over n steps.

Powers of transition matrix

- We write $P_{ij}^{(n)}$ for the probability to go from state i to state j over n steps.
- From the matrix point of view

$$\begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & \dots & P_{0M}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} & \dots & P_{1M}^{(n)} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{M0}^{(n)} & P_{M1}^{(n)} & \dots & P_{MM}^{(n)} \end{pmatrix} = \begin{pmatrix} P_{00} & P_{01} & \dots & P_{0M} \\ P_{10} & P_{11} & \dots & P_{1M} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_{M0} & P_{M1} & \dots & P_{MM} \end{pmatrix}^n$$

Powers of transition matrix

- We write $P_{ij}^{(n)}$ for the probability to go from state i to state j over n steps.
- From the matrix point of view

$$\begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & \dots & P_{0M}^{(n)} \\ P_{10}^{(n)} & P_{11}^{(n)} & \dots & P_{1M}^{(n)} \\ . & & & \\ . & & & \\ . & & & \\ P_{M0}^{(n)} & P_{M1}^{(n)} & \dots & P_{MM}^{(n)} \end{pmatrix} = \begin{pmatrix} P_{00} & P_{01} & \dots & P_{0M} \\ P_{10} & P_{11} & \dots & P_{1M} \\ . & & & \\ . & & & \\ . & & & \\ P_{M0} & P_{M1} & \dots & P_{MM} \end{pmatrix}^n$$

- If A is the one-step transition matrix, then A^n is the n-step transition matrix.

Questions

- What does it mean if all of the rows are identical?

Questions

- What does it mean if all of the rows are identical?
- Answer: state sequence X_i consists of i.i.d. random variables.

Questions

- What does it mean if all of the rows are identical?
- Answer: state sequence X_i consists of i.i.d. random variables.
- What if matrix is the identity?

Questions

- What does it mean if all of the rows are identical?
- Answer: state sequence X_i consists of i.i.d. random variables.
- What if matrix is the identity?
- Answer: states never change.

Questions

- What does it mean if all of the rows are identical?
- Answer: state sequence X_i consists of i.i.d. random variables.
- What if matrix is the identity?
- Answer: states never change.
- What if each P_{ij} is either one or zero?

Questions

- What does it mean if all of the rows are identical?
- Answer: state sequence X_i consists of i.i.d. random variables.
- What if matrix is the identity?
- Answer: states never change.
- What if each P_{ij} is either one or zero?
- Answer: state evolution is deterministic.

Outline

Outline

Simple example

- Consider the simple weather example: If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny. If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.

Simple example

- Consider the simple weather example: If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny. If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- Let rainy be state zero, sunny state one, and write the transition matrix by

$$A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$$

Simple example

- Consider the simple weather example: If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny. If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- Let rainy be state zero, sunny state one, and write the transition matrix by

$$A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$$

- Note that

$$A^2 = \begin{pmatrix} .64 & .35 \\ .26 & .74 \end{pmatrix}$$

Simple example

- Consider the simple weather example: If it's rainy one day, there's a .5 chance it will be rainy the next day, a .5 chance it will be sunny. If it's sunny one day, there's a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- Let rainy be state zero, sunny state one, and write the transition matrix by

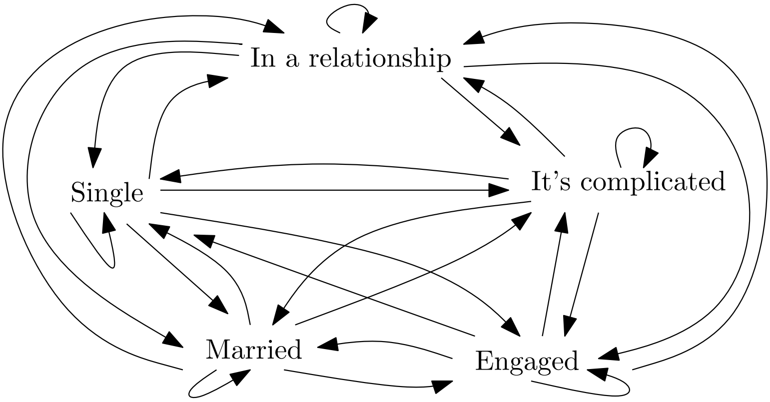
$$A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$$

- Note that

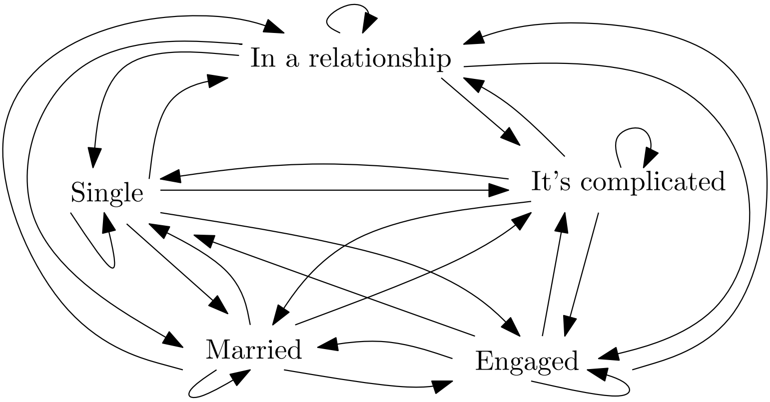
$$A^2 = \begin{pmatrix} .64 & .35 \\ .26 & .74 \end{pmatrix}$$

- Can compute $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix}$

Does relationship status have the Markov property?

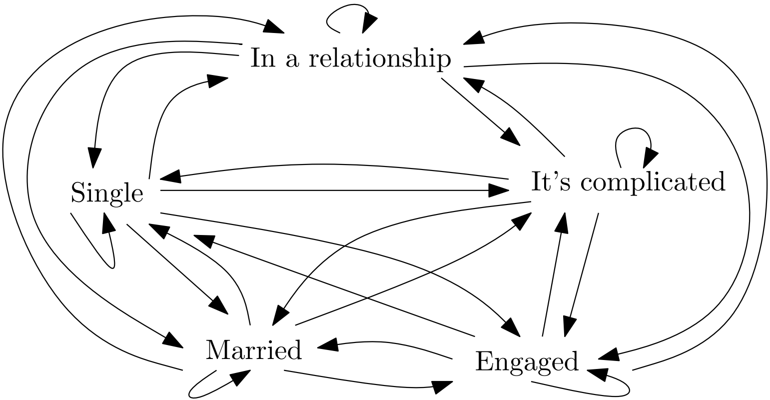


Does relationship status have the Markov property?



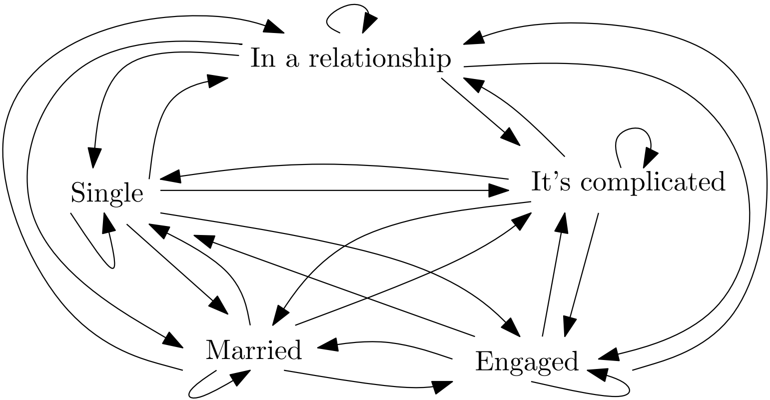
- Can we assign a probability to each arrow?

Does relationship status have the Markov property?



- Can we assign a probability to each arrow?
- Markov model implies time spent in any state (e.g., a marriage) before leaving is a geometric random variable.

Does relationship status have the Markov property?



- Can we assign a probability to each arrow?
- Markov model implies time spent in any state (e.g., a marriage) before leaving is a geometric random variable.
- Not true... Can we make a better model with more states?

Outline

Outline

Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.

Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.
- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the π_j are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.

Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.
- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the π_j are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.
- This means that the row vector

$$\pi = \begin{pmatrix} \pi_0 & \pi_1 & \dots & \pi_M \end{pmatrix}$$

is a left eigenvector of A with eigenvalue 1, i.e., $\pi A = \pi$.

Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.
- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the π_j are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.
- This means that the row vector

$$\pi = \begin{pmatrix} \pi_0 & \pi_1 & \dots & \pi_M \end{pmatrix}$$

  is a left eigenvector of A with eigenvalue 1, i.e., $\pi A = \pi$.
- We call π the *stationary distribution* of the Markov chain.

Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.
- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the π_j are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.
- This means that the row vector

$$\pi = \begin{pmatrix} \pi_0 & \pi_1 & \ldots & \pi_M \end{pmatrix}$$

  is a left eigenvector of A with eigenvalue 1, i.e., $\pi A = \pi$.
- We call π the *stationary distribution* of the Markov chain.
- One can solve the system of linear equations $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ to compute the values π_j. Equivalent to considering A fixed and solving $\pi A = \pi$. Or solving $(A - I)\pi = 0$. This determines π up to a multiplicative constant, and fact that $\sum \pi_j = 1$ determines the constant.

Simple example

- If $A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$, then we know

$$\pi A = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} = \pi.$$

Simple example

- If $A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$, then we know

$$\pi A = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} = \pi.$$

- This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_0 + \pi_1 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$.

Simple example

- If $A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$, then we know

$$\pi A = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} = \pi.$$

- This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_0 + \pi_1 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$.
- Indeed,

$$\pi A = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} = \pi.$$

Simple example

- If $A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$, then we know

$$\pi A = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} = \pi.$$

- This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_0 + \pi_1 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$.
- Indeed,

$$\pi A = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} = \pi.$$

- Recall that
$A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \\ 2/7 & 5/7 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}$