Worksheet 4: solution sets of matrix equations

1–4. For each of the matrices A below, decide (a) whether the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$; (b) whether the equation $A\vec{x} = 0$ has unique solution:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{1}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 6 & 2 \end{bmatrix}, \tag{2}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \tag{3}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{4}$$

Answers: 1. (a) False (b) False 2. (a) True (b) True 3. (a) False (b) True 4. (a) True (b) False

5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Describe the solution sets of the equations $A\vec{x} = 0$ and $A\vec{x} = \vec{b}$ in the parametric form. Find geometric interpretations for these sets.

Solution: The matrix $[A\ \vec{b}]$ is row reduced to

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}.$$

Therefore, the general solution of $A\vec{x} = \vec{b}$ has the form

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ c \in \mathbb{R},$$

and is a line (not passing through the origin). If we replace $\vec{b}$ by zero, the general solution has the form

$$c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ c \in \mathbb{R},$$

and is a line passing through the origin and parallel to the previous line.

6. Lay, 1.5.18.

Solution: For the equation $x_1 - 3x_2 + 5x_3 = 4$, the general solution is given by $x_1 = 4 + 3x_2 - 5x_3$ with x_2, x_3 free; therefore,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}.$$

This is a plane. The general solution of $x_1 - 3x_2 + 5x_3 = 0$ is given by the expression above without the vector $(4, 0, 0)$ and is a plane parallel to the previous plane.

7. Lay, 1.5.28.

Solution: No, it cannot. Indeed, if the solution set of $A\vec{x} = \vec{b}$ contained the origin, then we would have $A0 = \vec{b}$, which yields $\vec{b} = 0$, a contradiction.

8. Lay, 1.5.37.

Answer:

$$A = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The trick here is that the solution set of $A\vec{x} = \vec{b}$ is empty, while Theorem 6 assumes that the system $A\vec{x} = \vec{b}$ is consistent.

9. Lay, 1.5.38.

Solution: Since the equation $A\vec{x} = \vec{y}$ does not have a solution for some $\vec{y}$, the matrix A does not have a pivot in some row. Since A is a square matrix, this implies that it does not have a pivot in some column, producing a free variable. Therefore, whenever the system $A\vec{x} = \vec{z}$ is consistent, it has infinitely many solutions.