EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §5

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Recall the notation

$$\langle x\rangle := \sqrt{1+|x|^2}.$$

Exercise 5.1. **(a)** Let $s \in \mathbb{R}$. Show that

$$m(x,\xi) = \langle x\rangle^s, \quad m(x,\xi) = \langle \xi\rangle^s, \quad m(z) = \langle z\rangle^s$$

are order functions, where we denote $z = (x,\xi)$.

(b) Show that the order functions in part (a) satisfy $m \in S(m)$.

Exercise 5.2. Show that

$$a(x,\xi) = \sum_{|\alpha|\leq k} a_\alpha(x)\xi^\alpha$$

where each $a_\alpha(x)$ has all derivatives bounded, lies in $S(\langle\xi\rangle^k)$.

Exercise 5.3. **(a)** Let m_1, m_2 be order functions. Show that $m_1 m_2$ is an order function as well.

(b) Show that if $a_1 \in S(m_1), a_2 \in S(m_2)$, then $a_1 a_2 \in S(m_1 m_2)$.

Exercise 5.4.* **(a)** Arguing similarly to the proof in the lecture, show that if m is an order function and $a \in S(m)$, then $\mathrm{Op}_h(a)^*$ is a continuous operator on $\mathscr{S}(\mathbb{R}^n)$.

(b) Using part (a), show that $\mathrm{Op}_h(a)$ is a continuous operator on $\mathscr{S}'(\mathbb{R}^n)$.

Exercise 5.5.* Using Exercise 3.5(b) and following the proof for standard quantization given in the lecture, show that if m is an order function and $a \in S(m)$, then the Weyl quantization $\mathrm{Op}_h^{\mathrm{w}}(a)$ is a continuous operator on $\mathscr{S}(\mathbb{R}^n)$ and on $\mathscr{S}'(\mathbb{R}^n)$.

Exercise 5.6.* **(a)** Show that $a(x,\xi) = e^{-i\langle x,\xi\rangle}$ does not lie in $S(m)$ for any order function m.

(b) With a defined in part (a) and $h := 1$, show that $\mathrm{Op}_h(a)$ does not map $\mathscr{S}(\mathbb{R}^n)$ to itself, and it does not map $\mathscr{S}'(\mathbb{R}^n)$ to itself either.

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