THE ARITHMETIC STRUCTURE OF THE SPECTRUM OF A METRIC GRAPH

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"MICROLOCAL TO GLOBAL ANALYSIS"

MELROSE 75

JOINT WORK WITH PAVEL KURASOV.

GRAPH LAPLACIAN ;

G A FINITE CONNECTED GRAPH.

$V(G)$ ITS VERTICES ; $E(G)$ ITS EDGES

$|V(G)| = M$, $|E(G)| = N$.

THE LAPLACIAN Δ ON $\ell^2(V(G))$ IS GIVEN BY

$$\Delta f(v) = d(v) f(v) - \sum_{w \sim v} f(w)$$

$d(v)$ = DEGREE OF v.

$$\mathrm{SPEC}_\Delta(G) = \{ 0 < \lambda_1 \leq \lambda_2 \dots \leq \lambda_{M-1} \}$$

- THE $\lambda_j > 0$ ARE TOTALLY POSITIVE ALGEBRAIC INTEGERS WITH THEIR GALOIS CONJUGATES ALSO IN THE SET.

- THIS IS EXPLOITED TO PROVE THEOREMS IN ALGEBRAIC GRAPH THEORY.

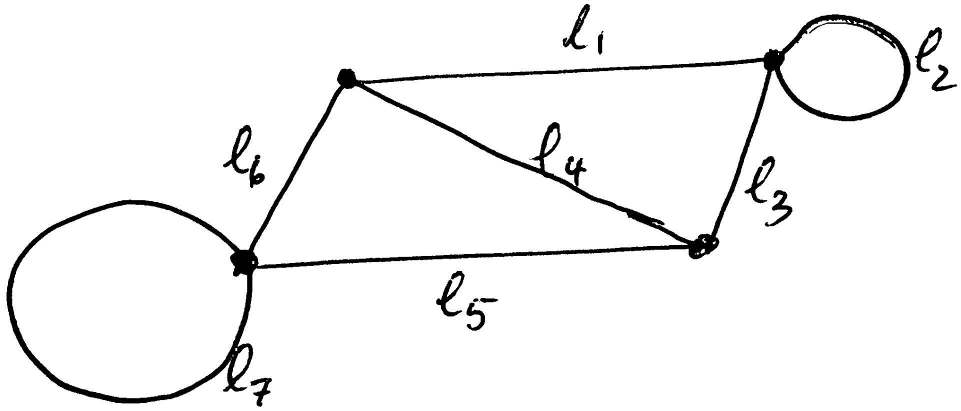
FOR EXAMPLE IN PRESCRIBING GAPS IN THE SPECTRA WHERE FEKETE'S THEOREM IMPOSES RESTRICTIONS (ALICIA KOLLAR — S 2021).

METRIC GRAPHS

THESE HAVE BEEN STUDIED SINCE THE 1930'S BY CHEMISTS, ENGINEERS PHYSICISTS AND MATHEMATICIANS.

KURASOV THINKS OF THEM AS VIBRATING SPIDERS.

$\Gamma_\ell :=$



G IS THE UNDERLYING GRAPH

FOR EACH EDGE OF G WE ASSIGN A LENGTH ℓ_j

$\ell_1, \ldots, \ell_N$ THE LENGTH SET

THE METRIC GRAPH $\Gamma = \Gamma_\ell$ ON G IS THIS SINGULAR (AT THE VERTICES) ONE DIMENSIONAL RIEMANNIAN MANIFOLD.

- G CARRIES THE TOPOLOGY OF Γ

$\pi_1(G)$ IS A FREE GROUP ON

$\beta_1(G) := N - M + 1$ GENERATORS

$H_1(G)$ IS A FREE ABELIAN GROUP ON $\beta_1(G)$ GENERATORS.

SCATTERING MATRIX OF G

ORIENT THE EDGES OF G TO GET $2N$ ORIENTED EDGES. DEFINE THE $2N \times 2N$ 'SCATTERING' MATRIX WHERE THE ROWS AND COLUMNS ARE LABELLED BY THE ORIENTED EDGES

$$S = (S_{fg}) \quad \text{WHERE} \quad S_{fg} = \begin{cases} -\delta_{fg} + \dfrac{2}{\deg(v)} & \text{IF } g \text{ FOLLOWS } f \text{ THROUGH THE VERTEX } v \\ 0 & \text{OTHERWISE} \end{cases}$$

NOTE.

- A VERTEX v OF DEGREE 2 IS A REMOVABLE SINGULARITY FOR Γ, SO WE ASSUME $d(v) \neq 2$ FOR ALL v.

- S IS UNITARY !

S IS A CENTRAL PLAYER IN STUDYING THE LAPLACIAN ON Γ.

LAPLACIAN ON Γ : "Δ"

IT IS $-\frac{d^2}{dx^2}$ ON THE EDGES OF Γ;

AT THE VERTICES WE IMPOSE NEUMANN BOUNDARY CONDITIONS: IF $\phi : \Gamma \to \mathbb{C}$

- ϕ IS CONTINUOUS ON Γ AND AT EACH v.
- $\sum_e \partial_e \phi(v) = 0$, WHERE THE SUM IS OVER ALL DIRECTED EDGES ENDING AT v, ∂_e IS DERIVATIVE ALONG THE EDGE.
- $-\frac{d^2}{dx^2}\phi(x) = k^2\phi(x)$ ON INTERIOR OF EDGES.

Δ IS SELF ADJOINT ON $L^2(\Gamma)$ AND HAS DISCRETE SPECTRUM $0 < \lambda_1 \leq \lambda_2 \cdots$, $\lambda_j \to \infty$.

SET

$$\mathrm{SPEC}_\Delta(\Gamma) := \{ \underbrace{0,0,\ldots,0}_{\beta_1+1 \text{ TIMES}}, \pm\sqrt{\lambda_j}, j \geq 1 \}$$

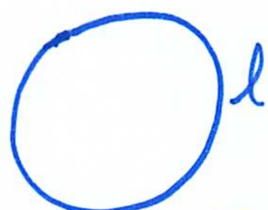
- THE ADJUSTMENT OF ZERO HAVING MULTIPLICITY EQUAL TO $\beta_1(G)+1$ MAKES THE TRACE FORMULA EXACT.

WEYL'S LAW: FOR T LARGE

$$\mathrm{SPEC}(\Gamma) \cap [-T,T] = \frac{2(\ell_1+\ell_2+\cdots\ell_N)}{\pi}T + O(1)$$

SO THE MULTIPLICITES ARE UNIFORMLY BOUNDED AND THE SPECTRUM IS A BOUNDED PERTURATION OF AN ARITHMETIC PROGRESSION.

EXAMPLE:



SMOOTH CIRCLE OF LENGTH ℓ.

$$\phi_m(x) = e^{2\pi i m x/\ell}, \quad m \in \mathbb{Z}$$ IS AN EIGENFUNCTION

$\mathrm{SPEC}(S_\ell)$ = THE ARITHMETIC PROGRESSION $\frac{2\pi m}{\ell}$, $m \in \mathbb{Z}$ EACH WITH MULTIPLICITY 2

IF $\mu := \sum_{k \in \mathrm{SPEC}(S_\ell)} \delta_k$; δ_ξ BEING A POINT MASS AT ξ

THEN $\hat{\mu}$ IS UP TO SCALE ITSELF.

THE POISSON SUMMATION FORMULA ASSERTS THAT

$$\widehat{\sum_{k \in \mathbb{Z}} \delta_k} = \sum_{m \in \mathbb{Z}} \delta_m .$$

TRACE FORMULA (ROTH, KOTTOS-SMILANSKY, KURASOV, ...)

Γ A METRIC GRAPH

$$\mu = \sum_{k \in \mathrm{SPEC}(\Gamma)} \delta_k \quad ; \quad \mu \text{ IS A POSITIVE TEMPERED MEASURE ON } \mathbb{R}.$$

$$\hat{\mu} = \frac{2(l_1+l_2+\cdots l_N)}{\pi}\delta_0 + \frac{1}{\pi}\sum_{p \in \underline{P}} \ell(\mathrm{prim}\, p)\left[S(p)\left(\delta_{\ell(p)} + \delta_{-\ell(p)}\right)\right]$$

WHERE:

- $\underline{P}$ IS THE SET OF ORIENTED PERIODIC PATHS IN G UP TO CYCLIC EQUIVALENCE (BACK-TRACKING IS ALLOWED)
- $\ell(p)$ IS THE LENGTH OF THE PATH
- PRIM (p) IS THE PRIMITIVE PART OF p (GOING AROUND ONCE)
- $S(p)$ IS THE PRODUCT OF THE SCATTERING COEFFICIENTS ENCOUNTERED WHEN TRAVERSING p.

$\hat{\mu}$ IS A SUM OF POINT MASSES SUPPORTED AT THE $\ell(p)$'s WHICH FORM A DISCRETE SUBSET OF $\mathbb{R}$ AS THEY ARE CONTAINED IN

$$\{ m_1 \ell_1 + m_2 \ell_2 + \dots + m_N \ell_N : m_j \geq 0 \text{ in } \mathbb{Z} \}$$

• ONE CAN SHOW THAT $|\hat{\mu}|$ IS ~~A~~ TEMPERED MEASURE: THE PAIR $\mu, \hat{\mu}$ IS A POSITIVE FOURIER QUASICRYSTAL (MEYER, LEV-OLEVSKII)

$$\mu = \sum_{k \in \Lambda} a_k \delta_k \quad , \quad \hat{\mu} = \sum_{\nu \in L} b_\nu \delta_\nu \quad , \qquad \text{——} (*)$$

$a_k \geq 0$ AND Λ AND L DISCRETE IN $\mathbb{R}$.

(*) GIVES A GENERALIZED POISSON SUMMATION FORMULA.

WE WILL SHOW THAT THESE COMING FROM THE SPECTRA OF METRIC GRAPHS ARE EXOTIC BEING FAR FROM ARITHMETIC PROGRESSIONS (DIRAC COMBS) AND RESOLVE VARIOUS QUESTIONS IN THE THEORY (MEYER, LEV-OLEVSKI, LAGARIAS, ..)

LOOPS

GIVEN A LOOP IN G OF LENGTH ℓ

SET $$\phi_j(x) = \begin{cases} \sin\left(\frac{2\pi j x}{\ell}\right) & \text{ON THE LOOP} \\ 0 & \text{ELSEWHERE} \end{cases}$$

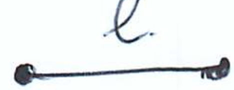
THEN $\Delta \phi_j = \frac{4\pi^2 j^2}{\ell^2} \phi_j$ AND WE GET THE ARITHMETIC PROGRESSION;

$$\frac{2\pi j}{\ell} \; ; \; j \in \mathbb{Z} \quad \text{IN SPEC}(\Gamma).$$

THAT IS EACH LOOP OF LENGTH ℓ_j FOR $j = 1, \ldots, \nu$ IF THERE ARE ν LOOPS, PRODUCES SUCH A PROGRESSION L_j. WE HAVE A DECOMPOSITION ACCORDING TO LOOPS

$$\text{SPEC}(\Gamma) = L_1 \sqcup L_2 \ldots \sqcup L_\nu \sqcup I(\Gamma)$$

WHERE THE UNION IS WITH MULTIPLICITIES AND THE LEFT OVER SPECTRUM $I(\Gamma)$ IS IRREGULAR AS WE WILL SHOW.

WE AVOID THE GRAPHS 

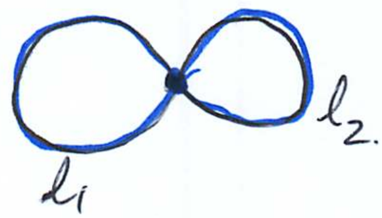
AND 

FIGURE EIGHT

WHOSE $I(\Gamma)$'S ARE ARITHMETIC PROGRESSIONS

$$\text{SPEC (FIGURE EIGHT)} = \left\{ \frac{2\pi j_1}{\ell_1} \right\}_{j_1 \in \mathbb{Z}} \sqcup \left\{ \frac{2\pi j_2}{\ell_2} \right\}_{j_2 \in \mathbb{Z}} \sqcup \left\{ \frac{2\pi j_3}{\ell_1 + \ell_2} \right\}_{j_3 \in \mathbb{Z}}$$

NOTATION :

TO STUDY $\mathbb{Q}$ LINEAR RELATIONS OF THE SPECTRA WE VIEW $\mathbb{R}$ AS A VECTOR SPACE OVER $\mathbb{Q}$.

- FOR $B \subset \mathbb{R}$, $\mathrm{DIM}_{\mathbb{Q}} B$ IS THE MAXIMAL SIZE OF A $\mathbb{Q}$-LINEARLY INDEPENDENT SUBSET OF B.

- THE TRANSCEDENCE DEGREE OVER $\mathbb{Q}$ OF A SUBSET B OF $\mathbb{R}$, IS THE SIZE MAXIMAL $\mathbb{Q}$-ALGEBRAICALLY INDEPENDENT SUBSET OF B.

- IF $\text{DIM}_{\mathbb{Q}}\{l_1,\dots,l_N\}=1$ THAT IS $(l_1,\dots,l_N)$ IS PROPORTIONAL TO A RATIONAL VECTOR, THEN $\text{SPEC}(\Gamma)$ IS A UNION OF ARITHMETIC PROGRESSIONS.

- WE ASSUME HENCEFORTH AS IS CUTOMARY THAT $l_1,\dots,l_N$ ARE LINEARY INDEPENDENT OVER $\mathbb{Q}$.

THEOREM 1: (KURASOV-S) $\mathbb{Q}$-LINEAR RELATIONS IN SPEC

THERE IS $C=C(N)$ (EFFECTIVE AND ITERATED EXPONENTIAL IN N) SUCH THAT FOR ANY METRIC GRAPH Γ_l AND $k_1,\dots,k_r$ r-DISTINCT POSITIVE ELEMENTS IN $I(\Gamma)$

$$\text{DIM}_{\mathbb{Q}}(k_1,\dots,k_r) \geq \frac{\log r}{C(N)}.$$

COROLLARY 1:

(a) $\text{DIM}_{\mathbb{Q}}(I(\Gamma))=\infty$

(b) FOR ANY ARITHMETIC PROGRESSION P IN $\mathbb{R}$

$$|I(\Gamma)\cap P| \leq B=B(N).$$

TRANSCENDENCE OF THE SPECTRUM

A SIMPLE CONSEQUENCE OF WEIESTRASS' EXTENSION OF LINDEMANN'S THEOREM :

(IF $z_1, \dots, z_N$ ARE ALGEBRAIC AND LINEARLY INDEPENDENT OVER $\mathbb{Q}$ THEN

$$\mathrm{TRANSDEG}_{\mathbb{Q}} \{ e^{z_1}, \dots e^{z_N} \} = N \text{)}$$

IS :

PROPOSITION: IF Γ_ℓ IS A METRIC GRAPH WITH ALGEBRAIC ($\mathbb{Q}$-LINEARLY INDEPENDENT) LENGTHS THEN EVERY NON-ZERO MEMBER OF SPEC(Γ) IS TRANSCENDENTAL.

SCHANUEL'S CONJECTURE

IF $z_1, \dots, z_N \in \mathbb{C}$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$ THEN

$$\mathrm{TRANSDEG}_{\mathbb{Q}} \{ z_1, \dots, z_N, e^{z_1}, \dots, e^{z_N} \} \geq N .$$

COROLLARY 2 (K-S) ALGEBRAIC INDEPENDENCE OVER $\mathbb{Q}$

ASSUME SCHANUEL'S CONJECTURE, THEN IF THE LENGTHS OF A METRIC GRAPH Γ ARE ALGEBRAIC AND LINEARLY INDEPENDENT OVER $\mathbb{Q}$, FOR $k_1, \dots, k_r$ DISTINCT POSITIVE MEMBERS OF $I(\Gamma)$

$$\mathrm{TRANSDEG}_{\mathbb{Q}} \{k_1, \dots, k_r\} \geq \frac{\log r}{\mathrm{DEG}(\ell) \cdot C(N)}$$

WHERE $\mathrm{DEG}(\ell)$ IS THE DEGREE OF THE EXTENSION $K = \mathbb{Q}[\ell_1, \dots, \ell_N]$.

COROLLARY 3 UNDER THE SAME ASSUMPTIONS

$$\mathrm{TRANSDEG}_{\mathbb{Q}} (I(\Gamma)) = \infty.$$

• HOW TO COMPUTE THE SPECTRUM ?

BARRA AND GASPARD'S SECULAR VARIETY :

$$T = T^N = (\mathbb{C}^*)^N \qquad \text{COMPLEX } N\text{-TORUS}$$

THE SECULAR POLYNOMIAL $\mathbb{P} = \mathbb{P}_G$

LET $U(z_1, \ldots, z_N)$ BE THE $2N \times 2N$ DIAGONAL MATRIX ON THE ORDERED EDGES

$$U(z_1, \ldots, z_N)_{fg} = z_f \, \delta_{fg} \qquad f, g \text{ ORDERED EDGES}$$

SET

$$\mathbb{P}(z_1, \ldots, z_N) := \mathrm{DET}_{2N \times 2N}\left(I - U(z_1, \ldots, z_N) S \right)$$

KEY PROPERTIES :

(a) $\mathbb{P}_G(z)$ IS OF DEGREE $2N$ AND DEGREE TWO IN EACH z_j.

(b) THE PAIR $\mathbb{P}(z_1, \ldots, z_N)$ AND $\mathbb{P}^{\iota}(z_1, \ldots, z_N) = \mathbb{P}(1/z_1, \ldots, 1/z_N)$ ARE BOTH STABLE, THAT IS THEY DON'T VANISH FOR ANY z WITH $|z_j| < 1$ FOR ALL j.

FOLLOWS FROM THE UNITARITY OF S !

THE CONNECTION TO SPEC(Γ) IS

$$\text{SPEC}(\Gamma) = \{\text{ZEROS WITH MULTIPLICITIES}\} \text{ OF}$$

$$k \longmapsto P_G(e^{ikl_1}, e^{ikl_2}, \ldots, e^{ikl_N})$$

(IT HAS REAL ROOTS ONLY)

THE ALGEBRAIC SECULAR VARIETY

$$Z_G = \{ z \in T^N : P_G(z) = 0 \} \subset T^N$$

PLAYS A CENTRAL ROLE.

• THE 'REAL' N-1 DIMENSIONAL TORUS

$$\Sigma = Z_G \cap (U(1))^N \qquad U(1) = \{ |z| = 1 \}$$

AND ITS DIFFERENTIAL GEOMETRY PLAYS A ROLE IN THE STUDY OF THE EIGENFUNCTIONS:

(A) BARRA-GASPARD USE IT AND THE EQUIDISTRIBUTION OF IRRATIONAL LINEAR FLOWS ON $U(1)^N$ TO STUDY THE CONSECUTIVE SPACING STATISTICS IN SPEC(Γ).

(B) COLIN-DE-VERDIERE USES THE GAUSS MAP ON Σ_i TO STUDY ALL POSSIBLE QUANTUM LIMITS ON Γ.

(C) ALON-BAND-BERKOLAIKO IDENDIFY FURTHER CODIMENSION ONE (REAL) ALGEBRAIC SUBSETS OF Σ_i THAT GOVERN THE NODAL COUNTS.

FOR US IT IS THE DIOPHANTINE/ALGEBRAIC GEOMETRY OF Z_G THAT IS CRITICAL.

<u>THEOREM 2</u> (KURASOV-S)

(a) $P_G(z) = Q_G(z) \prod_{e \text{ A LOOP}} (z_e - 1)$, WITH

$Q_G(z)$ ABSOLUTELY IRREDUCIBLE.

(b) Z_Q DOES NOT CONTAIN A TRANSLATE OF AN $(N-1)$ DIMENSIONAL SUBTORUS.

(*) (a) WAS CONJECTURED BY COLIN-DE-VERDIERE

(xx). THERE EXCEPTIONS TO THEOREM 2; THE FIGURE 8 ; AND  "WATERMELLONS".

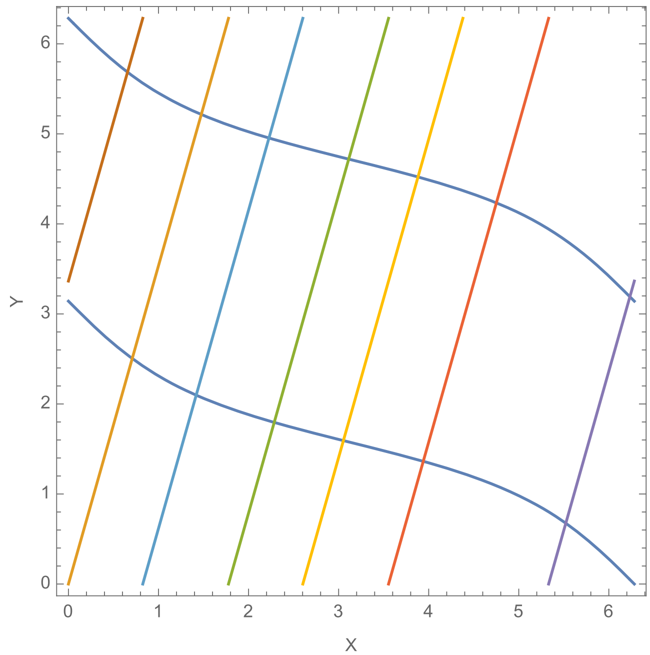


FIG. 1. Zero set for $L(x, y)$.

distance between the intersection points and the origin measured along the line. It is clear that $L(-x, -y) = -L(x, y)$ [which also follows from (15) and the fact that $F = G$ in the current example], implying that the zeroes are symmetric with respect to the origin.

The summation formula (27) takes the form

$$\sum_{\gamma_j} \hat{h}(\gamma_j) = (\xi_1 + 2\xi_2)h(0) - \sum_{\mathbf{n}=(n_1,n_2)\in\mathbb{Z}_+^2} c(n_1, 2n_2)(n_1\xi_1 + 2n_2b_2)(h(n_1\xi_1 + 2n_2\xi_2) + h(-(n_1\xi_1 + 2n_2\xi_2))), \tag{44}$$

where

- γ_j are solutions to the secular Eq. (43),
- $c(n_1, 2n_2)$ are given by (42), and
- $h \in C_0^\infty(\mathbb{R})$ is an arbitrary test function.

The difference between formula (44) and the general formula (27) is due to the fact that the stable polynomials just depend on z_2^2.

Both series on the left- and right-hand sides are infinite, but they have different properties depending on whether ξ_1 and ξ_2 are rationally dependent or not. This is related to the number of intersection points on the torus. The number of zeroes $i\gamma_j$ is also always infinite, and the number of intersection points on the torus may be finite. Indeed, if $\frac{\xi_1}{\xi_2} \in \mathbb{Q}$, then the line is periodic on the torus, implying that there are finitely many intersection points (on the torus). The points γ_j form a periodic sequence, implying that the obtained summation formula is just a finite sum of Poisson summation formulas with the same period and μ is a generalized Dirac comb.

Next, we assume that ξ_1 and ξ_2 are rationally independent,

$$\frac{\xi_1}{\xi_2} \notin \mathbb{Q}. \tag{45}$$

By Kronecker's theorem, the line covers the torus densely, and therefore, the intersection points $(\gamma_j\xi_1, \gamma_j\xi_2)$ cover densely the zero curve of L as well. We are interested in the rational dependence of $\gamma_j, j \in \mathbb{Z}$. In particular, we shall need the following:

Lemma 1. If ξ_1 and ξ_2 are rationally independent, then the secular Eq. (43),

$\Longrightarrow$ IF G CONTAINS NO LOOPS THEN ACCORDING TO THEOREM 1 $\mathrm{SPEC}(\Gamma) = \mathcal{I}(\Gamma)$ CONTAINS NO ARITHMETIC PROGRESSIONS OF LENGTH MORE THAN A FIXED NUMBER AND THIS SET IS VERY FAR FROM BEING A DIRAC COMB.

- WITH THE TRACE FORMULA THESE PROVIDE EXOTIC POSITIVE FOURIER QUASI-CRYSTALS.

- IN FACT ONE CAN USE ANY PAIR OF STABLE "LEE-YANG" POLYNOMIALS $P' = Q$ TO CARRY OUT SUCH CONSTRUCTIONS (K-S).

- REMARKABLY ONE CAN SHOW THE CONVERSE: OLEVSKII – ULANOVSKII (2021) LIOR ALON – CYNTHIA VINZANT – ALEX COHEN (2023):

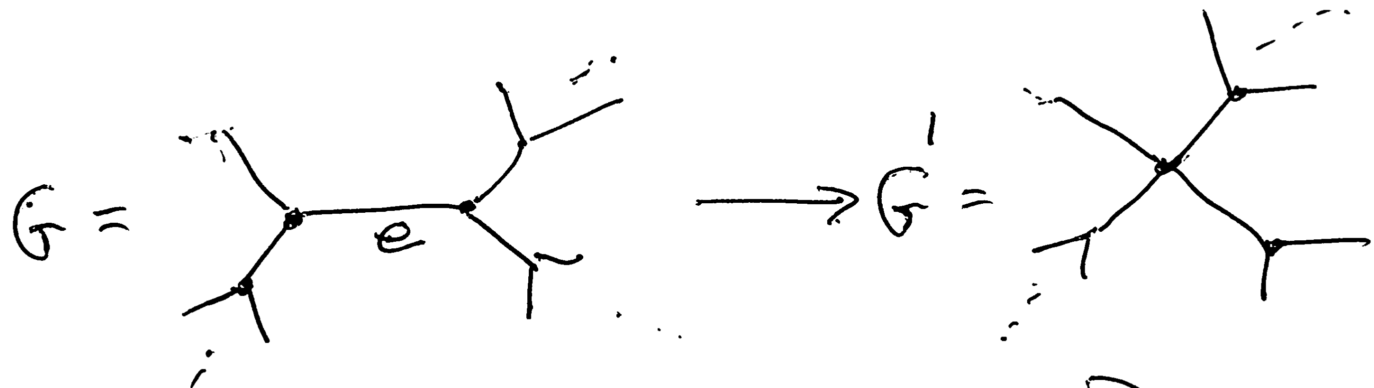
EVERY POSITIVE FOURIER QUASI-CRYSTAL μ WITH INTEGER COEFFICIENTS ARISES FROM SUCH A CONSTRUCTION WITH LEE-YANG POLYNOMIALS.

- THIS CLASSIFICATION DEMONSTRATES THE CENTRALITY OF SEVERAL VARIABLE HYPERBOLICITY IN FOURIER QUASI CRYSTALS (DYSON'S QUESTION)

OUTLINE OF PROOFS

- THEOREM 2 IS PURELY COMBINATORIAL INVOLVING AN ARBITRARY G AND ITS SECULAR POLYNOMIAL P_G.

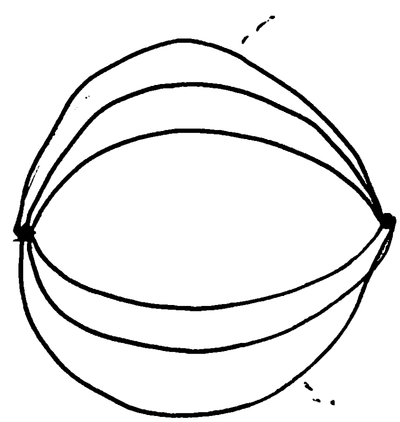
- THE PROOF IS BY REDUCTION OF THE SIZE OF G.
  VARIATIONS ON THE OPERATION OF EDGE CONTRACTION



REDUCE G TO G' WITH $P_{G'}$ OBTAINED FROM P_G BY SPECIALIZATION OF THE CORRESPONDING VARIABLE. THIS ALLOWS ONE TO COMPARE THE IRREDUCIBILITY OF P_G TO $P_{G'}$.

ONE AIMS TO NAVIGATE IN THIS WAY DOWN TO A FIXED SMALL LIST OF G' WHOSE P's ARE IRREDUCIBLE.

THIS STRATEGY DOES NOT QUITE WORK BUT A MODIFICATION WHICH ALLOWS SOME LIMITED EXTENSION OPERATIONS DOES. THESE PICK UP ALONG THE WAY THE WATERMELLON GRAPHS



FOR WHICH THE THEOREM FAILS BUT A SIMPLE MODIFICATION IS TRUE.

• THE PROOF OF THEOREM 1 MAKES USE OF ADVANCED DIOPHANTINE THEOREMS FOR TORI; SPECIFICALLY QUANTITATIVE VERSIONS OF "LANG'S "$\mathbb{G}_m$" CONJECTURES WHOSE RESOLUTION DEPEND ON W. SCHMIDT'S SUBSPACE THEOREM. THE LATTER IS A FAR REACHING SEVERAL VARIABLE EXTENSION OF THE THEOREMS OF THUE-SIEGEL-ROTH IN DIOPHANTINE APPROXMATION.

LET $W \subset T^N$ BE A SUBVARIETY AND Λ A FINITELY GENERATED SUBGROUP OF T^N.

THE <u>DIVISION</u> GROUP $\overline{\Lambda}$ OF Λ IS

$$\overline{\Lambda} = \{ \beta \in T^N : \beta^\nu \in \Lambda \text{ FOR SOME } \nu \geq 1,\ \nu \text{ AN INTEGER} \}$$

WE ARE CONCERNED WITH $\overline{\Lambda} \cap W$.

"LANG'S G_m" ASSERTS THAT THIS INTERSECTION IS LIMITED TO A FINITE NUMBER OF TRANSLATES OF SUBTORI CONTAINED IN W.

THE ULTIMATE UNIFORM VERSION IS DUE TO EVERTSE-SCHLICKEWEI-SCHMIDT.

THEOREM: THERE IS $C(W) < \infty$ (EFFECTIVE!) SUCH THAT IF Λ IS AS ABOVE, THEN THERE t TRANSLATES OF SUBTORI $B_1, \ldots, B_t$ ~~SUCH~~ CONTAINED ~~THAT~~ IN W SUCH THAT:

$$\overline{\Lambda} \cap W = \overline{\Lambda} \cap (B_1 \cup B_2 \ldots \cup B_t)$$

AND $t \leq C(W)^{\mathrm{RANK}(\Lambda)+1}$.

THE THEOREM SHOWS THAT THE INTERSECTION IS CONTROLLED BY LINEAR STRUCTURE!

IN OUR MAIN APPLICATION

$$W = Z_{Q_G}$$

IF $k_1, k_2, \ldots, k_r \in I(\Pi)$ SET
$$\rho(k_j) = \left(e^{ik_j \ell_1}, \ldots, e^{ik_j \ell_N}\right) \in W.$$

THE FOLLOWING LEMMAS ARE KEY AND ALLOW ONE TO COMBINE THE DIOPHANTINE THEOREM WITH THEOREM 2 AND GIVE LOWER BOUNDS FOR DIMENSIONS OF THE $\mathbb{Q}$ SPAN OF $k_1, \ldots, k_r$.

<u>LEMMA 1</u> THERE IS A SUBGROUP Λ OF RANK AT MOST $\mathrm{DIM}_{\mathbb{Q}}(k_1, \ldots, k_r)$ FOR WHICH $\overline{\Lambda}$ CONTAINS $\rho(k_1), \ldots, \rho(k_r)$.

<u>LEMMA 2</u> IF $\ell_1, \ell_2, \ldots, \ell_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$ AND B IS A TRANSLATE OF A SUBTORUS OF T^N OF DIMENSION AT MOST $N-2$ THEN

$$\left|\{k \in \mathbb{R} : \rho(k) \in B\}\right| \leq 1.$$

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