EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §9

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Throughout these exercises we assume that M is an n-dimensional manifold.

Exercise 9.1. Assume that X is a C^∞ vector field on M. It defines a first order differential operator on $C^\infty(M)$, by differentiating along the vector field. Show that $P := -ihX \in \Psi_h^1(M)$ and $\sigma_h(P) = p$ where

$$p(x,\xi) = \langle \xi, X(x)\rangle.$$

Use this example to explain why the principal symbol is a function on T^*M rather than on TM.

Exercise 9.2. In this exercise we assume that M is compact.

(a)* Show the following elliptic parametrix statement: if $a \in S^k(T^*M)$, $P \in \Psi_h^\ell(M)$, and there exists a constant $c > 0$ such that $|\sigma_h(P)| \geq c\langle\xi\rangle^\ell$ on $\operatorname{supp} a$, then there exist $Q, Q' \in \Psi_h^{k-\ell}(M)$ such that

$$\operatorname{Op}_h(a) = QP + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}, \quad \operatorname{Op}_h(a) = PQ' + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

(Hint: one way is to use a partition of unity on a to reduce to standard quantization on $\mathbb{R}^n$. Another way is to carry out the elliptic parametrix construction directly on M but then you have to take care of the supports of the resulting symbols.)

(b) Under the assumptions in part (a), show the following elliptic estimate: if $u \in \mathcal{D}'(M)$ and $Pu \in H_h^{s-\ell}(M)$, then $\operatorname{Op}_h(a)u \in H_h^{s-k}(M)$ and there exists C such that for all N

$$\|\operatorname{Op}_h(a)u\|_{H_h^{s-k}} \leq C\|Pu\|_{H_h^{s-\ell}} + \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}}.$$

(c) As an application of the elliptic estimate, show the following estimate for any Riemannian metric g on M, any N, any fixed $E \in \mathbb{R}$, and any $u \in \mathcal{D}'(M)$:

$$\|u\|_{H_h^{s+2}} \leq C\|(-h^2\Delta_g - E)u\|_{H_h^s} + C_N\|u\|_{H_h^{-N}}.$$

(Hint: take $a := 1 - \chi$ for a correct choice of $\chi \in C_c^\infty(T^*M)$. You can use that $\operatorname{Op}_h(1) = I$.)

Exercise 9.3.* This exercise defines the *wavefront set* of a semiclassical pseudodifferential operator. Morally speaking, this is the essential support of the full symbol of

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the operator (i.e. the set of points near which the full symbol is *not* $\mathcal{O}(h^\infty)$). However, one has to take caution since the full symbol of a pseudodifferential operator on a manifold is not invariantly defined. What saves us is that all the semiclassical asymptotic expansions have terms which are local in the symbols involved, so the wavefront set still makes invariant sense. (Caution: this definition is very different from the one of the wavefront set of a family of distributions in §10.) We only define wavefront sets for compactly microlocalized operators, for the general case (involving the fiber-radial compactification $\overline{T}^*M$) see §E.2.1 in the Dyatlov–Zworski book

Assume M is a compact manifold and $A \in \Psi_h^k(M)$. We say that A is *compactly microlocalized* and write $A \in \Psi_h^{\mathrm{comp}}(M)$, if for each cutoff chart (φ, χ), we have

$$(\varphi^{-1})^* \chi A \chi \varphi^* = \mathrm{Op}_h(a_{\varphi,\chi}), \quad a_{\varphi,\chi} \in S^k(\mathbb{R}^{2n}),$$

where $a_{\varphi,\chi} = \mathcal{O}(h^\infty)_{\mathscr{S}(\mathbb{R}^{2n})}$ outside of some h-independent compact set. For $A \in \Psi_h^{\mathrm{comp}}(M)$, we define its *wavefront set* $\mathrm{WF}_h(A) \subset T^*M$ as follows: a point $(x_0, \xi_0) \in T^*M$ does *not* lie in $\mathrm{WF}_h(A)$ if there exists a neighborhood U of (x_0, ξ_0) such that for each cutoff chart (φ, χ), we have $a_{\varphi,\chi} = \mathcal{O}(h^\infty)_{C^\infty}$ on $\widetilde{\varphi}(U)$.

(a) Show that $\Psi_h^{\mathrm{comp}}(M) \subset \Psi_h^\ell(M)$ for *all* ℓ.

(b) Let $A \in \Psi_h^{\mathrm{comp}}(M)$. Show that $\mathrm{WF}_h(A)$ is a closed subset of T^*M.

(c) Let $A \in \Psi_h^{\mathrm{comp}}(M)$. Show that $\mathrm{WF}_h(A) = \emptyset$ if and only if $A = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. (Hint: use the proof that Definition 2 $\Rightarrow$ Definition 1 in the lecture.)

(d) Assume that $a \in S^k(T^*M)$ is supported inside an h-independent compact set K. Using the change of variables formula, show that

$$\mathrm{Op}_h(a) \in \Psi_h^{\mathrm{comp}}(M), \quad \mathrm{WF}_h(\mathrm{Op}_h(a)) \subset K.$$

(e) Show that for $A, B \in \Psi_h^{\mathrm{comp}}(M)$,

$$\begin{aligned} \mathrm{WF}_h(A+B) &\subset \mathrm{WF}_h(A) \cup \mathrm{WF}_h(B), \\ \mathrm{WF}_h(AB) &\subset \mathrm{WF}_h(A) \cap \mathrm{WF}_h(B), \\ \mathrm{WF}_h(A^*) &= \mathrm{WF}_h(A). \end{aligned}$$

(Hint: use the proof of Proposition E.17 in the Dyatlov–Zworski book.)

Exercise 9.4.* This exercise establishes the basic properties of the symplectic form, the Poisson bracket, and Hamiltonian vector fields on T^*M. As a prerequisite it has the theory of differential forms.

(a) Define the *canonical 1-form* α on T^*M as follows: for $(x, \xi) \in T^*M$ and $v \in T_{(x,\xi)}(T^*M)$, define

$$\alpha(x, \xi)(v) := \xi(d\pi(x, \xi) \cdot v)$$

where $\pi : T^*M \to M$ is the canonical projection. Show that if N is another manifold, $\varphi : N \to M$ is a diffeomorphism, and

$$\widetilde{\varphi} : T^*N \to T^*M, \quad (y, \eta) \mapsto (\varphi(y), d\varphi(y)^{-T} \cdot \eta)$$

is the lifted map of the cotangent bundles, then $\widetilde{\varphi}^*\alpha_M = \alpha_N$ where α_M, α_N are the canonical 1-forms on M, N. ("The canonical 1-form does not depend on the choice of coordinates on M.") Define the *symplectic 2-form*

$$\omega := d\alpha.$$

(b) If $M \subset \mathbb{R}^n$ is an open set and we use coordinates (x, ξ) on $T^*\mathbb{R}^n$, show that

$$\alpha = \sum_{j=1}^n \xi_j \, dx_j, \quad \omega = \sum_{j=1}^n d\xi_j \wedge dx_j.$$

This explains the abbreviated notation $\alpha = \xi \, dx$, $\omega = d\xi \wedge dx$.

(c) Show that $d\omega = 0$. Next, show that the *symplectic volume form*

$$d \operatorname{vol}_\omega := \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ times}}$$

is nonvanishing. (Hint: use local coordinates.) These two properties mean that (T^*M, ω) is a *symplectic manifold*.

(d) Let $p : T^*M \to \mathbb{R}$ be a C^∞ function. Show that there exists a unique vector field H_p on T^*M, called the *Hamiltonian vector field* of p, such that

$$\iota_{H_p}\omega = -dp$$

where ι denotes the interior product, i.e. for any $(x, \xi) \in T^*M$ and any $v \in T_{(x,\xi)}(T^*M)$,

$$\omega(H_p(x, \xi), v) = -dp(x, \xi)(v).$$

If $M \subset \mathbb{R}^n$ is an open set, show that

$$H_p = \sum_{j=1}^n \big((\partial_{\xi_j} p)\partial_{x_j} - (\partial_{x_j} p)\partial_{\xi_j}\big).$$

(e) Using Cartan's formula for Lie derivative on differential forms, $\mathcal{L}_X = \iota_X d + d\, \iota_X$, show that $\mathcal{L}_{H_p}\omega = 0$. Conclude that the Hamiltonian flow $e^{tH_p} : T^*M \to T^*M$ is a symplectomorphism, i.e. $(e^{tH_p})^*\omega = \omega$.

(f) For $f, g \in C^\infty(T^*M; \mathbb{R})$, define the *Poisson bracket*

$$\{f, g\} := H_f g \in C^\infty(T^*M; \mathbb{R}).$$

Extend this notion to complex valued functions. If $M \subset \mathbb{R}^n$ is an open set, show that

$$\{f, g\} = \sum_{j=1}^n \big((\partial_{\xi_j} f)(\partial_{x_j} g) - (\partial_{x_j} f)(\partial_{\xi_j} g)\big).$$

(g) Show the following identities featuring the Poisson bracket:

$$\begin{aligned}
\{f, g\} &= -\{g, f\}, \\
\{f, gk\} &= \{f, g\}k + \{f, k\}g, \\
\{f, \{g, k\}\} + \{g, \{k, f\}\} + \{k, \{f, g\}\} &= 0, \\
[H_f, H_g] &= H_{\{f,g\}}.
\end{aligned}$$