§ 2. METHOD OF STATIONARY PHASE

We will study asymptotics as $h \to 0$ of

$$I(h) := \int_U e^{\frac{i\varphi(y)}{h}} a(y)\, dy, \quad 0 < h \le 1,$$

NOTATION: $I(h)$ is an oscillatory integral

- $U \subset \mathbb{R}^n$ is an open set
- $\varphi \in C^\infty(U; \mathbb{R})$, called "phase function"
- $a \in C_c^\infty(U)$, called "amplitude"

recall this means $a \in C^\infty(\bar{U})$ & supp $a \subset U$ is compact.

Definition $y \in U$ is called a critical (stationary) point of φ, if $d\varphi(y) = 0$.

§ 2.1. Method of nonstationary phase

Theorem. Assume supp a has no critical points of φ. Then $I(h) = O(h^\infty)$, i.e.

$$I(h) = O(h^N) \quad \forall N.$$

Proof We repeatedly integrate by parts using

$$L: f(y) \mapsto -i \sum_{j=1}^n \frac{\partial_{y_j} \varphi(y)}{|d\varphi(y)|^2} \partial_{y_j} f(y)$$

L is a 1^{st} order differential operator on U.
(Here we shrank U so that supp $a \subset U$, φ has no critical points in $\bar{U}$)

We compute

$$L\varphi(y) = -i\sum_{j=1}^{n} \frac{\partial_{y_j}\varphi(y)}{|d\varphi(y)|^2}\,\partial_{y_j}\varphi(y) = -i, \quad \text{thus}$$

$$e^{\frac{i\varphi}{h}} = h\cdot L\left(e^{\frac{i\varphi}{h}}\right).$$

We have for each $f \in C_c^\infty(U)$,

$$\text{(IBP)} \quad \int_U e^{\frac{i\varphi}{h}} f\,dy = h\int_U L\left(e^{\frac{i\varphi}{h}}\right) f\,dy = h\int_U e^{\frac{i\varphi}{h}} (L^t f)\,dy$$

where L^t is the first order differential operator given by $L^t f(y) = i\sum_{j=1}^{n} \partial_{y_j}\left(\frac{\partial_{y_j}\varphi(y)}{|d\varphi(y)|^2} f(y)\right)$

(the transpose of L, see exercises)

Now we apply (IBP) N times:

$$|I(h)| = h^N \left| \int_U e^{\frac{i\varphi}{h}} ((L^t)^N a)\,dy \right| = O(h^N). \quad \square$$

§2.2. Stationary phase

Gives a full expansion of $I(h)$ up to $O(h^\infty)$ remainder ~~for~~ under a nondegeneracy assumption.

Definition A critical point y_0 of φ is called nondegenerate, if the Hessian $d^2\varphi(y_0) = (\partial^2_{y_j y_k}\varphi(y_0))_{j,k=1}^{n}$ is invertible.

If all critical points are nondegenerate, we call φ a Morse function.

If y_0 is a nondegenerate critical point, then $d^2\varphi(y_0)$ has k positive & $n-k$ negative eigenvalues for some k. We define

$$\operatorname{sgn} d^2\varphi(y_0) := k - (n-k).$$

(signature of the Hessian)

The following result handles the contribution of a single nondegenerate critical point. (See exercises for how to use it to get an expansion for $I(h)$ when φ is any Morse function.)

Theorem (Method of stationary phase)

Assume φ, a are as above and φ has only one critical point y_0 ~~on~~ in $\overset{\circ}{\text{supp}}\, a$, which is moreover nondegenerate. Then as $h \to 0$,

$$\boxed{I(h) \sim e^{\frac{i\varphi(y_0)}{h}} \sum_{j=0}^{\infty} h^{\frac{n}{2}+j} L_{\varphi,j}\, a(y_0)} \qquad \text{(STPh)}$$

where each $L_{\varphi,j}$ is a differential operator of order $2j$ on U depending on φ, but not on a, and the leading term is given by

$$\boxed{L_{\varphi,0}\, a(y_0) = (2\pi)^{\frac{n}{2}} \left|\det d^2\varphi(y_0)\right|^{-\frac{1}{2}} e^{\frac{i\pi}{4}\operatorname{sgn} d^2\varphi(y_0)}\, a(y_0)}$$

__Note:__ one has a formula for all $L_{\varphi,j}$, see e.g. Hörmander, Vol I, Theorem 7.7.5 or Zworski, (3.4.11) (in dimension 1)

Remark: the expansion (STPh) is an asymptotic one: namely

$$\forall N, \quad \left| I(h) - \sum_{j=0}^{N-1} (\cdots) \right| = O(h^{\frac{n}{2}+N}).$$

Typically the series $\sum_{j=0}^{N-1} (\cdots)$ does not converge for any fixed $h>0$. Such style asymptotic expansions will appear a lot in the course.

In fact, we can say more about the remainder:

$$\forall N, \forall h \in (0,1], \ \left| I(h) - \sum_{j=0}^{N-1} (\cdots) \right| \leq C_{N,\varphi,y_0} \|a\|_{C^{2N+n+1}} h^{\frac{n}{2}+N}$$

where C_{N,φ,y_0} is some constant depending only on N, φ, y_0, (and on supp a) and $\|a\|_{C^{2N+n+1}} := \max_{|\alpha| \leq 2N+n+1} \sup |\partial^\alpha a|$.

See Zworski, Theorem 3.16
Hörmander, Vol I, Theorem 7.7.5

§2.3. Quadratic stationary phase

Here we consider a special case of the method of stationary phase, useful for the proof of the general case:

$$I(h) = \int_{\mathbb{R}^n} e^{\frac{i}{2h}\langle Qy,y\rangle} a(y)\, dy$$

where

- Q is an invertible symmetric real $n \times n$ matrix
- $a \in C_c^\infty(\mathbb{R}^n)$.

Note: $\varphi(y) = \frac{1}{2}\langle Qy, y\rangle$ is a Morse fn., the only critical point is $y=0$, and

$d^2\varphi(0) = Q$.

Zworski, Theorem 3.13

Theorem (Quadratic stationary phase)

We have $$I(h) \underset{h\to 0}{\sim} (2\pi h)^{\frac{n}{2}} \frac{e^{\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}} \sum_{j=0}^{\infty} \frac{h^j}{j!} \left(\frac{\langle Q^{-1} D_y, D_y\rangle}{2i}\right)^j a(0)$$

Here for an $n\times n$ matrix A we write

$$\langle A D_y, D_y\rangle = \sum_{k,\ell=1}^{n} A_{k\ell} D_{y_k} D_{y_\ell}, \quad D_{y_k} := -i\partial_{y_k}$$

The expansion has the same remainder estimate as the general stationary phase: $\forall N$

$$\left| I(h) - \sum_{j=0}^{N-1}(\dots) \right| \leq C_{N,Q}\, h^{\frac{n}{2}+N} \|a\|_{C^{2N+n+1}}$$

$C_{N,Q}$ depends on N, Q, and a compact set containing supp a.

Proof 1. We first express $I(h)$ in terms of the Fourier transform $\hat{a}$.

We only give here an outline, details might appear in the Fourier Transform Course

Start from the 1D formula: $\forall a \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} e^{-\frac{\lambda y^2}{2}} a(y)\,dy = (2\pi\lambda)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{\eta^2}{2\lambda}} \hat{a}(\eta)\,d\eta$$

(because the Fourier transform of $e^{-\frac{\eta^2}{2\lambda}}$ is $(2\pi\lambda)^{1/2} e^{-\frac{\lambda y^2}{2}}$)

Arguing by analytic continuation in λ

(putting $\lambda = \mp \frac{i\mu}{h}$), get for $\mu > 0$,

$$\int_{\mathbb{R}} e^{\pm \frac{i\mu y^2}{2h}} a(y)\, dy = \frac{h^{1/2}}{\sqrt{2\pi}} \cdot \frac{e^{\pm \frac{i\pi}{4}}}{\sqrt{\mu}} \int_{\mathbb{R}} e^{\mp \frac{ih\eta^2}{2\mu}} \hat{a}(\eta)\, d\eta$$

where $e^{\pm\frac{i\pi}{4}}$ come from taking $\left(\mp \frac{i\mu}{h}\right)^{-1/2}$.

Replacing the number μ by a matrix Q (e.g. taking tensor products + diagonalizing Q) we get

$\forall\, a \in \mathcal{S}(\mathbb{R}^n)$,

$$(\star) \quad \int_{\mathbb{R}^n} e^{\frac{i\langle Qy, y\rangle}{2h}} a(y)\, dy = \left(\frac{h}{2\pi}\right)^{\frac{n}{2}} \frac{e^{\frac{i\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}} \cdot J(h)$$

where $\displaystyle J(h) := \int_{\mathbb{R}^n} e^{\frac{h}{2i}\langle Q^{-1}\eta, \eta\rangle} \hat{a}(\eta)\, d\eta$

2. We now use the Taylor expansion of $e^{\frac{h}{2i}\langle Q^{-1}\eta,\eta\rangle}$ at $h \to 0$.

Here we do it formally, see exercises for how to get the precise remainder

$$e^{\frac{h}{2i}\langle Q^{-1}\eta,\eta\rangle} \underset{h\to 0}{\sim} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{h}{2i}\langle Q^{-1}\eta,\eta\rangle\right)^j \Rightarrow$$

$$\Rightarrow J(h) \underset{h\to 0}{\sim} \sum_{j=0}^{\infty} \frac{h^j}{j!} \underbrace{\int_{\mathbb{R}^n} \left(\frac{\langle Q^{-1}\eta,\eta\rangle}{2i}\right)^j \hat{a}(\eta)\, d\eta}_{\displaystyle = (2\pi)^n \left(\frac{\langle Q^{-1} D_y, D_y\rangle}{2i}\right)^j a(0)}$$

by properties of Fourier transform.

This gives the quadratic stationary phase expansion. $\square$

§2.4. Proof of general stationary phase

Reduces to quadratic stationary phase using

Theorem (Morse Lemma) Assume that $\varphi \in C^\infty(U;\mathbb{R})$ has a nondegenerate stationary point at $y_0 \in U$, and $d^2\varphi(y_0)$ has signature $k-(n-k)$. Then there exist:

- a neighborhood U' of y_0, and
- a diffeomorphism $F: V \to U'$, such that

$$\forall x \in V,\ \varphi(F(x)) = \varphi(y_0) + \tfrac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2)$$

Moreover, $\det dF(0) = |\det d^2\varphi(y_0)|^{-1/2}$.

Proof See Zworski, Theorem 3.15. $\square$

Returning to general stationary phase, write

$$I(h) = \int_U e^{\frac{i}{h}\varphi(y_0)} a(y)\,dy = \underbrace{\int_{U'} e^{\frac{i}{h}\varphi(y)} a_1(y)\,dy}_{= I_1(h)} + \underbrace{\int_U e^{\frac{i}{h}\varphi(y)} a_2(y)\,dy}_{= I_2(h)}$$

where $a = a_1 + a_2$, $\operatorname{supp} a_1 \subset U'$, $\operatorname{supp} a_2 \subset \operatorname{supp} a \setminus \{y_0\}$.

By nonstationary phase, $I_2(h) = O(h^\infty)$.

Changing variables in I_1, get

$$I_1(h) = \int_V e^{\frac{i}{h}\varphi(F(x))} a_1(F(x))\,|\det dF(x)|\,dx$$

can be handled by quadratic stationary phase!