C. CONTINUITY AND DISCONTINUITY
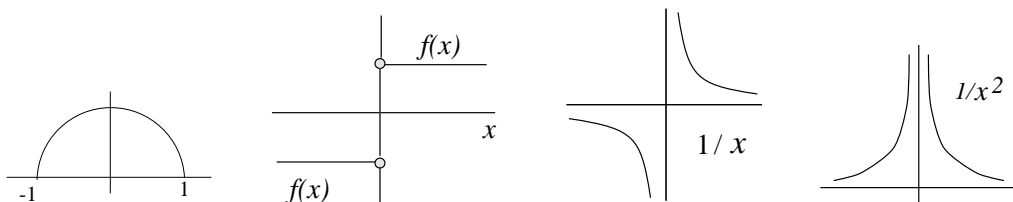
1. One-sided limits

We begin by expanding the notion of limit to include what are called *one-sided limits*, where x approaches a only from one side — the right or the left. The terminology and notation is:.

right-hand limit $\qquad \lim\limits_{x \to a^+} f(x) \qquad$ (x comes from the right, $x > a$)

left-hand limit $\qquad \lim\limits_{x \to a^-} f(x) \qquad$ (x comes from the left, $x < a$)

Since we use limits informally, a few examples will be enough to indicate the usefulness of this idea.

Ex. 1 $\qquad\qquad$ **Ex.2** $\qquad\qquad$ **Ex. 3** $\qquad\qquad$ **Ex.4**

Example 1. $\quad \lim\limits_{x \to 1^-} \sqrt{1 - x^2} = 0 \qquad \lim\limits_{x \to -1^+} \sqrt{1 - x^2} = 0$

(As the picture shows, at the two endpoints of the domain, we only have a one-sided limit.)

Example 2. Set $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0. \end{cases}$ $\quad$ Then $\quad \lim\limits_{x \to 0^-} f(x) = -1, \quad \lim\limits_{x \to 0^+} f(x) = 1.$

Example 3. $\qquad \lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty, \qquad \lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

Example 4. $\qquad \lim\limits_{x \to 0^+} \dfrac{1}{x^2} = \infty, \qquad \lim\limits_{x \to 0^-} \dfrac{1}{x^2} = \infty$

The relationship between the one-sided limits and the usual (two-sided) limit is given by

(1) $\qquad \lim\limits_{x \to a} f(x) = L \qquad \Longleftrightarrow \qquad \lim\limits_{x \to a^-} f(x) = L \quad \text{and} \quad \lim\limits_{x \to a^+} f(x) = L$

In words, the (two-sided) limit exists if and only if both one-sided limits exist and are equal. This shows for example that in Examples 2 and 3 above, $\lim\limits_{x \to 0} f(x)$ does not exist.

Students often say carelessly that $\lim\limits_{x \to 0} 1/x = \infty$, but this is not sloppy, it is simply wrong, as the picture for Example 3 shows. By contrast, $\lim\limits_{x \to 0} 1/x^2 = \infty$ is correct and acceptable terminology.

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2. Continuity

To understand continuity, it helps to see how a function can fail to be continuous. All of the important functions used in calculus and analysis are continuous except at isolated points. Such points are called **points of discontinuity**. There are several types. Let's begin by first recalling the definition of continuity (cf. book, p. 75).

(2) $\qquad\qquad f(x)$ is **continuous** at a if $\lim\limits_{x \to a} f(x) = f(a)$.

Thus, if a is a point of discontinuity, something about the limit statement in (2) must fail to be true.

Types of Discontinuity

$\dfrac{x^2-1}{x-1}$

$\sin(1/x)$

removable   *removable*   *jump*   *infinite*   *essential*
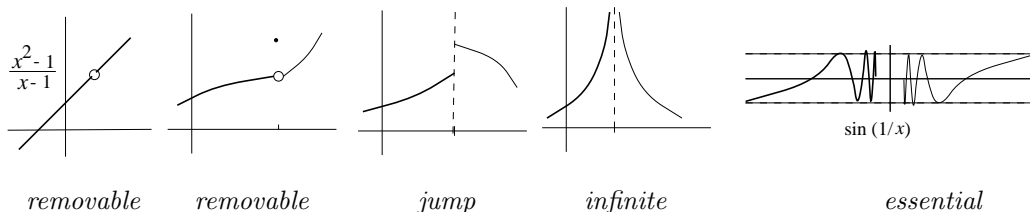
In a **removable** discontinuity, $\lim\limits_{x \to a} f(x)$ exists, but $\lim\limits_{x \to a} f(x) \neq f(a)$. This may be because $f(a)$ is undefined, or because $f(a)$ has the "wrong" value. The discontinuity can be removed by changing the definition of $f(x)$ at a so that its new value there is $\lim\limits_{x \to a} f(x)$. In the left-most picture, $\dfrac{x^2 - 1}{x - 1}$ is undefined when $x = 1$, but if the definition of the function is completed by setting $f(1) = 2$, it becomes continuous — the hole in its graph is "filled in".

In a **jump** discontinuity (Example 2), the right- and left-hand limits both exist, but are not equal. Thus, $\lim\limits_{x \to a} f(x)$ does not exist, according to (1). The *size* of the jump is the difference between the right- and left-hand limits (it is 2 in Example 2, for instance). Though jump discontinuities are not common in functions given by simple formulas, they occur frequently in engineering — for example, the square waves in electrical engineering, or the sudden discharge of a capacitor.

In an **infinite** discontinuity (Examples 3 and 4), the one-sided limits exist (perhaps as ∞ or $-\infty$), and at least one of them is $\pm\infty$.

An **essential** discontinuity is one which isn't of the three previous types — at least one of the one-sided limits doesn't exist (not even as $\pm\infty$). Though $\sin(1/x)$ is a standard simple example of a function with an essential discontinuity at 0, in applications they arise rarely, presumably because Mother Nature has no use for them.

We say a function is **continuous on an interval** $[a, b]$ if it is defined on that interval and continuous at every point of that interval. (At the endpoints, we only use the approrpiate one-sided limit in applying the definition (2).)

We say a function is **continuous** if its *domain is an interval*, and it is continuous at every point of that interval.

A **point of discontinuity** is always understood to be isolated, i.e., it is the only bad point for the function on some interval.
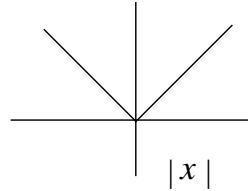
We illustrate the point of these definitions. (They are slightly different from the ones in your book, but are more consistent with standard terminology in calculus.)
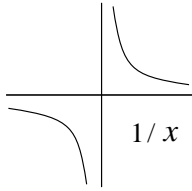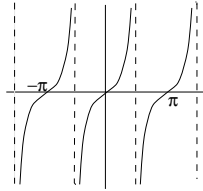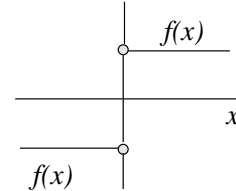

Example 5   Example 6   Example 7   Example 8

Example 5. The function $1/x$ is continuous on $(0, \infty)$ and on $(-\infty, 0)$, i.e., for $x > 0$ and for $x < 0$, in other words, at every point in its domain. However, it is not a continuous function since its domain is not an interval. It has a single point of discontinuity, namely $x = 0$, and it has an infinite discontinuity there.

Example 6. The function $\tan x$ is not continuous, but is continuous on for example the interval $-\pi/2 < x < \pi/2$. It has infinitely many points of discontinuity, at $\pm\pi/2, \pm 3\pi/2$, etc.; all are infinite discontinuities.

Example 7. $f(x) = |x|$ is continuous, but $f'(x)$ has a jump discontinuity at 0.

Example 8. The function in Example 2, $f(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0 \end{cases}$ does not have a continuous derivative $f'(x)$ — students often think it does since it seems that $f'(x) = 0$ everywhere. However this is not so: $f'(0)$ does not exist, since by definition,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x},$$

but $f(0)$ does not even exist. Even if $f(0)$ is defined to be say 1, or 0, the derivative $f'(0)$ does not exist.

Remember the important little theorem (Simmons p. 75)

(3) $f(x)$ differentiable at a $\Rightarrow$ $f(x)$ continuous at a

or to put it contrapositively,

$f(x)$ discontinuous at a $\Rightarrow$ $f(x)$ not differentiable at a

The function in Example 8 is discontinuous at 0, so it has no derivative at 0; the discontinuity of $f'(x)$ at 0 is a removable discontinuity.

Exercises: Section 1D