Problem Set 4 Solutions

Problem 4.1

(1) *Suppose that O is a* bounded *open subset, so $O \subset (-R, R)$ for some R. Show that the characteristic function of O, χ_O, is an element of $\mathcal{L}^1(\mathbf{R})$.*

Solution. Since O is open, O is the union of disjoint open intervals

$$O = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

where a_i, b_i are finite. We know that for each i $\chi_{(a_i,b_i)} \in \mathcal{L}^1(\mathbf{R})$, and thus

$$f_n = \sum_{i=1}^{n} \chi_{(a_i,b_i)} \in \mathcal{L}^1(\mathbf{R}).$$

Since $f_n \to \chi_O$ pointwise, andfor each n $|f_n| \leq \chi_{(-R,R)} \in L^1(\mathbf{R})$, theSi dominated convergence theorem implies that $\chi_O \in \mathcal{L}^1(\mathbf{R})$. □

(2) *If O is bounded define the length (or Lebesgue measure) of O to be $\ell(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is an at most countable union of bounded open sets such that $\sum_j \ell(O_j) < \infty$, then $\chi_U \in \mathcal{L}^1(\mathbf{R})$; again we set $\ell(U) = \int \chi_U$.*

Solution. Set

$$U_n = \bigcup_{j=1}^{n} O_j.$$

Since U_n is a bounded open set, $\chi_{U_n} \in \mathcal{L}^1(\mathbf{R})$. Furthermore, $\chi_{U_n} \to \chi_U$ pointwise and monotonically. Certainly

$$\chi_{U_n} \leq \sum_{j=1}^{n} O_j,$$

and thus

$$\int \chi_{U_n} \leq \sum_{j=1}^{n} \int \chi_{O_j} \leq \sum_{j=1}^{\infty} \ell(O_j) < \infty.$$

The monotone convergence theorem implies that $\chi_U \in \mathcal{L}^1(\mathbf{R})$. □

(3) *Conversely show that if U is open and $\chi_U \in L^1(\mathbf{R})$ then $U = \bigcup_j O_j$ is a countable union of bounded open sets with $\sum \ell(O_j) < \infty$.*

Solution. Write $U = \bigcup_{j=1}^{\infty} O_j$, where $O_j = (a_j, b_j)$ are disjoint open intervals, where perhaps a_j, b_j are infinite. We first show that neither of a_j, b_j are infinite. We will for simplicity only show that $(a_j, b_j) \neq (0, \infty)$. Suppose not. Then since

$$\chi_{(0,\infty)} = \chi_{(a_j,b_j)} \leq \chi_U \in \mathcal{L}^1(\mathbf{R}),$$

and the sequence $\chi_{(0,n)} \to \chi_{(0,\infty)}$ pointwise from below, the dominated convergence theorem shows that$\chi_{(0,\infty)} \in \mathcal{L}^1(\mathbf{R})$ with $\int \chi_{(0,\infty)} = \infty$, a contradiction.

Now, since all O_j are finite intervals, $\chi_{O_j} \in \mathcal{L}^1(\mathbf{R})$ for each j, with $f_n = \sum_{j=1}^{n} \chi_{O_j}$ converging monotonically to χ_U. An application of the monotone convergence theorem concludes that

$$\sum \ell(O_j) \leq \ell(U).$$

$\square$

(4) *Show that if $K \subset \mathbf{R}$ is compact then its characteristic function is an element of $\mathcal{L}^1(\mathbf{R})$.*

Solution 1. K is closed and bounded, so K is a closed subset of $(-R, R)$ for some $R > 0$. Thus

$$\chi_K = \chi_{(-R,R)} - \chi_{(-R,R)\setminus K}.$$

Since $(-R, R) \setminus K$ is a bounded open set, $\chi_{(-R,R)\setminus K} \in \mathcal{L}^1(\mathbf{R})$, and we conclude since $\mathcal{L}^1(\mathbf{R})$ is closed under subtraction. $\square$

Solution 2. Consider the sets

$$B_\varepsilon = K + (-\varepsilon, \varepsilon) = \{x + y \colon x \in K,\ |y| < \varepsilon\}.$$

Since K is bounded, B_ε is bounded. B_ε is certainly open. Thus $\chi_{B_\varepsilon} \in \mathcal{L}^1(\mathbf{R})$. Furthermore,

$$\bigcap_{n=1}^{\infty} B_{1/n} = K.$$

Indeed, z is in the intersection if and only if $z = x_n + y_n$ for $x_n \in K$ and $y_n \to 0$. Passing to a subsequence and using compactness, we may assume $x_n \to x$. Therefore $z \in K$. The reverse inclusion is clear. Thus,

$$\chi_{B_\varepsilon} \to \chi_K,$$

and we invoke the dominated convergence theorem to conclude. $\square$

Problem 4.2

Suppose $u_n \in \mathcal{C}_c(\mathbf{R})$ form an absolutely summable series with respect to the L^1 norm and set

$$E = \left\{ x \in \mathbf{R} \colon \sum_n |u_n(x)| = \infty \right\}.$$

(1) *Show that if $a > 0$ then the set*

$$\left\{ x \in \mathbf{R} \colon \sum_n |u_n(x)| \leq a \right\}$$

is closed.

Solution 1. Call the set E_a. Denote by E_N the set

$$\left\{ x \in \mathbf{R} \colon \sum_{n=1}^{N} |u_n(x)| \leq a \right\}.$$

Since u_n is continuous, each E_N is closed. Since $|u_n| \geq 0$, $E = \bigcap_N E_N$, we conclude. $\square$

Solution 2. We show instead that the complement is open. If x is in the complement, then there exists some N for which

$$\sum_{n=1}^{N} |u_n(x)| > a.$$

Since u_n is continuous, the same inequality holds true in a neighbourhood of x. Since the $|u_n|$ are all positive, $\sum_{n=1}^{\infty} |u_n| > a$ in this neighbourhood. Thus E_a is open. $\square$

(2) *Deduce that if $\varepsilon > 0$ is given then there is an open set $O_\varepsilon \supseteq E$ with $\sum_n |u_n| > 1/\varepsilon$ for each $x \in O_\varepsilon$.*

Solution. In the notation of the previous, set $O_\varepsilon = \mathbf{R} \setminus E_{1/\varepsilon}$. $\square$

(3) *Deduce that the characteristic function of OO_ε is in $\mathcal{L}^1(\mathbf{R})$ and that $\ell(OO_\varepsilon) \leq \varepsilon C$, $C = \sum_n \int |u_n(x)|$.*

Solution. Set

$$f = \sum_n |u_n| \in \mathcal{L}^1.$$

Then $f \in L^1(\mathbf{R})$ (we need to redefine f on E). Observe that by definition

$$\varepsilon \chi_{O_\varepsilon}(x) \leq f \text{ a.e.}$$

If $\chi_{O_\varepsilon} \in \mathcal{L}^1(\mathbf{R})$, then it follows that

$$\varepsilon \ell(O_\varepsilon) \leq C,$$

which is the desired inequality. To show that $\chi_{O_\varepsilon} \in \mathcal{L}^1(\mathbf{R})$, write O_ε as the union of disjoint open intervals, and argue as in Problem 4.1.3. □

(4) *Conclude that E has the standard property of a set of measure zero (mentioned last week) – for each $\varepsilon > 0$ it is covered by a countable collection of open intervals the sum of the lengths of which is less than ε.*

Solution. Choose O_ε above, and write it as the disjoint union of open sets. Then E is covered by a countable collection of open intervals, the sum of the lengths of which is less than $C\varepsilon$. Since C does not depend on ε, this suffices. □

Problem 4.3

Define $\mathcal{L}^\infty(\mathbf{R})$ as the set of functions $g \colon \mathbf{R} \to \mathbf{C}$ such that there exists $C > 0$ and a sequence $v_n \in \mathcal{C}(\mathbf{R})$ with $|v_n| \leq C$ an $v_n \to g$ a.e.

(1) *Show that $\mathcal{L}^\infty(\mathbf{R})$ is a linear space.*

Solution. $\mathcal{L}^\infty(\mathbf{R})$ inherits its linear structure from that of the set of function $\mathbf{R} \to \mathbf{C}$, so we need only verify closure under addition and scalar multiplication. Suppose $f, g \in \mathcal{L}^\infty(\mathbf{R})$, and $\lambda \in \mathbf{C}$. Suppose $u_n \to f$ a.e. and $v_n \to g$ a.e., with $|u_n| \leq B$ and $|v_n| \leq C$ and $u_n, v_n \in \mathcal{C}(\mathbf{R})$. Since the union of two sets of measure 0 is a set of measure 0, $\lambda u_n + v_n \to \lambda f + g$, and $|\lambda u_n + v_n| \leq |\lambda||B| + |C|$. Thus $\lambda f + g \in \mathcal{L}^\infty(\mathbf{R})$. □

(2) Show that

$$\|g\|_{L^\infty} = \inf\{\sup_{\mathbf{R}\setminus E} |g(x)| \colon E \text{ has measure zero and } \sup_{\mathbf{R}\setminus E} |g(x)| < \infty\}$$

is a seminorm on $\mathcal{L}^\infty(\mathbf{R})$ and that this makes $L^\infty(\mathbf{R}) = \mathcal{L}^\infty(\mathbf{R})/\mathcal{N}$ into a Banach space, where $\mathcal{N}$ is the space of null functions.

Solution. This really has several parts, so we break it up.

$\|g\|_{L^\infty}$ **is well-defined.** Since there is a uniformly bounded sequence $v_n \to g$ a.e., we deduce that g is bounded off of a set of measure zero.

$\|g\|_{L^\infty}$ **is a seminorm.** Homogeneity and non-negativeity are clear. We present two arguments fro triangle inequality. By definition, for $f, g \in \mathcal{L}^\infty(\mathbf{R})$, for all $\varepsilon > 0$ there exists sets D, E of measure 0 such that

$$\sup_{x \in \mathbf{R} \setminus D} |f(x)| \leq \|f\|_{L^\infty} + \varepsilon,$$

and similarly for g. Thus, since $D \cup E$ is a set of measure zero,

$$\sup_{x \in \mathbf{R} \setminus (D \cup E)} |f(x) + g(x)| \leq \sup_{x \in \mathbf{R} \setminus D} |f(x)| + \sup_{x \in \mathbf{R} \setminus E} |g(x)| \leq \|f\|_{L^\infty} + \|g\| + L^\infty + 2\varepsilon.$$

Taking the inf and then $\varepsilon \to 0$ shows triangle inequality. Alternatively, if $g \in \mathcal{L}^\infty(\mathbf{R})$, then for all n there exists a set E_n of measure zero for which

$$\|g\|_{L^\infty} \leq \sup_{x \in \mathbf{R} \setminus E_n} |g(x)| \leq \|g\|_{L^\infty} + 2^{-n}.$$

Set $E_g = \bigcup_n E_n$, which is a set of measure zero. Then the previous inequality implies that

$$\|g\|_{L^\infty} = \sup_{x \in \mathbf{R} \setminus E_g} |g(x)|,$$

i.e. the infimum is achieved. Now we argue as above with E_f, E_g replacing D, E, and obviate the need to approximate the L^∞ norm.

$L^\infty(\mathbf{R})$ **is a normed space.** We need only show that $\mathcal{N} \subseteq \mathcal{L}^\infty(\mathbf{R})$ and that $\|g\|_{L^\infty} = 0$ if and only if $g \in \mathcal{N}$. If $g \in \mathcal{N}$, then g is the a.e. imit a sequence of all 0, and hence $g \in L^\infty(\mathbf{R})$. It is also clear that $\|g\|_{L^\infty} = 0$. Conversely, if $\|g\|_{L^\infty} = 0$, then for all n there exists E_n such that

$$\sup_{x \in \mathbf{R} \setminus E_n} |g(x)| \leq 2^{-n}.$$

Since $E = \bigcup_{E_n}$ has measure 0, it follows that $g \equiv 0$ off of E. Alternatively, using E_g as defined above, we have that $g \equiv 0$ off of E_g.

$L^\infty(\mathbf{R})$ **is complete.** Suppose $[f_n] \in L^\infty(\mathbf{R})$ is a Cauchy sequence. Choose arbitrary representatives $f_n \in \mathcal{L}^\infty(\mathbf{R})$. Then for all $k \in \mathbf{N}$ there exists N such that if $n, m > N$,

$$\|f_n - f_m\|_{L^\infty(\mathbf{R})} < 2^{-k-1},$$

and so there exists sets of measure zero $E_{k,n,m}$ for which

$$\sup_{\mathbf{R} \setminus E_{k,n,m}} |f_n(x) - f_m(x)| < 2^{-k}.[1]$$

[1] In reality, one can pick the same set for each k, and not lose any error; however this is not necessary for the argument.

Let $E = \bigcup_{k,n,m} E_{k,n,m}$. Then E has measure zero, and f_n is uniformly Cauchy on the complement of E, and hence by the standard argument converges uniformly to some function f defined on $\mathbf{R} \setminus E$. Extend f to $\mathbf{R}$ by defining it to be 0 on E. By definition $f_n \to f$ in $L^\infty(\mathbf{R})$, so we just need to argue that $f \in \mathcal{L}^\infty(\mathbf{R})$.

Observe first that since f_n converge uniformly on E and are bounded on E, the f_n are uniformly bounded on E by some constant C, and thus f is bounded by C, too. Fix $a < b \in \mathbf{R}$. Then by Problem 4.3.3, below, $\chi_{[a,b)} f_n \in L^1(\mathbf{R})$. Observe that $C\chi_{[a,b)} \in \mathcal{L}^1(\mathbf{R})$, that $\chi_{[a,b)} f_n \to \chi_{[a,b)} f$ a.e. and that $|\chi_{[a,b)} f_n| \le C\chi_{[a,b)}$ a.e. The dominated convergence theorem implies that $\chi_{[a,b)} f \in \mathcal{L}^1(\mathbf{R})$. Since f is bounded, Lemma 1.1, below, shows that $\chi_{[a,b)} f \in \mathcal{L}^\infty(\mathbf{R})$. Then Lemma 1.2, below, shows that $f \in \mathcal{L}^\infty(\mathbf{R})$. Alternatively, one can replace $\chi_{[a,b)}$ with $g = (1 + |x|^2)^{-1}$ and prove variants of Lemmas 1.1 and 1.2. Since we will need both again, we prove them once below.

Instead of using Cauchy sequences, one can also use the criterion for completeness that absolute summability implies summability. The first part of the argument then needs the obvious modifications, and the second part goes through largely unchanged.

□

Lemma 1.1. *Suppose f is a.e. bounded and $f \in \mathcal{L}^1(\mathbf{R})$. Then $f \in \mathcal{L}^\infty(\mathbf{R})$.*

Proof. By assumption, there exist $u_n \in \mathcal{C}_c(\mathbf{R})$ converging to f a.e. Merging two sets of measure zero, Without loss of generality we may assume that u_n converge on the set where f is bounded, say by $C > 0$. Replacing u_n by "cut-off" versions u_n', we may assume that $|u_n'| \le C$. Indeed, simply set

$$u_n'(x) := \begin{cases} u_n(x), \ |u_n(x)| \le C \\ \frac{u_n(x)}{|u_n(x)|} C, \ |u_n(x)| \ge C. \end{cases} .$$

In effect, u_n' is u_n on the closed region where $|u_n| \le C$, and u_n continuous with the same complex phase, but is normalized on $|u_n| \ge C$.[2]The pasting lemma guarantees that u_n' is still continuous. Alternatively, the function

$$\mathbf{C} \ni z \mapsto \begin{cases} z, \ z \le C \\ \frac{z}{|z|}, \ |z| \ge C \end{cases}$$

is continuous, and u_n' is the composition of u_n with this map.

In any case, f is the a.e. limit of uniformly bounded continuous function, so $f \in L^\infty$. □

[2]If complex phases are confusing, simply break u_n up into real and imaginary parts, and then the phase becomes a sign.

Lemma 1.2. *Suppose $\chi_{[a,b)}f \in L^\infty(\mathbf{R})$ for all $a < b$, and f is a.e. bounded. Then $f \in L^\infty(\mathbf{R})$.*

Proof. The idea is to take $a = n, b = n+1$, $n \in \mathbf{Z}$, and piece together the limiting sequences for each $\chi_{[n,n+1)}f$. This runs into two problems, which we seek to remedy. The first is that the sequences may not be uniformly bounded. The second is that they may have behaviour away from $[n, n+1)$ which messes up the behaviour around other intervals. We remedy the first as we did in the proof of Lemma 1.1, using the fact that f is already bounded a.e. To remedy the second, suppose that $u_{n,k} \to \chi_{[n,n+1)}$ and $u_{n,k} \in \mathcal{C}(\mathbf{R})$. Let $\varphi_n \in \mathcal{C}(\mathbf{R})$ be any continuous function which is 1 on $[n, n+1)$, 0 on $(-\infty, n-1) \cup (n+2, \infty)$, and bounded by 1 everywhere. Then $\varphi_n u_{n,k} \to \chi_{[n,n+1)}f$ a.e., since the limit is anyway 0 wherever φ is not 1. Renaming $u_{n,k}$, we may thus assume that $u_{n,k}$ are uniformly bounded in n, k by some $C > 0$, and are supported in $[n-1, n+2]$. Set

$$u_k = \sum_n u_{n,k}.$$

This looks like a infinite sum, but since at most 3 summands are supported around any x, it is actually a finite sum around any x. In particular u_k is continuous. Since it is a sum of at most 3 functions, u_k is also uniformly bounded, say by $3C$. Since the sum is finite at any x,

$$u_k(x) \to \sum \chi_{[n,n+1)}(x) f(x) = f(x) \ a.e.$$

(of course, after taking the union in k of the sets of measure zero on which each sequence $u_{n,k}$ fails to converge). We conclude that $f \in \mathcal{L}^\infty(\mathbf{R})$. □

(3) *Show that if $g \in \mathcal{L}^\infty(\mathbf{R})$ and $f \in \mathcal{L}^1(\mathbf{R})$, then $gf \in \mathcal{L}^1(\mathbf{R})$, and that this defined a map*

$$L^\infty(\mathbf{R}) \times L^1(\mathbf{R}) \to L^1(\mathbf{R})$$

which satisfies $\|gf\|_{L^1(\mathbf{R})} \le \|g\|_{L^\infty}\|f\|_{L^1}$.

Solution. Assuming everything else, the map is well-defined on $L^\infty(\mathbf{R}) \times L^1(\mathbf{R})$ since null functions are in $\mathcal{L}^1(\mathbf{R}) \cap \mathcal{L}^\infty(\mathbf{R})$ with 0 L^∞ and L^1 norms. Take $u_n \to g$ a.e., with $u_n \in \mathcal{C}(\mathbf{R})$ bounded by C, and $v_n \to f$ a.e. with $f_n \in \mathcal{C}_c(\mathbf{R})$. Then $\|g\|_{L^\infty(\mathbf{R})} \le C$, and by the previous problem set, we may assume that there exists some $F \in \mathcal{L}^1(\mathbf{R})$ with $|v_n| \le F$ a.e. Thus $|u_n v_n| \le CF$ a.e., and so by the dominated convergence theorem $gf \in L^1$, and we have the estimate

$$\|gf\|_{L^1(\mathbf{R})} \le C\|F\|_{L^1(\mathbf{R})}.$$

There are two methods for improving this. The easiest is to notice that $|g| \le \|g\|_{L^\infty}$ a.e., and so

$$\|gf\|_{L^1(\mathbf{R})} = \int |gf| \le \|g\|_{L^\infty} \int |f| = \|g\|_{L^\infty}\|f\|_{L^1}.$$

Slightly more complicated is to use a trick like in the proof of Lemma 1.1 to show that we may take $|u_n| \leq \|g\|_{L^\infty}$, and thus $C = \|g\|_{L^\infty}$, and to also notice that we may take in fact $v_n \to f$ in L^1, too. $\square$

Problem 4.4

Define a set $U \subseteq \mathbf{R}$ to be (Lebesgue) measureable if its characteristic function $\chi_U \in \mathcal{L}^\infty(\mathbf{R})$. Letting $\mathcal{M}$ be the collection of measurable sets, show

(1) $\mathbf{R} \in \mathcal{M}$.

Solution. $\chi_{\mathbf{R}} = 1 \in \mathcal{C}(\mathbf{R})$. $\square$

(2) $U \in \mathcal{M} \Rightarrow \mathbf{R} \setminus U \in \mathcal{M}$.

Solution. $\chi_{\mathbf{R}\setminus U} = \chi_R - \chi_U \in \mathcal{L}^\infty(\mathbf{R})$. $\square$

(3) $U_j \in \mathcal{M}$ *for* $j \in \mathbf{N}$ *then* $\bigcup_{j=1}^\infty U_j \in \mathcal{M}$. Set

Solution. Set

$$V_k = \bigcup_{j=1}^k U_j.$$

Using Problem 4.3.3, we see that for each n $\chi_{[n,n+1)}\chi_{V_k} \in \mathcal{L}^1(\mathbf{R})$. Since $\chi_{[n,n+1)}\chi_{V_k} \to \chi_{[n,n+1)}\chi_U$ monotonically and are bounded by 1, the monotone convergence theorem implies that $\chi_{[n,n+1)}\chi_U \in \mathcal{L}^1(\mathbf{R})$. Lemmas 1.1 and 1.2 then show that $\chi_U \in \mathcal{L}^\infty(\mathbf{R})$. $\square$

(4) *If* $U \subseteq \mathbf{R}$ *is open, then* $U \in \mathcal{M}$.

Solution 1. Observe that

$$U = \bigcup_{N=1}^\infty (-N, N) \cap U.$$

and $\chi_{(-N,N)\cap U} \in \mathcal{L}^\infty(\mathbf{R})$ by Problem 4.1.1 and Lemma 1.1. Then use Problem 4.4.3 to conclude. $\square$

Solution 2. Write U as the disoint union of (possibly infinite) intervals. It is easy to see that each interval is in $\mathcal{M}$. Then use Problem 4.4.3 to conclude. $\square$

Problem 4.5

If $U \subseteq \mathbf{R}$ is measurable, show that

$$\int U f := \int \chi_U f \in \mathbf{C}$$

is well-defined. Prove that if $f \in \mathcal{L}^1(\mathbf{R})$ then

$$I_f(x) = \begin{cases} \int_{(0,x)} f, & x \geq 0 \\ -\int_{(x,0)} f, & x < 0 \end{cases}$$

is a bounded continuous function on $\mathbf{R}$.

Solution 1. $\chi_U f \in \mathcal{L}^1(\mathbf{R})$ follows from Problem 4.3.3, so the integral is well-defined. Also, $|\chi_U f| \leq |f|$, so $|I_f| \leq \|f\|_{L^1}$ and is therefore bounded. Take $x_n \to x$. Then $\chi_{(0,x_n)} f \to \chi_{(0,x)} f$, and the sequence is bounded by $|f|$. Thus the dominated convergence theorem implies that I_f is continuous. □

Solution 2. Argue that boundedness and that I_f is well-defined as in the first solution. Take a sequence $v_n \in \mathcal{C}(\mathbf{R})$ convering to f in L^1. Then

$$|I_f(x+h) - I_f(x)| \leq |I_f(x+h) - I_{v_n}(x+h)| + |I_f(x) - I_{v_n}(x)| + |I_{v_n}(x+h) - I_{v_n}|.$$

Using Problem 4.3.3, each of the first two terms is bounded by $\|f - v_n\|_{L^1}$. Using that the indicator function of $(x, x+h]$ (or $[x+h, x)$ if $h < 0$) is in $\mathcal{L}^1(\mathbf{R})$, and $v_n \in L^\infty(\mathbf{R})$, and then Problem 4.3.3, the third term is bounded by

$$|h| \|v_n\|_{L^\infty}.$$

So for fixed $\varepsilon > 0$, choose n large so that $\|f - v_n\|_{L^1} < \varepsilon/3$. Then if $|h| < \|v_n\|^{-1}\varepsilon/3$,[3]

$$|I_f(x+h) - I_f(x)| < \varepsilon/3,$$

and so I_f is continuous. □

[3] If $\|v_n\| = 0$, then h can be anything