Worksheet 21: Orthogonality

1–3. Given the subspace V with an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ and the vector $\vec{u}$,

(a) Find the orthogonal projection $\vec{v} = \operatorname{proj}_V \vec{u}$ of the vector $\vec{u}$ onto the subspace V.

(b) Compute the vector $\vec{w} = \vec{u} - \vec{v}$ and verify that it is orthogonal to $\vec{v}_1$ and $\vec{v}_2$.

(c) Find the coordinates of $\vec{v}$ in the basis $\{\vec{v}_1, \vec{v}_2\}$ of V.

(d) Find the distance from $\vec{u}$ to V.

$$\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \ \vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}; \tag{1}$$

$$\vec{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \ \vec{u} = \begin{bmatrix}0\\2\\1\end{bmatrix}; \tag{2}$$

$$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}, \ \vec{u} = \begin{bmatrix}1\\2\end{bmatrix}. \tag{3}$$

Answers: 1. (a) $(3/2, 3/2, 3)$ (b) $(-1/2, 1/2, 0)$ (c) $(3/2, 3)$ (d) $\sqrt{2}/2$

2. (a) $(0, 2, 1)$ (b) $(0, 0, 0)$ (c) $(-1, 1)$ (d) 0

3. (a) $(1, 2)$ (b) $(0, 0)$ (c) $(3/2, -1/2)$ (d) 0

4. Fix $\vec{v} = (1, 1)$. Define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula $T(\vec{u}) = \operatorname{proj}_{\vec{v}} \vec{u}$; that is, $T(\vec{u})$ is the orthogonal projection of $\vec{u}$ onto $\vec{v}$. (See also Lay, 6.2.33.)

(a) Assuming that T is linear, write its standard matrix A.

(b) Verify that $A^2 = A$ and use this fact to deduce possible eigenvalues of A.

(c) Using either the computed value of A or geometric considerations, find the bases of eigenspaces of A.

Solution: (a) We have for $\vec{u} = (u_1, u_2)$,

$$T(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{u_1 + u_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (u_1 + u_2)/2 \\ (u_1 + u_2)/2 \end{bmatrix}.$$

We then find

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(b) A direct computation shows that $A^2 = A$. Then, if λ is an eigenvalue of A and $\vec{x}$ is the corresponding eigenvector, we have $A\vec{x} = \lambda\vec{x}$ and $A^2\vec{x} = \lambda^2\vec{x}$; since $A^2 = A$, $\lambda\vec{x} = \lambda^2\vec{x}$ and $\lambda = \lambda^2$. Therefore, the possible eigenvalues of A are 0 and 1.

(c) The matrix A has eigenvalues 0 and 1. The eigenspace for the eigenvalue 0 is spanned by $(-1, 1)$ and it is orthogonal to $\vec{v}$. The eigenspace for the eigenvalue 1 is spanned by $(1, 1) = \vec{v}$.

5. Lay, 6.2.31.

Solution: We have

$$\text{proj}_{c\vec{u}}\, \vec{y} = \frac{(c\vec{u}) \cdot \vec{y}}{\|c\vec{u}\|^2}(c\vec{u}) = \frac{c(\vec{u} \cdot \vec{y})}{c^2\|\vec{u}\|^2} c\vec{u} = \frac{\vec{u} \cdot \vec{y}}{\|\vec{u}\|^2}\vec{u} = \text{proj}_{\vec{u}}\, \vec{y}.$$

6. Fix $\vec{v} = (1, 1)$. Define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula $T(\vec{u}) = \vec{u} - 2\,\text{proj}_{\vec{v}}\, \vec{u}$. Explain why $T(\vec{u})$ is the result of reflecting the vector $\vec{u}$ across the line $\text{Span}\{\vec{v}\}^\perp$. Then, do parts (a)–(c) from the previous problem for this transformation T. (You should use the equation $A^2 = I$ instead of $A^2 = A$ in (b). (See also Lay, 6.2.34.))

Solution: (a) We have for $\vec{u} = (u_1, u_2)$,

$$T(\vec{u}) = \vec{u} - 2\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - 2\frac{u_1 + u_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -u_2 \\ -u_1 \end{bmatrix}.$$

T is the reflection across the line $\text{Span}\{\vec{v}\}^\perp = \text{Span}\{(-1, 1)\}$. Then,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(b) A direct calculation shows that $A^2 = I$. Then, arguing similarly to the previous problem, we get that each eigenvalue λ of A satisfies $\lambda^2 = 1$ and thus the possible eigenvalues are 1 and -1.

(c) The matrix A has eigenvalues 1 and -1. The eigenspace for the eigenvalue 1 is spanned by $(-1,1)$ and is orthogonal to the eigenspace for the eigenvalue -1; the latter is spanned by $(1,1)=\vec{v}$.

7. Prove that if U is an orthogonal matrix, then its determinant is equal to either 1 or -1. (Hint: we have done this problem before.) If $\det U = 1$, we call U **orientation preserving**; if $\det U = -1$, we call U **orientation reversing**.

Solution: We have $U^T U = I$; therefore,

$$1 = \det(U^T U) = \det U^T \cdot \det U = (\det U)^2.$$

Thus, $\det U = 1$ or $\det U = -1$.

8. For the standard matrix A of the transformation T from problem 6, show that A is orthogonal. Is it orientation preserving or orientation reversing?

Solution: We check that $A^T A = I$ and $\det A = -1$; therefore, A is orthogonal and orientation reversing.

9.* (Orthogonal orientation preserving matrices in $\mathbb{R}^2$) Assume that A is a 2×2 orthogonal matrix such that $\det A = 1$. Prove that A must have the form

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

for some ϕ. (Hint: assume that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; write the identity $A^{-1} = A^T$ and use the formula for A^{-1} from Section 2.2.)

Solution: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix};$$

then, since $\det A = 1$,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^T = A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix};$$

therefore, $d = a$ and $b = -c$; A has the form

$$A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}.$$

Now, $1 = \det A = a^2 + c^2$; therefore, we can write $a = \cos\phi$ and $c = \sin\phi$ for some angle ϕ. It follows that A is the standard matrix of counterclockwise rotation by ϕ.

10.* (Orthogonal orientation preserving matrices in $\mathbb{R}^3$) Assume that A is a 3×3 orthogonal matrix such that $\det A = 1$.

(a) Prove that $I - A = (A^T - I)A$.

(b) Use part (a) and properties of determinants to prove that 1 is an eigenvalue of A. (In fact, combining this with the previous problem, one can show that A must be the standard matrix of rotation about some axis in $\mathbb{R}^3$.)

(c) Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1, x_2, x_3) = (x_2, x_3, x_1)$. Write down the standard matrix A of T and verify that it is orthogonal and orientation preserving. Find an eigenvector of A corresponding to the eigenvalue 1. Prove that $A^3 = I$. If I were to tell you that T is actually a rotation about some axis, what would the axis and the angle be?

Solution: (a) We have $(A^T - I)A = A^TA - A = I - A$.

(b) We have

$$\begin{aligned}\det(A - I) &= -\det(I - A) = -\det(A^T - I)\det A \\ &= -\det((A - I)^T) = -\det(A - I).\end{aligned}$$

in the first equality, we used the properties of determinants under row operations and A being a 3×3 matrix. We get $\det(A - I) = 0$; thus, 1 is an eigenvalue of A.

(c) We have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix};$$

we compute $A^TA = I$ and $\det A = 1$. An eigenvector for eigenvalue 1 is $(1, 1, 1)$. We can check that $A^3 = I$ directly or we can see that $T^2(x_1, x_2, x_3) = (x_3, x_1, x_2)$ and $T^3(x_1, x_2, x_3) = (x_1, x_2, x_3)$. Therefore, T has to be a 120 degree rotation around the line spanned by $(1, 1, 1)$.