NAME: ______________________________

Spring 2017 18.600 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$.

(a) Compute $E[3X^{13}]$

(b) Compute the conditional probability $P[X > 10|X > 5]$.

(c) Let $Y = X^2$. Compute the cumulative distribution function F_Y.

2. (10 points) Sally is a pleasant texting companion. Each of her texts consists of a string of five emoji, each chosen independently from the same probability distribution. Precisely, if $X = (X_1, X_2, X_3, X_4, X_5)$ represents one of her texts, then each X_i is independently equal to

Person Shugging with probability 1/4,
Face with Tears of Joy with probability 1/4,
Face Blowing a Kiss with probability 1/8,
Face with Rolling Eyes with probability 1/8,
Love Heart with probability 1/16,
Thinking Face with probability 1/16,
See-No-Evil Monkey with probability 1/16,
Hula-hooping Statue of Liberty with probability 1/16.

(a) Compute the entropy of one of Sally's emoji. That is, compute $H(X_1)$.

(b) Compute the entropy of an entire text, i.e., compute $H(X) = H(X_1, X_2, X_3, X_4, X_5)$.

(c) Suppose you try to figure out the value of X_1 by asking a series of yes or no questions to someone who knows the value. Assume you use the strategy that minimizes the expected number of questions you need. How many questions do you expect to ask?

3. Let X_1, X_2, X_3 be i.i.d. random variables, each with probability density function $\frac{1}{\pi(1+x^2)}$.

(a) Assume that a and b are fixed positive constants and write $Y = aX_1 + b$. Compute the probability density function f_Y.

(b) Compute the probability that $X_1 \in [-1, 1]$. Give an explicit number. (Recall spinning flashlight story.)

(c) Compute the probability density function for $Z = (X_1 + X_2 + X_3)/3$.

(d) Compute the probability density function for $W = (X_1 + 2X_2)/3$. [Hint: use (c).]

4. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables each of which is equal to 5 with probability 1/2 and -5 with probability 1/2. Write $Y_0 = 85$ and $Y_n = Y_0 + \sum_{i=1}^n X_i$ for $n > 0$.

(a) Find the probability that the sequence $Y_0, Y_1, Y_2, \ldots$ reaches 50 before it reaches 100.

(b) Let T be the smallest $n \geq 0$ for which Y_n is an integer multiple of 500. Compute $E[Y_T]$.

(c) Compute $E[3Y_{10} - 4Y_8 | Y_5]$ in terms of Y_5. In other words, recall that the conditional expectation $E[3Y_{10} - 4Y_8 | Y_5]$ can be understood as a random variable and express this random variable as a simple function of Y_5.

5. (10 points) Let X be a normal random variable with mean zero and variance one.

(a) Compute the moment generating function for X using the "complete the square" trick. (Show your work.)

(b) Fix positive constants a, b, and c and compute $E[aX^2 + bX + c]$ in terms of a, b, and c.

6. Ten people are taking a class together.

(a) Let K be the number of ways to divide the ten people into five groups, two people per group, so that each person in the class has a partner. Find K.

(b) Assume that one of these K ways is chosen uniformly at random on one day, and another is chosen uniformly at random (independently) on a subsequent day. Let N be the number of people who have the same partner on both days. What is $E[N]$?

(c) Compute $P(N = 6)$. You can use K in your answer if you did not compute an explicit value for K.

7. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the semi-circle $\{(x, y) : x^2 + y^2 \leq 1, x \geq 0$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$.

(b) Compute the conditional expectation $E[X|Y]$ as a function of Y.

(c) Express $E[X^3 \cos(Y)]$ as a double integral. You do not have to compute the integral.

8. (10 points) Let X, Y, Z be i.i.d. exponential random variables, each with parameter $\lambda = 1$. Compute the probability density functions for the following random variables.

(a) $2X + 1$

(b) $X + Y + Z$.

(c) $\min\{X, Y, Z\}$.

9. (10 points) A certain athletically challenged monkey has difficulty climbing a ladder. The ladder has seven rungs. At each given second, the monkey can be on any of the seven rungs of the ladder — or on the "zeroth" rung, meaning on the ground). If $0 \leq m \leq 6$ and the monkey is on the mth rung of the ladder at a given second, then at the next second the monkey will be at rung $m+1$ with probability $1/2$ and back at rung 0 with probability $1/2$. If the monkey is at level 7, then with probability 1 the monkey will go back to 0 at the next step.

(a) Give the 8 by 8 matrix describing the monkey's transition probabilities. (You don't have to write all 64 entries. Just write the entries that are non-zero.)

(b) If the monkey is at state zero at a given time, which is the probability that the monkey is on the 7th rung of the ladder seven seconds later. (That is, what is the probability the monkey goes straight from the bottom to the top without falling once?)

(c) Compute $(\pi_0, \pi_1, \ldots, \pi_7)$ where π_i is the fraction of the time the monkey spends in state i, over the long term. (Hint: first see if you can express each π_j as an integer multiple of π_{j+1} for $j < 7$. Then see if you can express each π_j as a multiple of π_7.)

10. (10 points) Let $X_1, X_2, \dots, X_{300}$ be independent random real numbers, each chosen uniformly on the interval $[0, 100]$. Let $S = \sum_{i=1}^{300} X_i$.

(a) Compute $E[S]$ and $\mathrm{Var}(S)$.

(b) Give a Poisson approximation for the probability that $3 \le X_j < 4$ for exactly 3 of the values $j \in \{1, 2, \dots, 300\}$.

(c) Give an interval $[a, b]$ such that $E[S] = (a + b)/2$ and $P(S \in [a, b]) \approx .95$. (You may use the fact that $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx .95$.)