A Chromatic Symmetric Function Conjecture

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Basic notation

G: simple graph with d vertices

V: vertex set of G

E: edge set of G

Coloring of G:

$$\text{any } \kappa : V \to \mathbb{P} = \{1, 2, \ldots\}$$

Proper coloring:

$$uv \in E \Rightarrow \kappa(u) \neq \kappa(v)$$

The chromatic symmetric function

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\text{proper } \kappa : V \to \mathbb{P}} x^\kappa,$$

the **chromatic symmetric function** of G, where

$$x^\kappa = \prod_{v \in V} x_{\kappa(v)} = x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots .$$

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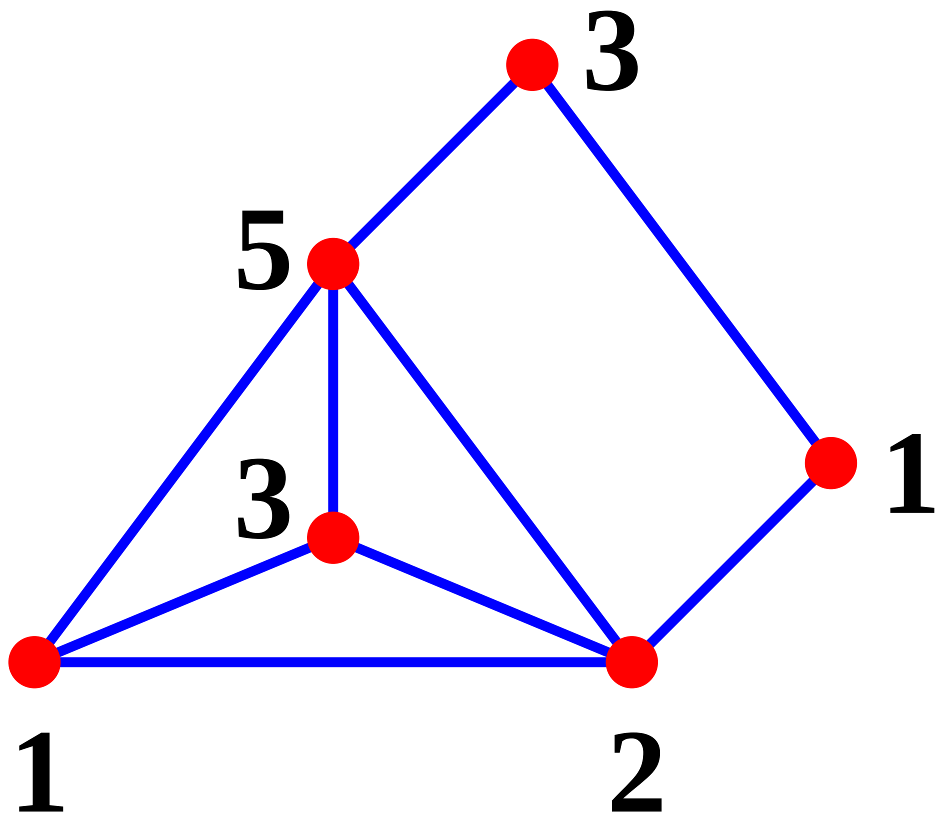
$$X_G(1^n) := X_G(\underbrace{1, 1, \ldots, 1}_{n\ 1'\text{s}}) = \chi_G(n),$$

the **chromatic polynomial** of G.

Example of a monomial



$$x^{\kappa} = x_1^2 x_2 x_3^2 x_5$$

Simple examples

$$X_{\text{point}} = x_1 + x_2 + x_3 + \cdots = e_1.$$

More generally, let

$$\boldsymbol{e_k} = \sum_{1 \le i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},$$

the kth **elementary symmetric function**. Then

$$\begin{aligned} X_{K_n} &= n!\, e_n \\ X_{G+H} &= X_G \cdot X_H. \end{aligned}$$

Acyclic orientations

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Theorem (RS, 1973). *Let* $\boldsymbol{a(G)}$ *denote the number of acyclic orientations of* G*. Then*

$$a(G) = (-1)^d \chi_G(-1).$$

Easy to prove by induction, by deletion-contraction, bijectively, geometrically, etc.

Fund. thm. of symmetric functions

Write $\boldsymbol{\lambda} \vdash \boldsymbol{d}$ if λ is a **partition** of d, i.e., $\lambda = (\lambda_1, \lambda_2, \ldots)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \;\; \sum \lambda_i = d.$$

Let

$$\boldsymbol{e_\lambda} = e_{\lambda_1} e_{\lambda_2} \cdots .$$

Fundamental theorem of symmetric functions. *Every symmetric function can be uniquely written as a polynomial in the* e_i*'s, or equivalently as a linear combination of* e_λ*'s.*

A refinement of $a(G)$

Note that if $\lambda \vdash d$, then $e_\lambda(1^n) = \prod \binom{n}{\lambda_i}$, so

$$e_\lambda(1^n)|_{n=-1} = \prod \binom{-1}{\lambda_i} = (-1)^d.$$

Hence if $X_G = \sum_{\lambda \vdash d} \boldsymbol{c_\lambda} e_\lambda$, then

$$a(G) = \sum_{\lambda \vdash d} c_\lambda.$$

Sinks

Sink of an acylic orientation (or digraph): vertex for which no edges point out (including an isolated vertex).

$\boldsymbol{a_k(G)}$: number of acyclic orientations of G with k sinks

$\boldsymbol{\ell(\lambda)}$: length (number of parts) of λ

The sink theorem

Theorem. *Let* $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$. *Then*

$$\sum_{\substack{\lambda \vdash d \\ \ell(\lambda) = k}} c_\lambda = a_k(G).$$

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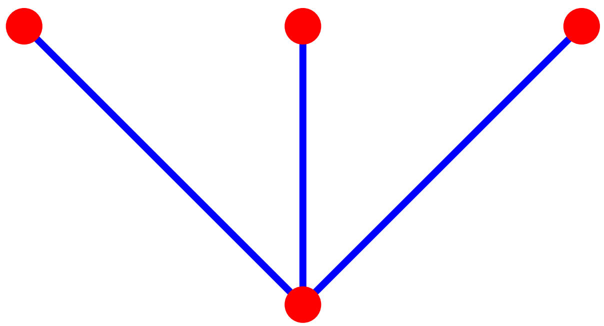
Proof based on quasisymmetric functions.

Open: Is there a simpler proof?

The claw

Example. Let G be the **claw** K_{13}.



Then

$$X_G = 4e_4 + 5e_{31} - 2e_{22} + e_{211}.$$

Thus $a_1(G) = 1$, $a_2(G) = 5 - 2 = 3$, $a_3(G) = 1$, $a(G) = 5$.

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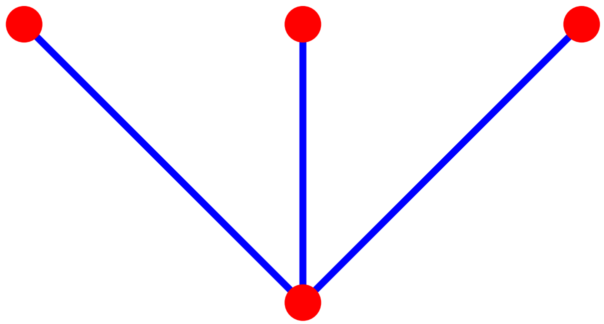

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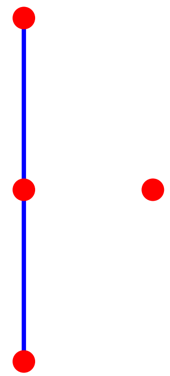
When is X_G ***e*-positive** (i.e., each $c_\lambda \geq 0$)?

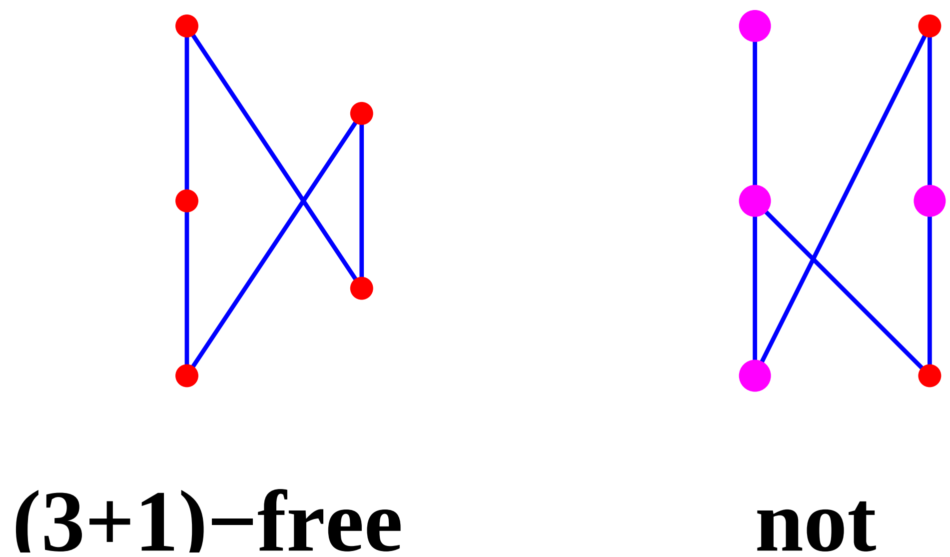
$\mathbf{3+1}$

Let P be a finite poset. Let $\mathbf{3+1}$ denote the disjoint union of a 3-element chain and 1-element chain:



$(3+1)$-free posets

P is **(3+1)-free** if it contains no **induced** $3+1$.



The main conjecture

$\mathbf{inc}(\boldsymbol{P})$: incomparability graph of P (vertices are elements of P; uv is an edge if neither $u \leq v$ nor $v \leq u$)

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Conjecture. *If P is $(\mathbf{3}+\mathbf{1})$-free, then $X_{\mathrm{inc}(P)}$ is e-positive.*

Two comments

- Suggests that for incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets, c_λ counts acyclic orientations of G with $\ell(\lambda)$ sinks and some further property depending on λ.

  Open: What is this property?

Two comments

- Suggests that for incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets, c_λ counts acyclic orientations of G with $\ell(\lambda)$ sinks and some further property depending on λ.

  Open: What is this property?

- True if P is $\mathbf{3-free}$, i.e., X_G is e-positive if G is the complement of a bipartite graph. More generally, X_G is e-positive if G is the complement of a triangle-free (or $\boldsymbol{K_3}\mathbf{-free}$) graph.

A simple special case

Fix $\boldsymbol{k} \geq 2$. Define

$$\boldsymbol{P_d} = \sum_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d},$$

where $i_1, \ldots, i_d$ ranges over all sequences of d positive integers such that any k consecutive terms are distinct.

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Conjecture. *P_d is e-positive.*

The case $k = 2$

$$P_d = \sum_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d},$$

where $i_j \geq 1$, $i_j \neq i_{j+1}$.

Theorem (**Carlitz**).

$$\sum P_d \cdot t^d = \frac{\sum_{i\geq 0} e_i t^i}{1 - \sum_{i\geq 1}(i-1)e_i t^i}.$$

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Theorem (**Carlitz**).

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Corollary. P_d *is* e*-positive for* $k = 2$.

The case $k = 3$

Ben Joseph (2001) probably had a complicated Inclusion-Exclusion proof.

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$$\sum P_d \cdot t^d =$$

$$\frac{\text{numerator}}{1 - \left(2e_3t^3 + 6e_4t^4 + 24e_5t^5 + (64e_6 + 6e_{51} - e_{33})t^6 + \cdots\right)}.$$

Schur functions

- **Schur functions** $\{s_\lambda\}$ forms a linear basis for symmetric functions.
- e_λ is s-positive.
- (**Gasharov**) X_G is s-positive if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset.
- **Conjecture** (**Gasharov**). If G is claw-free, then X_G is s-positive. (Need not be e-positive).

A final word

When G is a **unit interval graph** (special case of incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets), then **Haiman** found a close connection with Verma modules and Kazhdan-Lustzig polynomials.


That's all Folks!