EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §8

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Recall the Kohn–Nirenberg symbols

$$S^k(\mathbb{R}^{2n}) = \{a\colon \forall \alpha,\beta\ \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = \mathcal{O}(\langle\xi\rangle^{k-|\beta|})\}.$$

Exercise 8.1. **(a)** Assume that $a \in C^\infty(\mathbb{R}^{2n})$ is compactly supported in x and satisfies the following homogeneity condition:

$$a(x,\tau\xi) = \tau^k a(x,\xi) \quad \text{when} \quad \tau \geq 1,\ |\xi| \geq 1.$$

Show that $a \in S^k(\mathbb{R}^{2n})$.

(b) Show that $\langle\xi\rangle^k \in S^k(\mathbb{R}^{2n})$.

Exercise 8.2. **(a)** Assume that $a \in S^k(\mathbb{R}^{2n})$, $b \in S^\ell(\mathbb{R}^{2n})$. Show that $ab \in S^{k+\ell}(\mathbb{R}^{2n})$.

(b) Assume additionally that there exists a constant $c > 0$ such that $|b(x,\xi)| \geq c\langle\xi\rangle^\ell$ for all $(x,\xi) \in \operatorname{supp} a$. Show that $a/b \in S^{k-\ell}(\mathbb{R}^{2n})$.

Exercise 8.3. Assume that $U, V \subset \mathbb{R}^n$ are open sets, $\varphi : U \to V$ is a diffeomorphism, and $\chi \in C_c^\infty(U)$. Let $a \in S^k(\mathbb{R}^{2n})$. Show that $b(x,\xi) := \chi(x) a(\varphi(x), d\varphi(x)^{-T}\xi)$ lies in $S^k(\mathbb{R}^{2n})$ as well.

Exercise 8.4. Give the following extension of the elliptic parametrix construction of Exercise 7.1 to the Kohn–Nirenberg classes: assume that $a \in S^k$, $p \in S^\ell$, and there exists a constant c such that $|p| \geq c\langle\xi\rangle^\ell$ on $\operatorname{supp} a$. Construct $q, q' \in S^{k-\ell}$ such that

$$a = q\#p + \mathcal{O}(h^\infty)_{S^{-\infty}}, \quad a = p\#q' + \mathcal{O}(h^\infty)_{S^{-\infty}}$$

where $S^{-\infty} := \bigcap_N S(\langle\xi\rangle^{-N})$. (You may assume that Borel's Theorem is still valid, see §E.1.2 in the Dyatlov–Zworski book.)

Exercise 8.5.* Assume that $U \subset \mathbb{R}^n$ is an open set and

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha, \quad a_\alpha \in C^\infty(U),$$

is a (nonsemiclassical) differential operator of order k on U whose principal symbol

$$p_0(x,\xi) := \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$$

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is (nonsemiclassically) elliptic, namely $p_0(x,\xi) \neq 0$ for all $x \in U$, $\xi \in \mathbb{R}^n \setminus \{0\}$. (Examples include the Laplacian and, in dimension 2, the Cauchy–Riemann operator $\partial_{x_1} + i\partial_{x_2}$.)

We will show the following *elliptic regularity theorem*: if $u \in \mathcal{D}'(U)$ (i.e. u is a distribution on U, that is a continuous linear functional on $C_c^\infty(U)$; any element of $\mathscr{S}'(\mathbb{R}^n)$ would define such a distribution), then

$$Pu \in C^\infty(U) \quad \Longrightarrow \quad u \in C^\infty(U).$$

(a) Fix an arbitrary cutoff function $\chi \in C_c^\infty(U)$. Take $\chi' \in C_c^\infty(U)$ such that $\operatorname{supp}\chi \cap \operatorname{supp}(1-\chi') = \emptyset$ and define the rescaled cut off operator

$$P_h := h^k \chi' P.$$

Show that $P_h = \mathrm{Op}_h(p)$ for some $p \in S^k(\mathbb{R}^{2n})$ such that $p(x,\xi) = \chi'(x)p_0(x,\xi) + \mathcal{O}(h)_{S^{k-1}(\mathbb{R}^{2n})}$.

(b) Fix $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near 0, and put

$$a(x,\xi) := \chi(x)(1-\psi(\xi)) \ \in \ S^0(\mathbb{R}^{2n}).$$

Using Exercise 8.4, construct $q \in S^{-k}(\mathbb{R}^{2n})$ such that

$$\mathrm{Op}_h(a) = \mathrm{Op}_h(q)P_h + \mathrm{Op}_h(r), \quad r = \mathcal{O}(h^\infty)_{S^{-\infty}}. \tag{8.1}$$

(c) Put $v := \chi' u \in \mathscr{S}'(\mathbb{R}^n)$. Applying (8.1) to v, obtain

$$\chi\,\mathrm{Op}_h(a)v = \chi\,\mathrm{Op}_h(q)\chi' P_h u + \chi\,\mathrm{Op}_h(q)[P_h,\chi']u + \chi\,\mathrm{Op}_h(r)v.$$

Show that all three terms on the right-hand side are in $C_c^\infty(\mathbb{R}^n)$:

- for the first term, use the assumption $Pu \in C^\infty(U)$;
- for the second term, use the pseudolocality statement from the lecture and the fact that the coefficients of $[P_h,\chi']$ are supported away from $\operatorname{supp}\chi$;
- for the last term, use the properties of r to see that $\chi\,\mathrm{Op}_h(r) : \mathscr{S}'(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$.

(d) Now write

$$\chi^2 u = \chi\,\mathrm{Op}_h(\chi(x))v = \chi\,\mathrm{Op}_h(a)v + \chi\,\mathrm{Op}_h(\chi(x)\psi(\xi))v.$$

Using that $\mathrm{Op}_h(\chi(x)\psi(\xi)) : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ show that $\chi^2 u \in C_c^\infty(\mathbb{R}^n)$. Since χ was arbitrary, this gives $u \in C^\infty(U)$.

(Note: in the above arguments h was completely irrelevant, in fact we could have fixed $h := 1$.)