§8. CHANGE OF VARIABLES

This section will lay the groundwork for defining semiclassical quantization on manifolds

§8.1. Compactly supported symbols

Assume that we are given a diffeomorphism

$$\boxed{\varphi : U \to V,} \quad U, V \subset \mathbb{R}^n \text{ open sets.}$$

Let $a \in C_c^\infty(\mathbb{R}^{2n})$. We want to conjugate a by the pullback operator

$$\varphi^* : u \longmapsto u \circ \varphi,$$

i.e. study the operator

$$\varphi^* \mathrm{Op}_h(a) (\varphi^{-1})^*.$$

But for $u \in \mathcal{S}(\mathbb{R}^n)$, $(\varphi^{-1})^* u \in C^\infty(V)$ does not extend to a function on $\mathbb{R}^n$ because φ was only defined locally.

So we also fix a cutoff $\boxed{\chi \in C_c^\infty(U)}$

Then

$$\chi \varphi^* : C^\infty(V) \to C_c^\infty(U),$$

$$(\varphi^{-1})^* \chi : C^\infty(U) \to C_c^\infty(V)$$

naturally extend to operators $C^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ (and thus $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$)

The symbol will change by the map

$$\tilde{\varphi} : U_x \times \mathbb{R}^n_\xi \longrightarrow V_x \times \mathbb{R}^n_\xi \quad (\text{subsets of } \mathbb{R}^{2n})$$

$$\tilde{\varphi}(x, \xi) := (\varphi(x), d\varphi(x)^{-T} \cdot \xi)$$

inverse of the transpose of $d\varphi(x)$

Theorem Under the above assumptions,

$$\chi\varphi^* \underbrace{Op_h(a)}_{\text{"operator on } V\text{"}} (\varphi^{-1})^* \chi = \underbrace{Op_h(b)}_{\text{"operator on } U\text{"}}$$

for some $b(x,\xi;h) \in \mathcal{S}(\mathbb{R}^{2n})$ with an expansion

$$b \sim \sum_{j=0}^{\infty} h^j L_j(a \circ \tilde{\varphi})$$ where L_j are differential operators of order $2j$

and the leading term is

$$b(x,\xi;h) = \chi(x)^2 a(\tilde{\varphi}(x,\xi)) + O(h)_{\mathcal{S}(\mathbb{R}^{2n})}.$$

Proof 1. We use oscillatory testing: if $e_\xi(x) = e^{\frac{i}{h}\langle x,\xi\rangle}$ then

$$b(x,\xi;h) = e^{-\frac{i}{h}\langle x,\xi\rangle}\left(\chi\varphi^* Op_h(a)(\varphi^{-1})^*\chi e_\xi\right)(x)$$

$$= e^{-\frac{i}{h}\langle x,\xi\rangle}\chi(x)\left(Op_h(a)(\varphi^{-1})^*\chi e_\xi\right)(\varphi(x))$$

$$= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(-\langle x,\xi\rangle + \langle\varphi(x)-y,\eta\rangle)} \chi(x)a(\varphi(x),\eta)\left((\varphi^{-1})^*\chi e_\xi\right)(y)\,dy\,d\eta$$

$$= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(-\langle x,\xi\rangle + \langle\varphi(x)-y,\eta\rangle + \langle\varphi^{-1}(y),\xi\rangle)} \chi(x)a(\varphi(x),\eta)\chi(\varphi^{-1}(y))\,dy\,d\eta$$

(change of variables $y = \varphi(z)$, $J(z) = |\det d\varphi(z)|$)

$$= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle z-x,\xi\rangle + \langle\varphi(x)-\varphi(z),\eta\rangle)} \chi(x)\chi(z)a(\varphi(x),\eta)J(z)\,dz\,d\eta$$

2. For fixed (x,ξ), use stationary phase:

$\underline{\Phi} = \langle z-x,\xi\rangle + \langle\varphi(x)-\varphi(z),\eta\rangle$, integrating in z,η

$\partial_\eta = 0 \Leftrightarrow x = z$, $\partial_z = 0 \Leftrightarrow \xi = d\varphi(z)^T \cdot \eta$

Thus the only critical point is

$$z = x, \quad \eta = d\varphi(x)^{-T} \cdot \xi.$$

This gives an asymptotic expansion of the form stated above.

Let us compute the leading term.

The Hessian of the phase is

$$d^2\Phi = \begin{array}{c c} & \begin{array}{cc} z & \eta \end{array} \\ \begin{array}{c} z \\ \\ \eta \end{array} & \begin{pmatrix} \text{something} & -d\varphi(z)^T \\ & \\ -d\varphi(z) & 0 \end{pmatrix} \end{array} \Rightarrow \operatorname{sgn} d^2\Phi = 0$$

$$|\det d^2\Phi|^{1/2} = J(z) = |\det d\Phi(z)|$$

The value of Φ at the critical point is 0.

So the leading term is

$$b(x,\xi) = \chi(x)^2 a(\varphi(x), d\varphi(x)^{-T} \cdot \xi) + O(h)$$

3 It remains to get the expansion in $\mathcal{S}(\mathbb{R}^{2n})$.

- Higher derivatives: straightforward (st. phase uniform in parameters)
- $x \to \infty$: b is compactly supported in x
- $\xi \to \infty$: get $b = O(h^\infty \langle\xi\rangle^{-\infty})$ for large ξ by integrating by parts in z

See the book of Dyatlov-Zworski, Proposition E.10 for details. $\square$

§8.2. General symbols

Recall the expansion was

$$b(x,\xi;h) = \chi(x)^2 a(\varphi(x), d\varphi(x)^{-T}\cdot\xi) + \dots$$

Unfortunately, this operation does not preserve the class $S(1) = \{a : \forall\alpha, \partial^\alpha a \text{ is bounded}\}$.

Indeed, in 1D (for simplicity)

$$\partial_x b = \underbrace{\chi(x)^2 \partial_\xi a\left(\varphi(x), \frac{1}{\varphi'(x)}\cdot\xi\right)\cdot\left(\frac{1}{\varphi'(x)}\right)'}_{\text{only know this is bounded}} \cdot \xi$$

this is <u>not</u> bounded

To fix this, we need to require that derivatives in ξ decay by a power of ξ:

<u>Definition</u> Let $k\in\mathbb{R}$. We say $a(x,\xi;h)$ is in $S^k(\mathbb{R}^{2n})$, if $\forall\alpha,\beta\ \exists C_{\alpha\beta}\ \forall x,\xi,h$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha\beta}\langle\xi\rangle^{k-|\beta|}.$$

S^k are called <u>Kohn-Nirenberg symbols</u>.

Note that $S^k \subset S(\langle\xi\rangle^k)$.

<u>Definition</u> Assume that $a\in S^k(\mathbb{R}^{2n})$, $a_j\in S^{k-j}(\mathbb{R}^{2n})$, $j=0,1,\dots$

We write $a\sim\sum_{j=0}^\infty h^j a_j$, if $\forall N$,

$$a - \sum_{j=0}^{N-1} h^j a_j = O(h^N)_{S^{k-N}(\mathbb{R}^{2n})}$$

note we improve in $\langle\xi\rangle$ as well as in h.

We now revisit the calculus of §5:

Theorem (Composition Formula)

Let $a \in S^k(\mathbb{R}^{2n})$, $b \in S^\ell(\mathbb{R}^{2n})$. Then

$$Op_h(a)Op_h(b) = Op_h(a\#b), \quad a\#b \in S^{k+\ell}(\mathbb{R}^{2n}),$$

$$a\#b \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a \cdot \partial_x^\alpha b$$

Here the expansion is in $S^{k+\ell}$, so we get

Product Rule: $a\#b = ab + O(h)_{S^{k+\ell-1}}$

Commutator Rule: $a\#b - b\#a = -ih\{a,b\} + O(h^2)_{S^{k+\ell-2}}$

Why do we get an expansion with improved remainders?

An informal explanation is that the terms in the expansions decay faster in ξ, owing to the ξ-derivatives:

$$\partial_\xi^\alpha a \cdot \partial_x^\alpha b \in S^{k+\ell-|\alpha|}.$$

For the actual proof see Zworski's book, Theorem 9.5

Theorem (Adjoint Formula) Let $a \in S^k(\mathbb{R}^{2n})$. Then

$$Op_h(a)^* = Op_h(a^*), \quad a^* \in S^k(\mathbb{R}^{2n}),$$

$$a^* \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \bar{a} \leftarrow \text{expansion in } S^k$$

Adjoint Rule: $a^* = \bar{a} + O(h)_{S^{k-1}}$.

Theorem (Change of variables)

Assume that $U, V \subset \mathbb{R}^n$ are open sets, $\varphi : U \to V$ is a diffeomorphism, $\chi \in C_c^\infty(U)$, and $\tilde{\varphi}(x,\xi) := (\varphi(x), d\varphi(x)^{-T} \cdot \xi)$.
Let $a \in S^k(\mathbb{R}^{2n})$. Then

$$\chi \varphi^* Op_h(a) (\varphi^{-1})^* \chi = Op_h(b), \quad b \in S^k(\mathbb{R}^{2n}),$$

$$b \underset{\substack{\text{expansion} \\ \text{in } S^k}}{\sim} \sum_{j=0}^{\infty} h^j L_j (a \circ \tilde{\varphi})$$

L_j: diff. operator of order $2j$ mapping $S^k \to S^{k-j}$.

In particular

$$b(x,\xi;h) = \chi(x)^2 a(\tilde{\varphi}(x,\xi)) + O(h)_{S^{k-1}}.$$

Theorem (Pseudolocality)

Assume $a \in S^k(\mathbb{R}^{2n})$ and $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^{2n})$, $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$.
Then $\chi_1 Op_h(a) \chi_2 = O(h^\infty)_{S' \to S}$
namely it has the form $u \mapsto \int_{\mathbb{R}^n} K(x,y;h) u(y)\, dy$
where $\forall N\ \exists C_N\ \| \langle (x,y) \rangle^N K \|_{C^N_{x,y}(\mathbb{R}^{2n})} \le C_N h^N$

Proof Follows from the Composition Formula:
$\chi_1 Op_h(a) \chi_2 = Op_h(b)$ where $b = O(h^\infty)_{S(\langle x \rangle^{-N} \langle \xi \rangle^{-N})}$
for all N (the $\langle x \rangle^{-N}$ is because $\chi_1, \chi_2 \in C_c^\infty$) $\square$

Interpretation: $\operatorname{supp} u \subset U \Rightarrow Op_h(a) u = O(h^\infty)_{C^\infty}$ outside of U.