

18.600: Lecture 23

Conditional probability, order statistics, expectations of sums

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Outline

Conditional probability densities

Order statistics

Expectations of sums

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- ▶ Usually...

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- ▶ Then set $f_{X|Y=y}(a) = F'_{X|Y=y}(a)$. Consistent with definition from previous slide.

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- ▶ Conditioning on (X, Y) belonging to a $\theta \in (-\epsilon, \epsilon)$ wedge is very different from conditioning on (X, Y) belonging to a $Y \in (-\epsilon, \epsilon)$ strip.

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▶ Answer: $F_X(a) = \begin{cases} 0 & a < 0 \\ a^n & a \in [0, 1] \\ 1 & a > 1 \end{cases}$. And

$$f_X(a) = F'_X(a) = na^{n-1}.$$

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- ▶ Answer is beta distribution with parameters $(a, b) = (8, 3)$.
- ▶ Up to a constant, $f(x) = x^7(1 - x)^2$.
- ▶ General beta (a, b) expectation is $a/(a + b) = 8/11$. Mode is $\frac{(a-1)}{(a-1)+(b-1)} = 2/9$.

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- ▶ Choose Y uniformly on $[0, 1]$ and note that $g(Y)$ has the same probability distribution as X .
- ▶ So $E[X] = E[g(Y)] = \int_0^1 g(y) dy$, which is indeed the area under the graph of $1 - F_X$.