On a problem of conformal fill in by Poincare Einstein metrics

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§1. Conformal fill ins by Einstein manifolds

Given a compact manifold (M^n, h), when is it the boundary of a conformally compact Einstein manifold (X^{n+1}, g^+) with $\rho^2 g^+|_M = h$, where ρ is a defining function on X? This problem of finding "conformal fill in" is motivated by:

- The AdS/CFT correspondence in quantum gravity (proposed by Maldacena also Witten, around 1998)

- Geometric considerations to study the structure of non-compact asymptotically hyperbolic manifolds.

Outline of talk

1. Introduction and a brief survey.

2. Set-up of the compactness problem.

3. Compactness results for conformally compact Einstein manifolds of dimension 3+1.

4. Some existence Results.

5. Components of proofs.

§1. Conformally compact Einstein manifolds, Definition

- On a manifold X with boundary M, we call ρ a defining function on X, if $\rho > 0$ *on* X, $\rho = 0$ *on* M and $d\rho \neq 0$ on M.

- (X^{n+1}, g^+) is conformally compact if $(\bar{X}^{n+1}, \rho^2 g^+)$ is compact. Denote $h = \rho^2 g^+|_M$, we call $(M^n, [h])$ the conformal infinity of (X^{n+1}, g^+), where $[h]$ denotes the conformal class of metrics of h, i.e. the collection of metrics $\phi^2 h$ for some function ϕ on M.

- If $\mathrm{Ric}[g^+] = -n\ g^+$, we call (X^{n+1}, M^n, g^+) a conformally compact (Poincaré) Einstein (CCE) manifold.

- We remark on a CCE manifold, special r (called the geodesic defining function) can be chosen, with $|\nabla_{(r^2 g^+)} r| \equiv 1$ in an nbhd of $M \times (0, \epsilon)$ for some $\epsilon > 0$, so that $r^2 g^+$ is with totally geodesic boundary.

§1. Examples of CCE manifold

- Example 1.

On $(\mathbb{R}^{n+1}_+, \mathbb{R}^n, g_{\mathbb{H}})$, where $g_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}$, $x \in \mathbb{R}^n$, $\quad y > 0$. Choose $r = y$, then $(\mathbb{R}^{n+1}_+, dx^2 + dy^2)$ is not compact, but conformal to $g_{\mathbb{H}}$, with conformal infinity $(\mathbb{R}^n, [dx^2])$.

- Example 2.

On $(\mathbb{B}^{n+1}, \mathbb{S}^n, g_{\mathbb{H}})$, where $(\mathbb{B}^{n+1},\ g_{\mathbb{H}} = (\frac{2}{1-|y|^2})^2|dy|^2))$. Choose

$$r := 2\,\frac{1-|y|}{1+|y|},$$

$$g_{\mathbb{H}} = g^+ = r^{-2}\left(dr^2 + (1-\frac{r^2}{4})^2 g_c\right).$$

with $(\mathbb{S}^n, [g_c])$ as conformal infinity.
We remark that $r = e^{-2t}$, where $t(y) = dist_{g^+}(0, y)$.

§1. Examples of CCE manifold

- Example 3.

On $\mathbb{S}^1(\lambda) \times \mathbb{S}^2$ with the product metric, when $0 < \lambda < \frac{1}{\sqrt{3}}$, there are at least 3 different "conformal fill ins".

(a) One is when X is $(\mathbb{S}^1(\lambda) \times \mathbb{B}^3)$ with the fill in the hyperbolic metric $g^+ = f(y)dt^2 + g_{\mathbb{H}^3(y)}$.

(b) The other two: X is the AdS-Schwarzchild space $(\mathbb{R}^2 \times \mathbb{S}^2, g_m^+)$, where

$$g_m^+ = Vdt^2 + V^{-1}dr^2 + r^2 g_c,$$

$$V = 1 + r^2 - \frac{2m}{r}.$$

It turns out for $\lambda < \frac{1}{\sqrt{3}}$, there are two different choices of m. This is the famous "non-unique fill in" example of Hawking-Page '83.

§1. Some earlier existence results, Scattering theory on CCE manifolds

• "Ambient Metric" of Fefferman-Graham '85. On any compact manifold (M^n, h), h real analytic, there is a CCE metric on some $M^{n+1} \times (0, \epsilon)$ of M. Gursky-Székelyhidi '17, extend to smooth h .

• Graham-Lee '91: Any h in a small smooth neighborhood of h_c on $\mathbb{S}^n$. We remark that the fill in metrics constructed by Graham-Lee g^+ for h all exist in a small nbhd of the Hyperbolic metric, it turns out they are "unique" by a later result of C-Ge-Qing, '21.

• Gursky-Han '17 and Gursky-Han-Stolz '18 constructed many examples of boundary conformal classes that do not allow Poincaré-Einstein extensions on specified manifolds X^{4k} for $k \geqslant 2$.
Theorem (J, Lee '95). On CCE manifolds, if $R(h) > 0$, then $\lambda_1(-\Delta_{g^+}) \geqslant \frac{n^2}{4}$.
Corollary (J.Qing '03) On CCE manifolds, if $R(h) > 0$, then there exists a compactified metric g with $g|_M = h$ and $R(g) > 0$.

§1. Scattering theory on CCE manifolds

- Starting point of all

Theorem (Mazzeo-Melrose,' 87) On an AH manifolds (X^{n+1}, g^+), the essential spectrum of the $-\Delta_{g^+}$ includes $[\frac{n^2}{4}, \infty)$ and may be a finite points of point spectrum in $(0, \frac{n^2}{4})$.

Theorem (J. Lee '95). On a CCE manifold, if $R(h)$ is positive, then $\lambda_1(-\Delta_{g^+}) \geqslant \frac{n^2}{4}$.

In the proof of Lee, he studied solution of the Poisson equation:

$$(*) \quad -\Delta_{g^+} v + (n+1)v = 0 \;\; on \; X^{n+1}$$

with asymptotic behavior $v = r^{-1}(1 + f_2 r^2 + ...)$, where r denotes the geodesic defining function for h and when $R(h) > 0$, he used $v^{\frac{n}{2}}$ as a testing function to estimate $\lambda_1(-\Delta_{g^+})$.

- An observation of J. Qing is that in Lee's proof, $R(h) > 0$ implies the scalar curvature of metric $v^{-2}g^+$ is positive.

§2. Compactness of CCE manifolds – the set-up

- An open question: Does the entire class of metrics $(\mathbb{S}^3, h)$ with positive scalar curvature allow CCE fill in $\mathbb{B}^4$?
The class is path-connected by a result of F. Marques '12.
The index argument for non-existence of Gursky-Han, Gursky-Han-Stolz does not apply.

- We propose to study the "compactness" problem, and as an application some existence result for conformal fill in. More precisely, we ask the question:

Given a sequence of $(\mathbb{M}^n, [h_i])$ metrics with positive Yamabe constants, which are conformal infinity of CCE $(\mathbb{X}^{n+1}, g_i^+)$; when would

$$\{[h_i]\} \text{ forms a compact family on } \mathbb{M}^n$$

$$\Longrightarrow \{[g_i]\} \text{ forms a compact family on } X^{n+1}?$$

where g_i is some compactification of $\{g_i^+\}$ with $g_i|_M = h_i$.

§2. Compactness of CCE manifolds – an non-local inverse problem

The difficulty of the problem lies in the existence of an "non-local" term.

We will illustrate the case on (X^4, M^3, g^+) CCE manifold with (M^3, h) conformal infinity, recall the asymptotic behavior

$$g := r^2 g^+ = dr^2 + h + g^{(2)} r^2 + g^{(3)} r^3 + g^{(4)} r^4 + \cdots,$$

where $g^{(2)} = -\frac{1}{2}(Ric_h - \frac{1}{4} R_h h)$ determined by h (a local term), $Tr_h g^{(3)} = 0$, while

$$g^{(3)}_{\alpha,\beta} = -\frac{1}{3} \frac{\partial}{\partial n} (Ric_g)_{\alpha,\beta}$$

is a non-local term not determined by h.

We remark that h together with $g^{(3)}$ determines the asymptotic behavior of g. Fefferman-Graham '07, Biquard '08).

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§2. Conformal invariants

Yamabe constant

• On (M^n, h), compact closed manifold,
$Y(M, [h]) = \inf_{\tilde{h}\in[h]} \frac{\int_M R[\tilde{h}]dvol[\tilde{h}]}{Vol(M,\tilde{h})^{\frac{(n-2)}{n}}}$. We remark $Y(M, [h])$ corresponds to the "isoperimetric constant" of the Sobolev embedding of $W^{1,2}$ into $L^{\frac{2n}{n-2}}$.

• On compact manifold with boundary, there are two such constants. $(X^{n+1}, M^n, \bar{g})$

$$Y_a(X, M, [\bar{g}]) = \inf_{\tilde{g}\in[\bar{g}]} \frac{\int_X R[\tilde{g}]dvol[\tilde{g}] + c_n \int_M H[\tilde{g}|_M]d\sigma[\tilde{g}|_M]}{Vol(X, \tilde{g})^{\frac{(n-1)}{(n+1)}}}$$

$$Y_b(X, M, [\bar{g}]) = \inf_{\tilde{g}\in[\bar{g}]} \frac{\int_X R[\tilde{g}]dvol[\tilde{g}] + c_n \int_M H[\tilde{g}|_M]d\sigma[\tilde{g}|_M]}{Vol(M, \tilde{g}|_M)^{\frac{(n-1)}{n}}}.$$

Y_a and Y_b each corresponds to the (isoperimetric) constants in the Sobolev and Sobolev trace embeddings.

§2. Conformal invariants

• As we have mentioned before, it follows from result of J. Lee '95, and the observation by J. Qing, that on CCE setting, $Y(M,[h])>0$ implies that $Y_a(X,M,[g])\geqslant 0$.

• Combining works of Gursky-Han '17, X. Chen- M. Lai and F. Wang '18, Chang-Ge '21 we established that, there exists some constant c_n, such that

$$Y_a(X,M,[g])\geqslant C_nY(M,[h])^{\frac{n}{n+1}}.$$

Recall X. Chen-M. Lai and F. Wang

$$Y_b(X,M,[g])\geqslant C_nY(M,[h])^{\frac{1}{2}}$$

§2. Conformal invariants

- Another conformally invariant quantity is Weyl curvature W. $|W|[\tilde{g}] = \rho^{-2}|W|[g]$, if $\tilde{g} = \rho^2 g$. Thus $\int_X |W|^{\frac{n+1}{2}}[g]dv_g$ is a conformal invariant.
- On 4-manifold X, Bach tensor

$$B_{ij} = \nabla_l \nabla_k W_{kilj} + \frac{1}{2} Ric_{kl} W_{kilj}$$

is a conformally invariant. Bach flat metrics are the critical metric of the functional $g -> \int_X |W|^2[g]dv_g$. Einstein metrics are Bach flat, hence so are all metrics in the same conformal class of Einstein metric. Thus in a CCE setting (X, M, g^+), all compactified metrics of $[g^+]$ are Bach flat.
- We remark that it turns out we can re-write Bach flat condition as a 4th order system of PDE of elliptic type,

$$\Delta R_{ij} = c \nabla_i \nabla_j R + R_m * Ric,$$

which plays an important role in our estimates of the compactified metrics later. We also remark that for this PDE, the non-local tensor $-3g^{(3)} = \frac{\partial Ric}{\partial n}|_M$ is a natural matching boundary condition.

§2. Compactness of CCE manifolds – the set-up.

- For convenience, we choose $h = h^Y \in [h]$, the Yamabe metric on M. But what is a good choice of the compactified metric $g \in [g^+]$? A first attempt is to choose $g = g^Y$, a Yamabe metric among compactified metrics of g^+. The difficulty of this choice is we do not know how to control the behavior of $g^Y|_M$ in terms of h^Y.
- Instead, following the work of Lee, Graham-Zworski, '03 we will make a choice of a special representative metric , which we call scalar flat Adapted metrics on X obtained by solving the Poisson equation $(*)_s$ the boundary metric h with $R(h) > 0$ on M.

$$(*)_s \quad -\Delta_{g^+} v - s(n-s)v = 0, \;\; X^{n+1},$$

with Dirichlet data $f \equiv 1$. Choose $\rho = v^{\frac{1}{n-s}}$ and denote the adapted metric $g* = \rho^2 g^+$.
- Properties of $(*)_s$ has been studied in Fefferman-Graham '02, Chang-Gonzalez '11, Case-Chang '16, F. Wang '21-'22 and S. Lee '23 and others, Lee's metric is the adapted metric when $s = n+1$. In the statement of the theorems below, we choose $s = \frac{n}{2} + 1$, call it the scalar flat adapted metric.

§2. Properties of the adapted metric

On $(X, M, g+)$ CCE, for a given metric we have the adapted metric g^*, $g^*|_M = h$, with the key properties:
(1) $R[g*] = 0$ on X.
(2) $R[h] > 0$ on M implies the mean curvature $H > 0$ on M.
(3) Denote $g^* = \rho^2 g^+$, $|\nabla_{g*}\rho| \leqslant 1$.
(4) Gauss Bonnet formula

$$8\pi^2\chi = \int_X (\frac{1}{4}|W|^2 - \frac{1}{2}|E|^2) + \oint_M (\frac{4}{3}R[h]H - \frac{2}{27}H^3).$$

Hence Hence under the assumption $R[h] > 0$,

$$\int_X |E|^2 + \oint_M H^3 \leqslant C(\int_X |W|^2 + \oint_M (R[h])^3,)$$

where E denote the traceless Ricci.

§3. A compactness result on 4-manifold

Compactness Theorem (C and Yuxin Ge)

Let $\{X, M = \partial X, g_i^+\}$ be a family of 4-dimensional CCE manifolds. g_i is a sequence of adapted metrics. Denote $h_i = g_i|_M$. Assume

1. The boundary metric (M, h_i) is compact in $C^{k,\alpha}$ norm with $k \geqslant 6$; and there exists some positive constant $C_1 > 0$

$$Y(M, [h_i]) \geqslant C_1;$$

2. There exists some positive constant $C_2 > 0$ such that

$$\int |W[g_i]|^2 \leqslant C_2$$

3. $H_2(X, \mathbb{Z}) = 0$ and $H_1(X, \mathbb{Z}) = 0$.

Then, the sequence g_i is compact in $C^{k,\alpha'}$ norm for any $\alpha' \in (0, \alpha)$ up to a diffeomorphism fixing the boundary.

§4, An Existence Result

• Recall Graham-Lee '91: Any h in a small smooth neighborhood of h_c on $\mathbb{S}^3$ allows a CCE fill in, which are in a small nbhd of the Hyperbolic metirc on B^4, thus has the small L^2 norm of its Weyl tensor.
On the other hand, we also have the following result:

• When $n = 3$, on a CCE manifold (X^4, M^3, g^+) if $Y(M, [h]) > 0$, and

$$(*) \int_X |W|^2_{g^+} dv_{g^+} \leqslant cY_a^2$$

for some $c \leqslant \frac{1}{12^2}$, then any metric in some small nbhd of h allows a (unique) CCE fill in.

The natural question we then ask is can one impose conditions on the boundary metric h which will ensure $(*)$ to happen? As an application of our compactness result, we partially answer the question above.

§4. Statement of an existence result

Existence Theorem Let $(X = B^4, M = S^3)$ and $h \in C^{6,\alpha}$ be a metric with the positive scalar curvature on S^3. Given the positives constants $\bar{C}_4, \delta > 0$, such that

1. $\|h\|_{C^4} \leqslant \bar{C}_4$,
2. $Y(M, [h|_M]) \geqslant \delta$;
3. $vol(h) = 1$.

Then there exists some constant $C(\bar{C}_4, , \delta) > 0$ and some (small) positive constant ε so that denote $E(h)$ the traceless Ricci of h, if

$$||E(h)||_2 \leqslant \varepsilon$$

then for some dimension constant c_0, we can find a CCE fill in metric with the conformal infinity $[h]$ satisfying

$$c_0||W||_2 \leqslant \sqrt{\varepsilon} C(\bar{C}_4, \delta) \leqslant \frac{1}{4} Y_a.$$

Moreover, such solution with the above bound is unique.

§5. Some outline of proof of the existence theorem

The strategy of proof is as follows: Denote $g = g^*$, and $S = g^{(3)}$ the non-local term, under assumptions of the theorem.

- Step 1: Apply Bach flat equation to g , control $||W||_2$ by the norm of S and $\hat{E}$, where $\hat{E} = E(h)$. More precisely, We apply the Bach equation to g to obtain

$$(Y_a - c_0(||W||_2 + ||E||_2))(||W||_4^2 + ||E||_4^2) \leqslant C \oint_{S^3} S\hat{E}.$$

where c_0 and C are some dimension constant.

- Step 2: Under assumption (**) ($\frac{5}{18} Y_a - c_0(||W||_2 + ||E||_2) > 0$),

$$Y_b ||S||_3 \leqslant C(\bar{C}_4, \delta).$$

(This is the hard step, which we will supplement later.)

Combine step (1) and (2) we have if $||\hat{E}||_{\frac{3}{2}} \leqslant \epsilon$ then under $(**)$, we have

$$c_0(||W||_2 + ||E||_2) \leqslant \frac{1}{4} Y_a.$$

§5. Outline of proof of the existence theorem

• Step 3 We now run a continuity argument connecting h to h_c in S^3. Note for metrics close to h_c, the fill in metric always exists and $||W||_2$ tends to zero so $(**)$ condition is always satisfied. It turns out we can find such a path via the Ricci flow due to some recent work of E. Chen, G. Wei and R. Ye '24, here we quote a special case $n = 3$ of their work.

Theorem On (S^3, h), assume $R(h) > 0$, there exists a constant $\delta(3)$ sufficient small, so that

$$||\hat{E}(h))||_{\frac{3}{2}} + ||R(h) - \bar{R}(h)||_{\frac{3}{2}} \leqslant \delta(3),$$

where $\bar{R}(h)$ denotes the average of $R(h)$, then along the normalized Ricci flow the family $h(t)$ converges smoothly to h_c.

• We remark that under the assumption $||\hat{E}(h)||_2$ small and $vol(h) = 1$, the condition in the theorem above is satisfied by an earlier result of Y. Ge and G. Wang '14.

• Combining the three steps, along this path, under the assumptions of the Existence theorem, (**) is automatic and we reached the estimate in Step 2 and finished the proof of the

§6. More outline of proof of Step 2

- Step 2

Estimate of S-tensor: Recall $S = \frac{\partial}{\partial n_g} Ric_g$. To estimate S, we first recall a fact which was used in the work of S. Bando, A. Kasue, H. Nakajima [BKN]'89 to derive ALE decay of sequence of Einstein metrics. In the special case of 4-manifold, if g^+ is an Einstein metric, denote W^+ the Weyl tensor of g^+, then there is a Kato inequality

$$|\nabla_{g^+} W^+|^2 \geqslant \frac{5}{3} |\nabla_{g^+} |W^+||^2 \tag{1}$$

From these, one can derive

$$-\triangle_{g^+} |W^+|^{1/3} \leqslant c |W^+|^{4/3} + \frac{1}{6} R_{g^+} |W^+|^{1/3}$$

In work of [BKN], when scalar curvature $R_{g^+} = 0$, on a region $\int_A |W^+|^2_{g^+} dv_{g^+}$ is small, [BKN] derive the decay estimate

$$|W^+|^{1/3}(x) \lesssim \frac{1}{|x|^{2^-}} \text{ when } x \in A \text{ and } |x| \to \infty$$

Our Lemma is an application of (1) in conformal Einstein setting.

§5. More outline of proof of Step 2

Lemma 1 Let g^+ be CCE, $g = \rho^2 g^+$ be a compactification, define $U = U_g := \left(\frac{|W|_g}{\rho}\right)^{1/3}$, then

$$-\triangle_g U \leqslant c|W|_g U + \frac{1}{6} R_g U \tag{2}$$

Lemma 2 Denote $\tilde{r}(x) = \mathrm{dist}_g(x, M)$, $x \in X$, $g = g^*$, then $|W|_g^2 = e_2 \tilde{r}^2 + e_3 \tilde{r}^3 + O(\tilde{r}^4)$, where
$e_2 = 8|S|^2 + 4|\hat{C}|^2$, $\hat{C}$ is the Cotton tensor on M^3,
$e_3 = -4S_{\alpha\beta}(\hat{\nabla}_\gamma \hat{C}_{\alpha\beta\gamma} + \hat{\nabla}_\gamma \hat{C}_{\beta\alpha\gamma}) + 4H|S|^2 +$ some other lower order terms.

§5. More outline of proof of Step 2

Lemma 3

$$U_g^6 = \frac{|W|_g^2}{\rho_g^2} = \frac{|W|_g^2}{r^2}\frac{r^2}{\rho_g^2}$$

where

$$\rho_g = \tilde{r} - \frac{H}{18}\tilde{r}^2 + O(\tilde{r}^3) \tag{3}$$

$$U_g^6|_{\partial X} = e_2 \tag{4}$$

$$\frac{\partial U_g^6}{\partial r} = \frac{1}{9}He_2 + e_3 \tag{5}$$

§5. More outline of proof of Step 2

We then use the estimates

$$Y_a \left(\int_X U^{12} \right)^{1/2} \leqslant \int_X |\nabla U^3|^2 \tag{6}$$

$$Y_b \left(\oint_{\partial X} U^9 \right)^{1/3} \leqslant \int_X |\nabla U^3|^2 \tag{7}$$

while

$$\frac{5}{9} \int_X |\nabla U^3|^2 dv_g = - \int_X (\triangle_g U) U^5 + \frac{1}{6} \oint_{\partial X} \frac{\partial U^6}{\partial r} \tag{8}$$

$$\leqslant c \int_X |W|_g U^6 + \frac{1}{6} \oint_{\partial X} \frac{\partial U^6}{\partial r} \tag{9}$$

$$\leqslant c \left(\int_X |W|_g^2 \right)^{1/2} \left(\int_X U^{12} \right)^{1/2} + \frac{1}{6} \oint_{\partial X} \frac{\partial U^6}{\partial r} \tag{10}$$

§6. More outline of proof of Step 2

Combine (6) and (7) and estimate in (4) and (5) of U_g^6 and $\frac{\partial U_g^6}{\partial r}$ on ∂X, we get

$$\left(\frac{5}{18}Y_a - c\|W\|_2\right)\left(\int_X U^{12}\right)^{1/2} + Y_b\|S\|_3^2 \tag{11}$$

$$\lesssim \int_X \left|S\hat{\nabla}\hat{C}\right| + \|\hat{E}\|_{3/2}^2 + \|\hat{Ric}\|_2\|\hat{\nabla}\hat{C}\|_2 + \|\hat{Ric}\|_4^2\|\hat{\nabla}\hat{C}\|_4^2 \tag{12}$$

Thus under the assumption

$$(**) \quad \frac{5}{18}Y_a - c\|W\|_2 > 0 \tag{13}$$

we get

$$\|S\|_3 \leqslant C(\bar{C}_4, \delta)$$

where $\bar{C}_4$ is C^4 norm of h and $Y(M,[h]) \geqslant \delta > 0$, since $Y_b \gtrsim \sqrt{\delta}$.

Congratulations, Richard,
for your fantastic life long achievement!
May you have many more productive years to come!!