“Just foolin’ around”

or

The unifying power of graded rings

Haynes Miller

Fields Institute, June 7, 2025

Agenda

I'll report on an assortment of projects undertaken with undergraduates from MIT, Birzeit University, An-Najah National University, and the Arab American University, Jenin, under the Palestine UROP program.

- generalities
- commutative monoids
- categories
- level algebras
- alternative algebras
- divided power algebras

Quillen homology

In *Homotopical Algebra*, 1967, Daniel Quillen provided a definition of what is to be meant by "homology" and "cohomology" in a very general algebraic setting. In later work he spelled out the special case of commutative rings. That case was also explored by Michel André. In many cases the simplicial or cotriple resolution due to Michael Barr and Jonathan Beck serve to make this definition explicit. Some underlying categorical and naturality issues were clarified much later by Martin Frankland in his MIT thesis.

Coefficients for Quillen cohomology lie in categories of "Beck modules."

Beck modules

Let $\mathbf{V}$ be an "algebraic" category (models for a Lawvere theory).

The category $\mathrm{Ab}(\mathbf{V})$ of abelian objects in $\mathbf{V}$ is an abelian category, and the forgetful functor $\mathrm{Ab}(\mathbf{V}) \to \mathbf{V}$ has a left adjoint, "abelianization."

Given $A \in \mathbf{V}$, the overcategory $\mathbf{V}/A$ is again algebraic. The category of "Beck A-modules" is

$$\mathbf{Mod}_A = \mathrm{Ab}(\mathbf{V}/A)$$

There is often a small projective generator, whose endomorphism ring (or its opposite) is the "universal enveloping algebra" $U_{\mathbf{V}}(A)$, and

$$\mathbf{Mod}_A = \mathbf{LMod}_{U_{\mathbf{V}}(A)}$$

Beck modules: examples

Our "K-algebras" will generally lack a unit. To adjoin one, form

$$A_+ = A \oplus K$$

With $\mathbf{V} = \mathbf{CAlg}_K$, $\mathbf{Mod}_A$ is the usual category of A_+-modules: $U_{\mathbf{CAlg}_K}(A) = A_+$.

With $\mathbf{V} = \mathbf{Alg}_K$, $U_{\mathbf{Alg}_K}(A) = A_+ \otimes_K A_+^{op}$.

With $\mathbf{V} = \mathbf{Gp}$, $U_{\mathbf{Gp}}(G) = \mathbb{Z}G$.

With $\mathbf{V} = \mathbf{Lie}_K$, $U_{\mathbf{Lie}_K}(L) = U(L)$.

With $\mathbf{V} = \mathbf{Ab}$, $\mathbf{Mod}_A = \mathbf{Ab}$: $U_{\mathbf{Ab}}(A) = \mathbb{Z}$.

Kähler differentials

We will write $\mathrm{Ab}_A : \mathbf{V}/A \to \mathbf{Mod}_A$ for the abelianization functor. The category $\mathbf{Mod}_A$ contains a canonical object, the "module of Kähler differentials":

$$\Omega_A^{\mathbf{V}} = \mathrm{Ab}_A(\mathrm{id} : A \downarrow A)$$

With this definition, differentials corepresent derivations:

$$\mathrm{Hom}_A(\Omega_A^{\mathbf{V}}, M) = \Gamma(M \downarrow A)$$

Given $f : B \to A$ in $\mathbf{V}$, there is an adjoint pair

$$f_* : \mathbf{Mod}_B \rightleftarrows \mathbf{Mod}_A : f^*$$

and

$$\mathrm{Ab}_A(f : B \downarrow A) = f_* \Omega_B^{\mathbf{V}}$$

Kähler differentials: examples

In commutative associative K-algebras, $\Omega_A^{\mathbf{CAlg}_K}$ is the usual module of Kähler differentials $\Omega_{A/K}$.

In associative K-algebras, $\Omega_A^{\mathbf{Alg}_K} = \ker(\mu : A \otimes A \to A)$.

In groups, $\Omega_G^{\mathbf{Gp}} = \ker(\epsilon : \mathbb{Z}G \to \mathbb{Z})$.

In Lie algebras, $\Omega_L^{\mathbf{Lie}_K} = \ker(\epsilon : U(L) \to K)$.

In abelian groups, $\Omega_A^{\mathbf{Ab}} = A$.

Quillen homology and cohomology

Quillen established a model category structure on $s\mathbf{V}/A$. Let $p : P_\bullet \to A$ be a cofibrant replacement. In all our cases this can be constructed using a cotriple, following Barr and Beck. Define the *cotangent complex* to be the simplicial object in $\mathbf{Mod}_A$:

$$\mathbf{L}_A = p_* \Omega^{\mathbf{V}}_{P_\bullet} = \mathrm{Ab}_A P_\bullet$$

The (absolute) Quillen homology is then the graded Beck module

$$HQ_*(A) = \pi_*(\mathbf{L}_A)$$

For a Beck A-module M,

$$HQ^*(A; M) = \pi^*(\mathrm{Hom}_A(\mathbf{L}_A, M))$$

Example 1: Commutative monoids

This is a well-studied case: Books and papers by Pierre Grillet, work by Kurdiani and Pirashvili, and many others.
Given a commutative monoid X, the *Leech category* L_X associated to X has as objects the underlying set of X, and

$$L_X(x, y) = \{z : x + z = y\} \quad : \quad z : x \to y$$

It's easy to see that

$$\mathbf{Mod}_X = \mathrm{Fun}(L_X, \mathbf{Ab})$$

The abelian object $M \downarrow X$ corresponding to $F : L_X \to \mathbf{Ab}$ has $M_x = F(x)$; the commutative monoid structure covering that of X is

$$\begin{aligned} M_x \times M_y &\to M_{x+y} \\ (a, b) &\mapsto y_* a + x_* b \end{aligned}$$

The abelian structure $M \times_X M \to M$ over x is the addition in $F(x)$.

Grillet's computations

In a series of papers spanning almost half a century (1974–2022), Pierre Grillet showed that a certain chain complex – much simpler than the one arising from the cotriple resolution – successfully computed the Quillen cohomology of a commutative monoid X: In degree n his chain complex consisted in functions $s : X^n \to M$ satisfying certain symmetry conditions, with a differential resembling the Hochschild differential. Grillet's mysterious symmetry conditions:

$$\mathbf{n = 2} : s(a,b) = s(b,a)$$
$$\mathbf{n = 3} : s(a,b,c) + s(c,b,a) = 0 \quad , \quad s(a,b,c) + s(b,c,a) + s(c,a,b) = 0$$
$$\mathbf{n = 4} : s(a,b,c,d) - s(b,a,c,d) + s(b,c,a,d) - s(b,c,d,a) = 0\,,$$
$$s(a,b,c,d) + s(d,c,b,a) = 0 \quad , \quad s(a,b,b,a) = 0\,.$$

He made conjectures in dimensions 5 and 6 as well.

Enter graded rings

A commutative monoid X determines a symmetric monoidal structure on the category $\mathbf{Ab}^X$ of X-graded abelian groups:

$$(A_\bullet \otimes B_\bullet)_z = \bigoplus_{x+y=z} A_x \otimes B_y$$

The initial X-graded commutative ring $\widetilde{\mathbb{Z}}X$ has $(\widetilde{\mathbb{Z}}X)_x = \mathbb{Z}\iota_x$ for every $x \in X$, and product determined by

$$\iota_x \iota_y = \iota_{x+y}$$

Observation (with Bhavya Agrawalla and Nasief Khlaif)

There is an additive equivalence between $\mathbf{Mod}_X$ and the category of left $\widetilde{\mathbb{Z}}X$-modules, under which $\Omega_X^{\mathbf{CMon}}$ and $\Omega_{\widetilde{\mathbb{Z}}X}^{\mathbf{CAlg}^X}$ correspond.

Theorem (with Bhavya Agrawalla and Nasief Khlaif)

$$HQ^*_{\mathbf{CMon}}(X; M) = HQ^*_{\mathbf{CAlg}^X}(\widetilde{\mathbb{Z}}X; M)$$

Constant coefficients

A monoid homomorphism $\alpha : X \to Y$ induces an adjoint pair

$$\alpha_{\#} : \mathbf{Mod}_{\widetilde{\mathbb{Z}}X} \rightleftarrows \mathbf{Mod}_{\widetilde{\mathbb{Z}}Y} : \alpha^{\#}$$

in which

$$(\alpha^{\#} N)_x = N_{\alpha(x)} \quad , \quad (\alpha_{\#} M)_y = \bigoplus_{\alpha(x)=y} M_x$$

With $\alpha : X \to *$, $\alpha_{\#}\widetilde{\mathbb{Z}}X = \mathbb{Z}X$ and $\alpha^{\#}N$ is the "constant" module, and we find

Corollary (Kurdiani and Pirashvili, 2016)

For $N \in \mathbf{Mod}_{\mathbb{Z}X}$*,*

$$HQ^*_{\mathbf{CMon}}(X; \alpha^{\#}N) = HQ^*_{\mathbf{CAlg}}(\mathbb{Z}X; N)$$

But this case didn't shed much light on Grillet's work.

Harrison cohomology

Attempts to construct a cohomology theory for commutative rings began in the early 1960's with David Harrison's modification of the Hochschild complex. For a commutative ring A, the Hochschild complex is a commutative differential graded algebra with divided powers. Harrison proposed to look at cochain that kill decomposables; this gives "Harrison cohomology."

Michael Barr showed that if you are working over $\mathbb{Q}$ this coincides with Quillen cohomology, but examples due to Michel André showed that this is not true without that assumption, already in dimension 4. Barr surmounted André's example by killing divided power decomposables as well, but Sarah Whitehouse showed that Barr's refinement failed in the next dimension.

Grillet explained

Observation (with Bhavya Agrawalla and Nasief Khlaif)

Under the identification $\mathbf{Mod}_X = \mathbf{Mod}_{\widetilde{\mathbb{Z}}X}$*, Grillet's cochain complex is exactly the Harrison-Barr cochain complex. Grillet's symmetry conditions are precisely those demanded by Harrison and Barr.*

So the real import of Grillet's work is that Harrison and Quillen agree, working over $\mathbb{Z}$, in degrees less than 5, at least for graded algebras of the form $\widetilde{\mathbb{Z}}X$.

I don't know whether this identification – in these dimensions – holds for more general commutative algebras.

On the other hand, Whitehouse's example shows that Grillet's conjectural identification doesn't work in dimension 5, even for this special class of algebras.

Example 2: Categories

The Quillen cohomology of categories with fixed object set has been studied by Baues and Wirsching, Dwyer and Kan, and others. First, Beck C-modules are "natural systems," that is,

$$\mathbf{Mod}_C = \mathrm{Fun}(\mathrm{Fac}_C, \mathbf{Ab})$$

where Fac_C is the "factorization category" of C: objects are morphisms in C, and given $f : X' \to X$ and $g : Y' \to Y$,

$$\mathrm{Fac}_C(f,g) = \left\{ \begin{tikzcd} X' \arrow[d, "f"] & Y' \arrow[l, dotted] \arrow[d, "g"] \\ X \arrow[r, dotted] & Y \end{tikzcd} \right\}$$

Category grading

The category structure of C determines a (non-symmetric) monoidal structure on the category of $\mathrm{Mor}(C)$-graded abelian groups:

$$(A_\bullet \otimes B_\bullet)_h = \bigoplus_{f,g \text{ s.t. } fg=h} A_f \otimes B_g$$

The initial monoid with respect to this monoidal structure has $(\widetilde{\mathbb{Z}}C)_f = \mathbb{Z}\iota_f$, with

$$\iota_f \iota_g = \iota_{fg}$$

Observation

$\mathbf{Mod}_C$ *is equivalent to the category of* $\widetilde{\mathbb{Z}}C$*-bimodules.*

Under this identification, the cochain complex used by Hans Baues and Günther Wirsching their "cohomology of small a category with coefficients in a natural system" is simply the Hochschild complex computing the Hochschild cohomology $HH^*(\widetilde{\mathbb{Z}}C; M)$.

Quillen cohomology of categories

Let $A_\bullet$ be C-graded unital associative K-algebra. There is a canonical Hochschild complex

$$A_\bullet \leftarrow A_\bullet \otimes A_\bullet \Leftarrow A_\bullet \otimes A_\bullet \otimes A_\bullet \Lleftarrow \cdots$$

that is split exact as left or right $A_\bullet$ modules. It expresses the Kähler differentials as a kernel:

$$0 \leftarrow A_\bullet \leftarrow A_\bullet \otimes A_\bullet \leftarrow \Omega_{A_\bullet}^{\mathbf{Alg}^C} \leftarrow 0$$

Following Quillen, this quickly implies

Theorem

Suppose A_f is K-flat for all $f \in \mathrm{Mor}(C)$. Then $\mathbf{L}_{A_\bullet}$ is discete.

Corollary (Dwyer and Kan)

Quillen and Baues-Wirsching cohomology for small categories coincide.

Interlude: Varieties of binary algebras

The most general kind of binary K-algebra is "magmatic": simply a K-linear map $A \otimes A \to A$. $KMag(S)$ is the free magmatic K-algebra on the set S. An "equation" is an element $\omega \in KMag(S)$ for some $S = S(\omega)$. A collection Ω of equations cuts out a "variety" $\mathbf{V}$ of algebras, namely those algebras A such that for every $\omega \in \Omega$ and every $a : S(\omega) \to A$, $\omega(a) = 0$.
(a induces $KMag(S(\omega)) \to KMag(A) \to A$, and $\omega(a)$ is the image of ω.)

Every $\mathbf{V}$-algebra A has an abelian category of Beck modules, which is equivalent to a category of modules over a "universal enveloping algebra" $U_{\mathbf{V}}(A)$.

There's a beautiful process of "noncommutative partial differentiation" that converts equations for a variety to relations defining $U_{\mathbf{V}}(A)$ (spelled out in work with Nishant Dhankhar, Ali Tahboub, Victor Yin).

Example 3: Level algebras

Definition (Chataur and Livernet 2005)

A *level K-algebra* is a K-module equipped with a bilinear product satisfying

$$ab = ba\,, \quad (ab)(cd) = (ac)(bd)$$

For example if A is a commutative associative algebra (write $a \cdot b$) and $\varphi : A \to A$ an algebra endomorphism, then A with product given by

$$ab = \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$$

is a level algebra. The "Carlsson module" is a level algebra of this form.

Lemma

$$U_{\mathbf{Lev}}(A) = \mathrm{Tens}(A)/([ab|c] - [ca|b])$$

This is a graded quadratic algebra.

Koszul duality

If the level algebra A contains identity element, it is associative:

$$(ab)c = (ab)(1c) = (a1)(bc) = a(bc)$$

Theorem (with Abdallateef Irdniya and Zian Shang)

Suppose K is a field and A contains an identity. (So A is any commutative associative unital K-algebra.) Then $U_{\mathbf{Lev}}(A) = (tA[t])_+$, and is Koszul. Suppose $\dim_K A < \infty$, pick a basis $\{e_i : i \in I\}$, and say

$$e_i e_j = \sum_k \lambda_{i,j}^k e_k$$

Then the Koszul dual is

$$\mathrm{Tens}(A^\vee)/(\{\sum_{i,j} \lambda_{i,j}^k e_i^\vee e_j^\vee : k \in I\})$$

Another level algebra

Example (Irdniya)

$\{a, b, c\}$ with $aa = a, ab = c$ and permutations of these is a "level set." Its linear extension to $A = K\langle a, b, c\rangle$ has

$$U_{\mathbf{Lev}}(A) = \mathrm{Tens}\langle x, y, z\rangle/(xx - yy, xy - yz \text{ and permutations })$$

It also is Koszul, with dual

$$\mathrm{Tens}\langle u, v, w\rangle/(u^2 + v^2 + w^2, uv + vw + wu, uw + wv + vu)$$

Example 4: Alternative algebras

Definition

A K-algebra is *alternative* if $(ab)c = a(bc)$ as long as b coincides with either a or c.

The algebra of octonions is the famous example of a non-associative alternative algebra. Again there is a large literature about alternative algebras (Emil Artin, Richard Shafer, Kevin McCrimmon,. . .) Artin showed that an algebra is alternative iff every two-generator subalgebra is associative.

A surprisingly interesting example is given by the trivial product on any K-module. In this case the weight filtration on $U_{\mathbf{Alt}}(A)$ splits: $U_{\mathbf{Alt}}(A)$ is graded.

Abelian alternative algebras

Theorem (with Nishant Dhankhar, Ali Tahboub, Victor Yin)

$U_{\mathbf{Alt}}(K^n)$ is free as a K-module, with

$$\operatorname{rank}_K U_{\mathbf{Alt}}(K^n)_k = \begin{cases} 1 & \text{if } k = 0 \\ 2n & \text{if } k = 1 \\ \frac{3n^2-n}{2} & \text{if } k = 2 \\ 2\binom{n}{k} & \text{if } k \geq 3 \end{cases}$$

Corollary

For any alternative K-algebra A, free of rank n, $U_{\mathbf{Alt}}(A)$ has a set of K-module generators with at most

$$2 \cdot 2^n + \frac{n^2+n-2}{2}$$

elements

Solvability and nilpotence of alternative algebras

The basis we give (using Gröbner basis methods) provides the following interesting example.

Corollary (Dorofeev, 1960)

There is an alternative K-algebra E such that $(uv)(xy) = 0$ for all u, v, x, y (it's "solvable") but E contains elements $x_1, x_2, \ldots,$ such that for all n

$$(\cdots((x_1x_2)x_3)\cdots x_{n-1})x_n \neq 0$$

(but not "nilpotent").

Namely, take E to be the alternative K-algebra over $A = K^\infty$ corresponding to a free $U_{\mathbf{Alt}}(A)$-module of rank 1.

Example

Victor Yin also studied the standard octonions $\mathbb{O}$, and found that if $1/2 \in K$ then $U_{\mathbf{Alt}}(\mathbb{O})$ is free of rank 65, while over a field of characteristic 2, $\dim U_{\mathbf{Alt}}(\mathbb{O}) = 113$.

Example 5: Divided power algebras

"Divided powers" model $x \mapsto x^n/n!$. Extending work of Norbert Roby (1966) and Ioannis Dokas (2021):

Theorem (with Aseel Kmail and Julia Kozak)

Abelian DP algebras over a commutative ring K are the same as modules over

$$U_{\mathbf{DP}}(0) = K[\phi_p : p \text{ prime}]/(p\phi_p)$$

with $\phi_p r = r^p \phi_p$. For any DP K-algebra A,

$$U_{\mathbf{DP}}(A) = A_+ \otimes_K U_{\mathbf{DP}}(0)$$

with $(1 \otimes \phi_p)(a \otimes 1) = 0$.

DP Kähler differentials

Theorem (with Aseel Kmail and Julia Kozak)

For any DP K-algebra A, $\Omega^{\mathbf{CAlg}}_{A/K}$ admits a DP Beck A-module structure, natural in A, making it a model for $\Omega^{\mathbf{DP}}_{A/K}$.

We simplify and extend arguments of Dokas, who assumed K is a field.

Corollary (Roby)

Let $A = \Gamma(V)$ be the free DP K-algebra on an K-module V. Then as A_+-modules

$$\Omega^{\mathbf{CAlg}}_{A/K} = A_+ \otimes_K U_{\mathbf{DP}}(0) \otimes_K V$$

Proof: General principles imply that $\Omega^{\mathbf{DP}}_A = U_{\mathbf{DP}}(A) \otimes_K V$.

Collaborators through https://purop.birzeit.edu

Hadeel AbuTabeekh, An-Najah National University, Nablus
Bhavya Agrawalla, MIT
Mohamed Badarin, Palestinian Polytechnic University, Hebron
Mohammad Damaj, Birzeit University
Nishant Dhankhar, MIT
Kim Eppling, MIT
Abdalateef Irdniya, An-Najah National University, Nablus
Nasief Khlaif, Birzeit University
Aseel Kmail, Arab American University, Jenin
Julia Kozak, MIT
Heidi Lei, MIT
Zian Shang, MIT
Kenta Suzuki, MIT
Ali Tahboub, Birzeit University
Victor Yin, MIT
Honglin Zhu, MIT