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Lecture 15	
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1 Matroid Matching

The Matroid matching Problem: Given a matroid $M = (S, \mathcal{I})$, let E be a set of pairs on S. The matroid matching problem is to find disjoint set of pairs $F \subseteq E$, such that $\bigcup F \in \mathcal{I}$ and $|F|$ is maximum. The maximum cardinality of the matching F is denoted by $\nu(M)$.

The following are a few illustrations of the matroid matching problem.

Examples (Matroid matching):

1. Let M be the trivial matroid on a set S, i.e., $M = (S, 2^S)$. Let E be a collection of pairs on S which define a graph $G = (S, E)$. Then the matroid matching problem is equivalent to finding a maximum size matching in $G = (S, E)$.

2. Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids on the ground set S. Then the matroid intersection problem can be formulated using the matroid matching problem in the following manner. Let S' be an identical copy of S where for every $a \in S$ there is a corresponding $a' \in S'$. Define M_1 on S and M_2 on S', so that $\mathcal{I}_1$ is defined on S and $\mathcal{I}_2$ is defined on S'. Define M and E as follows.

$$\begin{aligned} M =&(S \cup S',\ \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}) \\ E =&\{(a, a') : a \in S, a' \in S'\} \end{aligned}$$

   With the above definition, the matroid matching problem for M is equivalent to finding a maximum independent set in $M_1 \cap M_2$.

3. Consider the graphic matroid $M(G)$ of a graph $G = (V, E)$. Partition the edge set E into pairs. Then the matroid matching problem is to find the maximum forest consisting of the pairs in the partition of the edge set E.

4. Finding a maximum forest in a 3-uniform hypergraph. Consider the problem of finding a maximum forest in a 3-uniform hypergraph. In other words, the problem is to find a maximum subgraph without cycles. Recall that a cycle in a hypergraph is a sequence of hyperedges $h_1, h_2 \dots h_T$ such that, $\exists \{s_i : i = 1, 2 \dots T\}$, and $s_i s_{i+1} \in h_i$ for $i = 1, 2 \dots T$ (with $s_{T+1} = s_1$). The problem can be formulated as a matroid matching problem by creating a graph G and having two edges (a, b) and (b, c) for each hyperedge $\{a, b, c\}$, creating a pair for these 2 edges, and considering the cycle matroid of G. choosing any two pairs in each of the hyperedges to construct the set of pairs.

1.1 Is the matroid matching problem solvable in polynomial time?

We will first construct an example to show that the matroid matching problem is not solvable in polynomial time.

We show this by using an independent set testing oracle, which can check whether a given $T \in \mathcal{I}$ is independent. Let $M = (S, \mathcal{I})$ be a matroid, and let E be partition of S into pairs. Let the collection of independent sets be as follows.

$$\mathcal{I} = \{I : |I| \leq 2k - 1\} \cup \{I : |I| = 2k, I \text{ is not a union of } k \text{ pairs in } E\}.$$

It is easy to check that M, with $\mathcal{I}$ defined as above, is a matroid. To see this, let $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$. If $I_1 \leq 2k-1$, then I_2 can be trivially augmented using elements from $I_2 \setminus I_1$. If $I_1 = 2k$, then I_1 intersects at least $k+1$ pairs in E, and thus, I_2 can again be augmented without creating exactly k pairs. Note that $\nu(M) = k-1$. Now take any $F \subseteq E$ such that $|F| = k$. Define M_F as

$$M_F = (S, \mathcal{I} \cup \{\bigcup F\})$$

which, by the same reasoning, is a matroid for every choice of F. Clearly, $\nu(M_F) = k$. If it is known that the matroid is M or any of the M_F's, The number of oracle calls required to check if there is a matching of size k is at least $\binom{|E|}{k}$ since all the possible k-subsets from E have to checked.

The following construction also shows that the matroid matching need not be polynomial time solvable even when the matroid is given more explicitly. Suppose we are given a graph G whose *vertex* set is E. Let $M = (S, \mathcal{I})$ be a matroid with $\mathcal{I}$ defined as as

$$\begin{aligned}\mathcal{I} =& \{I : |I| \leq 2k-1\} \cup \{I : |I| = 2k, I \text{ is not a union of } k \text{ pairs in } E\} \\ & \cup \{I : |I| = 2k, I \text{ is a union of } k \text{ pairs in } E \text{ such that the pairs form a clique in } G\}\end{aligned}$$

Now clearly,

$$\nu(M) = \begin{cases} k-1 & \text{if there is no clique of size } k \\ k & \text{o.w.} \end{cases}$$

Thus, checking whether $\nu(M) = k$ is not possible in polynomial time unless $P = NP$.

1.2 Min-max relation for matroid matching

Lovász derived a min-max relationship for matroid matching for special class of matroids, namely linear matroids. He also gave a polynomial time algorithm for the problem. For example, the maximum forest problem in a 3-uniform hypergraph can be solved in polynomial time using Lovász' algorithm.

We next extend the definition of matroid for which one can apply Lovász' min-max theorem on matroid matching. The notion of infinite matroid is a generalization of linear spaces.

Definition 1 (Infinite matroid) *The matroid $M = (S, \mathcal{I})$ is an infinite matroid if the following properties hold:*

1. $I \in \mathcal{I},\ J \subseteq I \Rightarrow J \in \mathcal{I}$,
2. $J \in \mathcal{I}\ \forall (J \subseteq I, |J| < \infty) \Rightarrow I \in \mathcal{I}$,
3. *If $I, J \in \mathcal{I}$ and $|I| < |J| < \infty$, then $\exists j \in J \setminus I$ such that $I + j \in \mathcal{I}$.*

Note that the second property is essential to a matroid being an infinite matroid. □

Before we state the min-max theorem, recall that a flat in a matroid $M = (S, \mathcal{I})$ is defined as all $F \subseteq S$ such that $F = \text{span}(F)$. For linear matroids, flats are precisely the linear subspaces.

Theorem 1 (Lovász) *Let $M = (S, \mathcal{I})$ be a linear matroid (finite or infinite), let r be the rank function, and let E be a finite set of pairs in S. Then*

$$\nu(M) = \min_F \left[r(F) + \sum_{i=1}^{k} \lfloor \frac{1}{2}(r(F_i) - r(F)) \rfloor \right] \ , \qquad (1)$$

where the minimization is carried over the set

$$\{F : F \subseteq F_1 \cap F_2 \ldots \cap F_k;\ F_1, F_2, \ldots F_k \, \textit{are flats}; \forall (e \in E)\ \exists (F_i) \textit{ such that } e \in F_i\}.$$

One can check that our examples in Section 1.1 are not linear. We next discuss a few examples where Theorem 1 can be applied.

Examples (Application of Theorem 1):

1. **Berge-Tutte formula:** Let $M = (S, 2^S)$ be the trivial matroid (in which all sets are independent) and let the edges in the graph $G = (S, E)$ define the set of pairs E in S. Clearly,

   $$\nu(M) = \text{maximum size matching in } G \ .$$

   Now we proceed to compute the RHS of (1). In this case

   $$\text{RHS of (1)} = \min_F \left[|F| + \sum_{i=1}^{k} \lfloor \frac{1}{2}(|F_i| - |F|) \rfloor \right]. \tag{2}$$

   (For the trivial matroid, all sets are flats.) First, note that the minimization can be restricted to the all flats F_i's such that the sets $F_i \setminus F$ are disjoint. To see this, observe that, if for some i and j, $(F_i \cap F_j) \setminus F \neq \emptyset$, then, we can replace F_i and F_j by a single flat $F_i \cup F_j$ and that will reduce the sum in (2). Thus, we assume the minimization in (2) is carried over flats such that $F_i \setminus F$ are disjoint. Thus it means that $F, F_1 \setminus F, F_2 \setminus F \ldots F_k \setminus F$ is a partition of S. Moreover, all edges of G must belong to $E(F_i)$ for some i. If all the quantities $|F_i| - |F|$ were even, then (2) boils down to minimization over $(1/2)(|F| + |S|)$. Taking into account the fact that some of the $|F_i| - |F|$ can be odd, we can write (2) as

   $$\frac{1}{2} \min_F \left[|F| + |S| - |\{i : |F_i \setminus F| \text{ odd}\}| \right] \ ,$$

   which is precisely the Berge-Tutte formula since $(F_i \setminus F)$ can be seen to be a connected component of $G \setminus F$.

2. **Graphic matroid:** Let $G = (V, E)$ be a graph and P be a partition of the edges into pairs. The matroid matching problem is to find the maximum size forest that only contains pairs in the partition P. We will derive the min-max relation given by Theorem 1 in this special case.

   We first see what the flats correspond to in this case. Let Q be a partition of V into classes. The flats are all edges contained within the classes of Q. Thus for a flat F, if Q is the corresponding partition, then

   $$r(F) = |V| - |Q| \ .$$

   Now we can form super-flats by merging some of the classes in Q to form larger classes. Now partition E into classes $E_1, E_2 \ldots E_k$ such that each E_i only consists of pairs in P in the statement of the problem. Thus, in this case, the maximum size of a forest only consisting of pairs in P (which is 2×RHS of (1) in Theorem 1) equals

   $$\min_{Q, E_1, \cdots, E_k} 2 \left[|V| - |Q| + 2 \sum_{i=1}^{k} \lfloor \frac{1}{2} \delta_Q(E_i) \rfloor \right] \ ,$$

   where $\delta_Q(E_i)$ is the size of the largest forest in the graph (V, E_i) after shrinking the classes of Q.

Comments of the linearity condition in Theorem 1:

The min-max relationship given by (1) in Theorem 1 holds under a more general condition. Let $M = (S, \mathcal{I})$ be a matroid and let $\mathcal{C}$ be the set of all the circuits. Then Theorem 1 holds if M and all its contractions satisfy the relationship that

$$r\left(\bigcap_{C\in\mathcal{C}'}\operatorname{span}(C)\right) > 0, \tag{3}$$

where

$$\mathcal{C}' = \{\text{circuit } C : C \subseteq C_1 \cup C_2, r(C) = |C_1 \cup C_2| - 2\},$$

for any two circuits C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$.

We next show that linear matroid satisfy the condition given by (3).

Proposition 2 *If $M = (S, \mathcal{I})$ is a linear matroid, then it satisfies the condition given by (3).*

Proof: Note that $C_1 \setminus C_2 \in \mathcal{I}$ and $C_1 \cap C_2 \in \mathcal{I}$. Since $\operatorname{span}(C_1 \setminus C_2)$, $\operatorname{span}(C_1 \cap C_2)$, and $\operatorname{span}(C_1))$ are linear subspaces, and further since,

$$r(C_1 \setminus C_2) + r(C_1 \cap C_2) = |C_1 \setminus C_2| + |C_1 \cap C_2| = |C_1| > r(C_1)$$

it follows that

$$P = \operatorname{span}(C_1 \setminus C_2) \cap \operatorname{span}(C_1 \cap C_2) \neq \emptyset\ .$$

Thus, $\exists p \neq 0 \in P$. We next argue that $p \in \operatorname{span}(C)$ for every $C \subseteq C_1 \cup C_2$ with $r(C) = |C_1 \cup C_2| - 2$. Suppose not, i.e., $p \notin \operatorname{span}(C)$. Now,

$$p \in \operatorname{span}(C_1 \setminus C_2) \Rightarrow C_1 \setminus C_2 \nsubseteq C \Rightarrow \exists s \in C_1 \setminus C_2,\ s \notin C\ .$$

and similarly

$$p \in \operatorname{span}(C_1 \cap C_2) \Rightarrow \exists t \in C_1 \cap C_2,\ t \notin C\ .$$

Now, $\operatorname{span}(C_2) = \operatorname{span}(C_2 - t)$ (this is always true for an element of a circuit) implies

$$t \in \operatorname{span}(C_2 - t) \subseteq \operatorname{span}(C_1 \cup C_2 \setminus \{s,t\})\ ,$$

as $C_2 \setminus \{t\} \subseteq (C_1 \cup C_2) \setminus \{s,t\}$. Therefore

$$s \in \operatorname{span}(C_1 - s) \subseteq \operatorname{span}((C_1 \cup C_2) \setminus \{s\}) = \operatorname{span}((C_1 \cup C_2) \setminus \{s,t\}),$$

as $t \in \operatorname{span}((C_1 \cup C_2) \setminus \{s,t\})$. Thus

$$\{s,t\} \subseteq \operatorname{span}(C_1 \cup C_2 \setminus \{s,t\})\ .$$

Since $|C| > r(C)$ (as C is a circuit), $r(C) = |C_1 \cup C_2| - 2$ (by assumption) and $C \subseteq (C_1 \cup C_2) \setminus \{s,t\}$, we obtain

$$|C| > r(C) = |C_1 \cup C_2| - 2 = |(C_1 \cup C_2) \setminus \{s,t\}| \geq |C|,$$

and we have reached a contradiction. $\square$