

NAME: \_\_\_\_\_

**Spring 2016 18.600 Final Exam: 100 points**

**Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.**

1. (10 points) Suppose that  $X_1, X_2, \dots$  is an i.i.d. sequence of normal random variables, each of which has mean 1 and variance 1.

(a) Compute the mean and variance of  $Y = X_1 - 2X_2 + 3X_3 - 4X_4$ .

(b) Compute the probability density function for  $Y$ .

(c) Compute  $P(Y > 0)$  in terms of the function  $\Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ .

2. (10 points) Suppose that  $X_1, X_2, X_3, \dots$  is an infinite sequence of i.i.d. normal random variables, this time with mean 0 and variance 1.

(a) Write  $Y_n = \sum_{i=1}^n X_i$ . Is  $Y_n$  a martingale?

(b) Write  $Z_n = Y_n^2 - n$ . Compute  $E[Z_n - Z_{n-1} | X_1, X_2, \dots, X_{n-1}]$ . You can use the following calculation to help you get started:

$$\begin{aligned} Z_n - Z_{n-1} &= (Y_n^2 - n) - (Y_{n-1}^2 - (n-1)) = Y_n^2 - Y_{n-1}^2 - 1 \\ &= (Y_{n-1} + X_n)^2 - Y_{n-1}^2 - 1 \\ &= 2Y_{n-1}X_n + X_n^2 - 1. \end{aligned}$$

(c) Is  $Z_n$  a martingale?

3. Suppose that  $X_1, X_2, \dots$  is an infinite sequence of i.i.d. Cauchy random variables, so that each has probability density function  $\frac{1}{\pi(1+x^2)}$ . Let  $W$  be an independent normal random variable with mean zero and variance one.

(a) Write  $S = X_1 + X_2 + X_3 + X_4 + X_5$ . Give an explicit formula for the probability density function  $f_S$ .

(b) Write the probability density function for  $A = X_1 + W$ . (Your answer will involve an integral that you do not have to try to evaluate explicitly.)

(c) Compute (as an explicit rational number) the probability that  $X_1 + X_2 > X_3 + 3$ . (Hint: remember the spinning flashlight story.)

4. Suppose that  $n$  people toss their shoes into a bin (two shoes—one left and one right—per person) and then the shoes are randomly shuffled and returned to the  $n$  people, with all ways of returning two shoes to each person being equally likely. Let  $B$  be the number of people who get one left and one right shoe (regardless of whether they match).

(a) Compute the expectation  $E[B]$ .

(b) Compute  $E[B^2]$ .

5. (10 points) Suppose that the pair  $(X, Y)$  is uniformly distributed on the triangle  $\{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$ .

(a) Compute the joint probability density  $f_{X,Y}(x, y)$ .

(b) Compute the conditional expectation  $E[X|Y]$  as a function of  $Y$ .

(c) Compute  $E[X]$ .

6. (10 points) Let  $X$  and  $Y$  be independent uniform random variables on  $[0, 1]$ .

(a) Compute the moment generating function for  $Z = X + Y$ .

(b) Compute the probability density function for  $W = X^3$ .

7. (10 points) Let  $X, Y, Z$  be i.i.d. uniform random variables on  $[0, 5]$ .

(a) Set  $A = \min\{X, Y, Z\}$ . Compute the probability density function  $f_A$ .

(b) Let  $B$  be the second largest of the three values in  $\{X, Y, Z\}$ . Compute  $E[B]$ .

(c) Let  $C = \max\{X, Y, Z\}$ . Compute the probability  $P(1 < C < 4)$ .

8. (10 points)

(a) Let  $X$  be the number of heads that come up when three independent fair coins are tossed. Compute the entropy  $H(X)$ .

(b) Suppose that  $X$  and  $Y$  are two (not necessarily independent or identically distributed) random variables, each of which takes values in the set  $\{1, 2, 3, 4, 5, 6\}$ . Which of the following is *necessarily* true? (Just circle the corresponding letters.)

(i)  $H(X, Y) \geq H(X) + H(Y)$

(ii)  $H(X, Y) = H(X) + H(Y)$

(iii)  $H(X) \leq H(X, Y)$ .

(iv)  $H(X + Y) \leq \log(11)$ .

(v)  $H(X - Y) \geq 0$ .

9. (10 points) A certain country has three distinct types of leaders: liberal, conservative, and insane. Every four years they elect a new leader.

(i) If the current leader is liberal, there is a  $2/3$  chance the next leader will be liberal also, a  $1/4$  chance the next leader will be conservative, and a  $1/12$  chance the next leader will be insane.

(ii) If the currently leader is conservative, there is a  $2/3$  chance the next leader will be conservative also, a  $1/4$  chance the next leader will be liberal, and a  $1/12$  chance the next leader will be insane.

(iii) If the currently leader is insane, then there is a  $1/2$  chance the next leader will be conservative and a  $1/2$  chance the next leader will be liberal. (They never allow two consecutive terms of insanity.)

(a) Use  $L, C, I$  to denote the three states. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by  $L, C$ , and  $I$ .

(b) If the current leader is insane, what is the probability that, after two elections, the leader will be insane again?

(c) Over the long term, what fraction of the time does the country spend under each of the three types of leaders?

10. (10 points) A certain town (with constant climate) has had an average of one house fire per year for the past century. At the beginning of one calendar year, Jill moves to town, and during that year there are 6 house fires. A trial is held to determine whether the increase in fires is due to Jill being a witch. During the trial the judge asks a math expert the following (which you should answer):

- (a) Suppose that house fire times in this town are a Poisson point process with parameter  $\lambda$  equal to 1 per year. Under this assumption, let  $p$  be the probability that there will be exactly 6 house fires during a single given year. What is  $p$ ?
  
  
  
  
  
  
  
  
  
  
- (b) Under the same assumption, what is the probability that, during the course of a century, there will be *at least* 1 calendar year during which there are exactly 6 house fires? Compute your answer in terms of the  $p$  computed in (a).

When Jill first moved to town, Nora thought that there was a  $1/100$  chance that Jill was a witch. She also thought if Jill *wasn't* a witch the number of fires that year would be Poisson with parameter 1, and that if Jill *was* a witch the number would be 6 with probability 1. (Arranging for there to be exactly 6 fires per year is what arsonist witches in this world do.)

- (c) *Given* that the number of observed fires during Jill's first year in town is 6, what is Nora's assessment of the *conditional* probability that Jill is a witch? Compute your answer in terms of the  $p$  computed in (a).