EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §3

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Recall the standard and Weyl quantization formulas (valid as convergent integrals when $a \in \mathscr{S}(\mathbb{R}^{2n})$, $u \in \mathscr{S}(\mathbb{R}^n)$)

$$\mathrm{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) u(y)\, dyd\xi, \tag{3.1}$$

$$\mathrm{Op}_h^{\mathrm{w}}(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\Big(\frac{x+y}{2},\xi\Big) u(y)\, dyd\xi. \tag{3.2}$$

We use the notation $\langle x\rangle := \sqrt{1+|x|^2}$.

Exercise 3.1. Fill in the details of the proof in lecture that the formula

$$\mathrm{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)\hat u\Big(\frac{\xi}{h}\Big)\, d\xi \tag{3.3}$$

implies that

(a) if $a \in \mathscr{S}(\mathbb{R}^{2n})$ then $\mathrm{Op}_h(a) : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ (hint: use that $\xi \mapsto e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)$ is a Schwartz function all of whose seminorms are rapidly decaying in x);

(b) if $a \in C^\infty(\mathbb{R}^{2n})$ and $|a(x,\xi)| \le C\langle x\rangle^N \langle \xi\rangle^N$ for some C, N then we may define $\mathrm{Op}_h(a) : \mathscr{S}(\mathbb{R}^n) \to \langle x\rangle^N L^\infty(\mathbb{R}^n)$ (hint: the integral (3.3) converges).

Exercise 3.2. Using (3.3) and properties of the Fourier transform, verify that if a is a polynomial in the ξ variables

$$a(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha$$

where each $a_\alpha \in C^\infty(\mathbb{R}^n)$ is polynomially bounded, then $\mathrm{Op}_h(a)$ is a semiclassical differential operator:

$$\mathrm{Op}_h(a) = \sum_{|\alpha|\le k} a_\alpha(x)(hD_x)^\alpha, \quad D_x := -i\partial_x$$

Here $\alpha = (\alpha_1,\dots,\alpha_n)$ denotes multiindices and

$$\xi^\alpha := \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}, \quad |\alpha| := \alpha_1+\dots+\alpha_n, \quad (hD_x)^\alpha = h^{|\alpha|} D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}.$$

Exercise 3.3. This exercise establishes some basic properties of the Weyl quantization.

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(a) Verify that $\mathrm{Op}_h^{\mathrm{w}}(a)^* = \mathrm{Op}_h^{\mathrm{w}}(\overline{a})$ for all $a \in \mathscr{S}(\mathbb{R}^{2n})$, that is for all $u, v \in \mathscr{S}(\mathbb{R}^n)$

$$\langle \mathrm{Op}_h^{\mathrm{w}}(a)u, v\rangle_{L^2(\mathbb{R}^n)} = \langle u, \mathrm{Op}_h^{\mathrm{w}}(\overline{a})v\rangle_{L^2(\mathbb{R}^n)}.$$

(b) For $a \in \mathscr{S}(\mathbb{R}^{2n})$ and $u, v \in \mathscr{S}(\mathbb{R}^n)$, show that

$$\langle \mathrm{Op}_h^{\mathrm{w}}(a)u, v\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} a(x,\xi) W_{u,v}(x,\xi)\, dx d\xi \tag{3.4}$$

where the function $W_{u,v}(x,\xi)$ is defined as follows:

$$W_{u,v}(x,\xi) := (\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{2i}{h}\langle w,\xi\rangle} u(x-w)\overline{v(x+w)}\, dw.$$

(For $u = v$, $W_{u,v}$ is called the *Wigner function* of u.)

(c) For $u, v \in \mathscr{S}(\mathbb{R}^n)$, show that $W_{u,v} \in \mathscr{S}(\mathbb{R}^{2n})$. (Hint: write $W_{u,v}$ as the rescaled Fourier transform in w of the function $B(x,w) = u(x-w)\overline{v(x+w)}$ which lies in $\mathscr{S}(\mathbb{R}^{2n})$.) Using this, show that for $a \in \mathscr{S}'(\mathbb{R}^{2n})$ we may define $\mathrm{Op}_h^{\mathrm{w}}(a) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ via (3.4).

(d) Show that $\mathrm{Op}_h^{\mathrm{w}}(1) = I$.

Exercise 3.4.* Finish the proof of oscillatory testing from the lecture: assuming that $e_\xi(x) = e^{i\langle x,\xi\rangle}$ (we remove h for simplicity, since it does not matter for this part), $B : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is continuous, and $Be_\xi = 0$ for all ξ, show that $B = 0$. (Hint: by approximation it suffices to show that $Bu = 0$ for each $u \in \mathscr{S}(\mathbb{R}^n)$. Write by Fourier inversion formula

$$u = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) e_\xi\, d\xi,$$

and use that the Riemann sums of the above integral converge to u in $\langle x\rangle L^\infty(\mathbb{R}^n)$.)

Exercise 3.5. **(a)** Show the following product formulas for the standard quantization when $a \in \mathscr{S}(\mathbb{R}^{2n})$:

$$x_j \,\mathrm{Op}_h(a) = \mathrm{Op}_h(x_j a), \tag{3.5}$$

$$\mathrm{Op}_h(a) x_j = \mathrm{Op}_h(x_j a - ih\partial_{\xi_j} a), \tag{3.6}$$

$$(hD_{x_j})\,\mathrm{Op}_h(a) = \mathrm{Op}_h(\xi_j a - ih\partial_{x_j} a), \tag{3.7}$$

$$\mathrm{Op}_h(a)(hD_{x_j}) = \mathrm{Op}_h(\xi_j a). \tag{3.8}$$

(Hint: use the formula (3.1). For (3.6), integrate by parts in ξ_j. For (3.8), integrate by parts in y_j.)

(b) Show the following product formulas for the Weyl quantization when $a \in \mathscr{S}(\mathbb{R}^{2n})$:

$$x_j \operatorname{Op}_h^{\mathrm{w}}(a) = \operatorname{Op}_h^{\mathrm{w}}(x_j a + \tfrac{ih}{2}\partial_{\xi_j} a), \tag{3.9}$$

$$\operatorname{Op}_h^{\mathrm{w}}(a) x_j = \operatorname{Op}_h^{\mathrm{w}}(x_j a - \tfrac{ih}{2}\partial_{\xi_j} a), \tag{3.10}$$

$$(hD_{x_j}) \operatorname{Op}_h^{\mathrm{w}}(a) = \operatorname{Op}_h^{\mathrm{w}}(\xi_j a - \tfrac{ih}{2}\partial_{x_j} a), \tag{3.11}$$

$$\operatorname{Op}_h^{\mathrm{w}}(a)(hD_{x_j}) = \operatorname{Op}_h^{\mathrm{w}}(\xi_j a + \tfrac{ih}{2}\partial_{x_j} a). \tag{3.12}$$

(Hint: use the formula (3.2). For (3.9)–(3.10), integrate by parts in ξ_j. For (3.12), integrate by parts in y_j.)

(c) Using (3.9)–(3.12) (which are still valid for $a \in \mathscr{S}'(\mathbb{R}^{2n})$ via approximating it by Schwartz functions), show that $\operatorname{Op}_h^{\mathrm{w}}(x_j\xi_j) = x_j(hD_{\xi_j}) - \frac{ih}{2}$.