18.600 Midterm 2, Spring 2017: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

1. (10 points) Suppose that X, Y, and Z are independent random variables, each of which is equal to 1 with probability 1/3 and 5 with probability 2/3. Write $W = X + Y + Z$.

(a) Compute the moment generating function M_X.

(b) Compute the moment generating function M_W.

2. (15 points) Let $X_1, X_2, \ldots, X_7$ be independent normal random variables, each with mean 0 and variance 1. Write $Z = \sum_{j=1}^{7} X_j$.

(a) Give the probability density function for Z.

b) Compute the probability that that the random variables are in increasing order, i.e., that $X_1 < X_2 < X_3 < \ldots < X_7$.

3. (15 points) Let X_1, X_2, X_3 be independent exponential random variables with parameter $\lambda = 1$.

(a) Compute the probability density function for $\min\{X_1, X_2, X_3\}$.

(b) Compute the expectation $E[\max\{X_1, X_2, X_3\}]$.

4. (10 points) Suppose that X and Y are independent random variables, each of which has a probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$.

(a) Give the probability density function for $A = (X + Y)/2$.

(b) Give the probability density function for $B = X - Y$.

5. (15 points) Let C be fraction of students in a very large population that will say (when asked) that they prefer curry to pizza. Imagine that you start out knowing nothing about C, so that your Bayesian prior for C is uniform on $[0, 1]$. Then you select a student uniformly from the population and ask what that student prefers. You independently repeat this experiment two more times. You find that two students prefer curry and one prefers pizza.

(a) *Given* what you have learned from these three answers, give a revised probabilty density function f_C for the unknown quantity C (i.e., a Bayesian posterior).

NOTE: If you remember what this means, you may use the fact that a Beta (a, b) random variable has expectation $a/(a+b)$ and density $x^{a-1}(1-x)^{b-1}/B(a,b)$, where $B(a,b) = (a-1)!(b-1)!/(a+b-1)!$.

(b) According to your Bayesian prior, the expected value of C was $1/2$. *Given* what you learned from the three answers, what is your *revised expectation* of the value C?

6. (15 points) Let $X_1, X_2, \ldots, X_{100}$ be independent random variables, each of which is equal to 1 with probability 1/2 and 0 with probability 1/2. Write $S = \sum_{j=1}^{100} X_j$.

(a) Compute $E[S]$ and $\mathrm{Var}[S]$.

(b) Use a normal random variable to approximate $P(S > 60)$. You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer.

7. (20 points) The residents of a certain planet are careful with nuclear weapons, but regional nuclear wars still occur. The times at which these wars occur form a Poisson point process with rate λ equal to one per thousand years. So the expected number of nuclear wars during any 1000 year period is 1.

(a) Compute the probability that there will be exactly three nuclear wars during the next 2000 years.

(b) Let X be the number of millenia until the third nuclear war. (In other words, $1000X$ is the number of years until the third nuclear war.) Give the probability density function for f_X.

(c) Let Y be the number of nuclear wars that will occur during the next 5000 years. Compute the variance of Y.