18.600 Midterm 2, Spring 2016: Solutions

1. (20 points) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 2/3 and tails with probability 1/3. Let X_i be 1 if the ith toss comes up heads and 0 otherwise. Write $S_n = \sum_{i=1}^n X_i$.

(a) Compute $E[X_1]$ and $\mathrm{Var}[X_1]$. **Answer:** 2/3 and 2/9

(b) Compute $E[S_n]$ and $\mathrm{Var}[S_n]$ as functions of n. **Answer:** $(2/3)n$ and $(2/9)n$

(c) Compute the covariance of S_5 and S_{10}. **Answer:**

$$\mathrm{Cov}\Big(\sum_{i=1}^{5} X_i, \sum_{j=1}^{10} X_j\Big) = \sum_{i=1}^{5}\sum_{j=1}^{10} \mathrm{Cov}(X_i, X_j).$$

Terms with $i \neq j$ are zero, so this is $\sum_{i=1}^5 \mathrm{Cov}(X_i, X_i) = 5\mathrm{Var}(X_i) = 10/9$.

(d) Using a normal approximation, estimate the probability that $S_{300} \leq 220$. You may use the function $\Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **Answer:** $\Phi(a)$ where $a = (220 - 200)/\sqrt{300 \times 2/9} = 20\sqrt{9/600} = \sqrt{3600/600} = \sqrt{6}$.

2. (20 points) Suppose that X and Y are the outcomes of independent fair die rolls. So each takes a value in $\{1, 2, 3, 4, 5, 6\}$, with all values being equally likely. Write $Z = X + Y$.

(a) Compute the moment generating function for X. **Answer:**

$$M_X(t) = E[e^{tX}] = \frac{e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}}{6}.$$

(b) Compute the moment generating function for Z. **Answer:**

$$M_Z(t) = [M_X(t)]^2 = \Big(\frac{e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}}{6}\Big)^2.$$

(c) Compute $E[Y|Z]$. (That is, express the random variable $E[Y|Z]$ as a function of the random variable Z.) **Answer:** $E[Y|Z] + E[X|Z] = E[Z|Z] = Z$ and by symmetry $E[Y|Z] = E[X|Z]$ so $E[Y|Z] = Z/2$. **Alternative answer:** check by hand that given $Z = k \in \{2, 3, \ldots, 12\}$, conditional law of Y is uniform on a set of consecutive integers centered at $k/2$.

3. (20 points) Let X be a uniformly random variable on $[0, 5]$. Let Y be an independent uniformly random variable on $[0, 10]$. Write $Z = \min\{X, Y\}$.

(a) Compute the joint density function $f(x, y)$ for X and Y. **Answer:** $f(x, y) = 1/50$ if $(x, y) \in [0, 5] \times [0, 10]$, and 0 otherwise.

(b) Compute $P(Z > 0)$ and $P(Z > 3)$ and $P(Z > 5)$. **Answer:** $P(Z > 0) = 1$ and $P(Z > 3) = P(X > 3)P(Y > 3) = (2/5)(7/10) = 7/25$ and $P(Z > 5) = 0$.

(c) Compute the cumulative distribution function $F_Z(a)$. **Answer:** For $a \in [0,5]$, have $1 - F_Z(a) = P(X > a)P(Y > a) = (\frac{5-a}{5})(\frac{10-a}{10}) = \frac{(5-a)(10-a)}{50}$. So

$$F_Z(a) = \begin{cases} 0 & a < 0 \\ 1 - \frac{(5-a)(10-a)}{50} & a \in [0,5] \\ 1 & a > 5 \end{cases}.$$

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4. (20 points) Alice's Pastry Shop is open from 7:00 a.m. until 10:00 p.m. Throughout those 900 minutes, Alice has an extremely steady business: customers show up according to a Poisson point process with parameter $\lambda = 1$, where time is measured in minutes. (That is, the expected number of customers per minute is one.) Let N be the total number of customers that arrive during the day.

(a) Compute the probability that there are exactly 3 customers during the first three minutes. **Answer:** $e^{-(\lambda t)}(\lambda t)^k/k! = e^{-3}3^3/6 = (9/2)e^{-3}$.

(b) Write a probability density function for the time it takes from the store opening until the arrival of the second customer. (Imagine that customers keep arriving after closing, so that with probability one a second customer comes *eventually*. In other words, don't worry about the 900 minute upper bound for this part of the problem.) **Answer:** this is Γ with parameters $\lambda = 1$ and $\alpha = 2$. Density function is $f(x) = xe^{-x}/\Gamma(2) = xe^{-x}$ for $x \geq 0$ (and zero if $x < 0$).

(c) Compute $E[N]$ and $\text{Var}[N]$. **Answer:** $E[N] = \lambda = 900$ and $\text{Var}[N] = \lambda = 900$.

(d) Compute the probability that the entire day goes by without a single customer. **Answer:** e^{-900}. (Probability λ-exponential random variable exceeds T is $e^{-\lambda T}$.)

5. (10 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent exponential random variables with parameter $\lambda = 1$.

(a) Write $Y = \min\{X_1, X_2, \ldots, X_n\}$. Compute the density function f_Y. **Answer:** This is exponential with rate $\lambda = n$ so $f_Y(y) = \lambda e^{-\lambda y} = ne^{-ny}$.

(b) Compute $E[Y^k]$ as a function of n and k. You may assume that n and k are positive integers. **Answer:** Recall that if X is exponential with parameter 1 we have $E[X^k] = \int_0^\infty x^k e^{-x} dx = k!$. (This is one of the definitions of the factorial.) Note that Y has same law as X/n, so $E[Y^k] = E[(X/n)^k] = E[X^k]/n^k = k!/n^k$.

6. (10 points) Suppose that $X_1, X_2, X_3, X_4, \ldots, X_n$ are independent random variables, each of which has a probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$. Compute the probability that $X_1 + X_2 + \ldots + X_n \geq n$. **Answer:** Each X_i is Cauchy and we seek the probability that $Z = \frac{1}{n}\sum_{i=1}^n X_i \geq 1$. Since Z is itself Cauchy, spinning flashlight story gives probability $P(Z) > 1 = (\pi/4)/\pi = 1/4$. (The angle between segment $[(0,1),(1,0)]$ and horizontal line $y = 1$ is $\pi/4$.)