PROBLEM SET 4 FOR 18.102
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RICHARD MELROSE

Hint; It might be wise to remind yourself (if necessary) of the structure of an open subset of $\mathbb{R}$.

Note: If you wish to you may use the Monotonicity Lemma, Fatou's Lemma and Dominated Convergence in this problem set.

Problem 4.1

(1) Suppose that $O \subset \mathbb{R}$ is a *bounded* open subset, so $O \subset (-R, R)$ for some R. Show that the characteristic function of O

$$\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases} \tag{1}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

(2) If O is bounded and open define the length (or Lebesgue measure) of O to be $l(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is a (n at most) countable union of bounded open sets such that $\sum\limits_j l(O_j) < \infty$ then $\chi_U \in \mathcal{L}^1(\mathbb{R})$; again we set $l(U) = \int \chi_U$.

(3) Conversely show that if U is open and $\chi_U \in \mathcal{L}^1(\mathbb{R})$ then $U = \bigcup_j O_j$ *is* the union of a countable collection of bounded open sets with $\sum\limits_j l(O_j) < \infty$.

(4) Show that if $K \subset \mathbb{R}$ is compact then its characteristic function is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 4.2

Suppose $u_n \in \mathcal{C}_c(\mathbb{R})$ form an absolutely summable series with respect to the L^1 norm and set

$$E = \{x \in \mathbb{R}; \sum_n |u_n(x)| = \infty\}. \tag{2}$$

(1) Show that if $a > 0$ then the set

$$\{x \in \mathbb{R}; \sum_n |u_n(x)| \le a\} \tag{3}$$

is closed.

(2) Deduce that if $\epsilon > 0$ is given then there is an open set $O_\epsilon \supset E$ with $\sum\limits_n |u_n(x)| > 1/\epsilon$ for each $x \in O_\epsilon$.

(3) Deduce that the characteristic function of O_ϵ is in $\mathcal{L}^1(\mathbb{R})$ and that $l(O_\epsilon) \le \epsilon C$, $C = \sum\limits_n \int |u_n(x)|$.

(4) Conclude that E has the standard property of a set of measure zero (mentioned last week) – for each $\epsilon > 0$ it is covered by a countable collection of open intervals the sum of the lengths of which is less than ϵ.

Problem 4.3

Define $\mathcal{L}^\infty(\mathbb{R})$ as the set of functions $g : \mathbb{R} \longrightarrow \mathbb{C}$ such that there exists $C > 0$ and a sequence $v_n \in \mathcal{C}(\mathbb{R})$ with $|v_n(x)| \le C$ and $v_n(x) \to g(x)$ a.e.

(1) Show that $\mathcal{L}^\infty(\mathbb{R})$ is a linear space.
(2) Show that

$$\|g\|_{L^\infty} = \inf\{\sup_{\mathbb{R}\setminus E} |g(x)|; E \text{ has measure zero and } \sup_{\mathbb{R}\setminus E} |g(x)| < \infty\}$$

is a seminorm on $\mathcal{L}^\infty(\mathbb{R})$ and that this makes $L^\infty(\mathbb{R}) = \mathcal{L}^\infty(\mathbb{R})/\mathcal{N}$ into a Banach space, where $\mathcal{N}$ is the space of null functions.
(3) Show that if $g \in \mathcal{L}^\infty(\mathbb{R})$ and $f \in \mathcal{L}^1(\mathbb{R})$ then $gf \in \mathcal{L}^1(\mathbb{R})$ and that this defines a map

$$L^\infty(\mathbb{R}) \times L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$$

which satisfies $\|gf\|_{L^1} \le \|g\|_{L^\infty} \|f\|_{L^1}$.

Hint: About completeness of L^infty. First show that for a Cauchy sequence $[f_]]$, f_n converges pointwise a.e. and so defines a bounded function f – just define it as zero otherwise. One way to show that $f \in L^\infty(\mathbb{R})$ is to use the final part of the problem, which does not depend on completness. Look at say $g = (1 + |x|^2)^{-1}$ which is in $\mathcal{L}^1(\mathbb{R})$ and is positive and continuous. Now $gf_n \to gf$ a.e. and LDC shows $gf \in \mathcal{L}^1(\mathbb{R})$, and therefore is the pointwise limit of continuous functions, hence so is f. Maybe after passing to the real part use boundedness to cut off this sequence and conclude $f \in \mathcal{L}^1(\mathbb{R})$ and that $f_n \to f$ in $\mathcal{L}^\infty$.

Problem 4.4

Define a set $U \subset \mathbb{R}$ to be (Lebesgue) measureable if its characteristic function

$$\chi_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

is in $\mathcal{L}^\infty(\mathbb{R})$. Letting $\mathcal{M}$ be the collection of measureable sets, show

(1) $\mathbb{R} \in \mathcal{M}$
(2) $U \in \mathcal{M} \Longrightarrow \mathbb{R} \setminus U \in \mathcal{M}$
(3) $U_j \in \mathcal{M}$ for $j \in \mathbb{N}$ then $\bigcup_{j=1}^\infty U_j \in \mathcal{M}$
(4) If $U \subset \mathbb{R}$ is open then $U \in \mathcal{M}$

Hint: It might be useful to you to know (and if you need it, show,) that a bounded L^1 function is in L^∞ and that a bounded function f such that $f\chi_{[-R,R]} \in L^1$ for all R is in L^∞. This allows you to use LDC and monotonicity to prove (3) for instance.

Problem 4.5

If $U \subset \mathbb{R}$ is measureable and $f \in \mathcal{L}^1(\mathbb{R})$ show that

$$\int_U f = \int \chi_U f \in \mathbb{C}$$

is well-defined. Prove that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$I_f(x) = \begin{cases} \int_{(0,x)} f & x \ge 0 \\ -\int_{(x,0)} f & x < 0 \end{cases}$$

is a bounded continuous function on $\mathbb{R}$.

Problem 4.6 – Extra

Recall (from Rudin's book for instance) that if $F : [a, b] \longrightarrow [A, B]$ is an increasing continuously differentiable map, in the strong sense that $F'(x) > 0$, between finite intervals then for any continuous function $f : [A, B] \longrightarrow \mathbb{C}$, (Rudin shows it for Riemann integrable functions)

$$\int_A^B f(y)dy = \int_a^b f(F(x))F'(x)dx. \tag{4}$$

Prove the correspondng identity for every $f \in \mathcal{L}^1((A, B))$, which in particular requires the right side to make sense.

Problem 4.7 – Extra

Show that if $f \in \mathcal{L}^1(\mathbb{R})$ and I_f in Problem 4.5 vanishes identically then $f \in \mathcal{N}$.

Hint: Show that the integral $\int fu = 0$ where first u is the characteristic function of any interval, then a finite linear combination of such functions (a step function). Then use the basis of Riemann integrability of a continuous function (here of compact support) that it is the uniform limit of such step functions to show this holds for $u \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$. Then use the definition of L^∞ above (and Lebesgue Dominated Convergence) to show that $\int fu = 0$ for every $u \in L^\infty(\mathbb{R})$. Finally show that $g = \bar{f}/|f|$ or 0 where $f = 0$, is in L^∞.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu