Worksheet 20: Inner product

1. Given $\vec{u} = (1,2,3)$ and $\vec{v} = (2,3,1)$, compute $\vec{u} \cdot \vec{v}$, $\|\vec{u}\|$, $\|\vec{v}\|$.
Answer: $\vec{u} \cdot \vec{v} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 11$, $\|\vec{u}\| = \|\vec{v}\| = \sqrt{14}$.

2. Given $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $\|\vec{u}\| = 1$, $\vec{u} \cdot \vec{v} = 2$, $\|\vec{v}\| = 3$, compute $\|\vec{u} + \vec{v}\|$ and $\|\vec{u} - \vec{v}\|$.
Solution: Using the properties of inner products, we compute

$$\|\vec{u} \pm \vec{v}\|^2 = \|\vec{u}\|^2 \pm 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2.$$

Using the known values of $\|\vec{u}\|, \|\vec{v}\|, \vec{u} \cdot \vec{v}$, we get

$$\|\vec{u} + \vec{v}\| = \sqrt{14}, \ \|\vec{u} - \vec{v}\| = \sqrt{6}.$$

3. Lay, 6.1.24.
Solution: We calculate

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2\vec{u} \cdot \vec{v}, \\ \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\vec{u} \cdot \vec{v}. \end{aligned}$$

Adding these two equalitites up, we get the parallelogram identity.

4. Find all unit vectors lying in $\text{Span}\{(3,4)\}$.
Solution: Every element of $\text{Span}\{(3,4)\}$ has the form $t(3,4) = (3t, 4t)$, where $t \in \mathbb{R}$. This is a unit vector if and only if $\|(3t, 4t)\| = 1$; in other words, if $(3t)^2 + (4t)^2 = 1$. This equation has two solutions, $t = \pm 1/5$. Therefore, $\text{Span}\{(3,4)\}$ has two unit vectors, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

5. Describe the set of all unit vectors in $\mathbb{R}^2$.
Solution: A vector in $\mathbb{R}^2$ has the form (x, y); this is a unit vector if and only if $x^2 + y^2 = 1$. Therefore, the set of all unit vectors in $\mathbb{R}^2$ is the circle of radius 1 centered at the origin.

6. Let $W \subset \mathbb{R}^3$ be the subspace spanned by the vectors $(1, 0, -1)$ and $(1, -1, 0)$. Using Theorem 6.1.3, find a basis for $W^\perp$.

Solution: We have $W = \operatorname{Col} A$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Then $W^\perp = \operatorname{Nul} A^T$, where

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

A basis for $\operatorname{Nul} A^T$ is given by $\{(1, 1, 1)\}$.

7. Find all values of t for which the vectors $(1, 2)$ and $(1, t)$ are orthogonal.

Solution: We have $(1, 2) \cdot (1, t) = 1 + 2t$; therefore, the vectors in question are orthogonal if and only if $t = -1/2$.

8. Let $\vec{w} \in \mathbb{R}^n$ be nonzero. Explain why $\{\vec{w}\}^\perp$ cannot equal to the whole $\mathbb{R}^n$. (Hint: find a specific vector in $\mathbb{R}^n$ which cannot lie in $W^\perp$.)

Solution: We argue by contradiction. Assume that $\{\vec{w}\}^\perp = \mathbb{R}^n$. Then in particular $\vec{w} \in \{\vec{w}\}^\perp$; so, $\vec{w} \cdot \vec{w} = 0$. However, this can only happen when $\vec{w} = \vec{0}$, a contradiction.

9. Given $\vec{u} = (1, 1)$, $\vec{v} = (-1, 1)$, $\vec{w} = (0, 1)$, prove that $\{\vec{u}, \vec{v}\}$ form an orthogonal set. Then, find the orthogonal projections $\vec{w}_u$ and $\vec{w}_v$ of $\vec{w}$ onto $\vec{u}$ and $\vec{v}$, respectively. Draw the vectors $\vec{u}, \vec{v}, \vec{w}, \vec{w}_u, \vec{w}_v$ and verify that $\vec{w} = \vec{w}_u + \vec{w}_v$ by the parallelogram rule. Find the coordinates of $\vec{w}$ in the basis $\mathcal{B} = \{\vec{u}, \vec{v}\}$ of $\mathbb{R}^2$.

Solution: We compute $\vec{u} \cdot \vec{v} = 0$ and

$$\vec{w}_u = \frac{\vec{w} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{w}_v = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}, \quad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

10. Given $\vec{u} = (1, 0)$, $\vec{v} = (-1, 1)$, $\vec{w} = (1, 2)$, find the orthogonal projections $\vec{w}_u, \vec{w}_v$ of $\vec{w}$ onto $\vec{u}$ and $\vec{v}$, respectively. Draw the vectors $\vec{u}, \vec{v}, \vec{w}, \vec{w}_u, \vec{w}_v$ and verify that $\vec{w} \neq \vec{w}_u + \vec{w}_v$.

Solution: We compute

$$\vec{w}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{w}_v = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}; \quad \vec{w}_u + \vec{w}_v = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \neq \vec{w}.$$

100.* Let A be an $m \times n$ matrix.

(a) Prove that there exists unique $n \times m$ matrix B with the following property: for each $\vec{u} \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^m$,

$$(A\vec{u}) \cdot \vec{v} = \vec{u} \cdot (B\vec{v}); \tag{1}$$

in fact, $B = A^T$. (Hint: for the uniqueness part, try substituting columns of the $n \times n$ and $m \times m$ identity matrices in place of $\vec{u}$ and $\vec{v}$.)

(b) Use part (a) to prove that $(\operatorname{Col} A)^\perp = \operatorname{Nul} A^T$.

(c) Use part (a) to prove that $(AC)^T = C^T A^T$.

Solution: (a) (The proof is quite technical, so it might be helpful to run it on some specific example to understand how it works.) First, assume that B is a matrix such that (1) holds for all $\vec{u}$ and $\vec{v}$. Let $\vec{e}_i$ be the i-th column of the $n \times n$ identity matrix and $\vec{f}_j$ be the j-th column of the $m \times m$ identity matrix. Take arbitrary i, j and put $\vec{u} = \vec{e}_i$, $\vec{v} = \vec{f}_j$; then

$$(A\vec{e}_i) \cdot \vec{f}_j = \vec{e}_i \cdot (B\vec{f}_j). \tag{2}$$

A direct calculation shows that the left-hand side is the element in the j-th row and i-th column of A, while the right-hand side is the element in the i-th row and j-th column of B. Since (2) holds for all i, j, we get $B = A^T$.

We have just proved uniqueness of the solution of (1); now, a direct calculation shows that $B = A^T$ solves (1).

(b) Let $\vec{v} \in \mathbb{R}^m$. Then

$$\begin{aligned} \vec{v} \in \operatorname{Nul} A^T &\text{ if and only if} \\ A^T\vec{v} = 0 &\text{ if and only if (see problem 8)} \\ \forall \vec{u} \in \mathbb{R}^n \colon \vec{u} \cdot (A^T\vec{v}) = 0 &\text{ if and only if} \\ \forall \vec{u} \in \mathbb{R}^n \colon (A\vec{u}) \cdot \vec{v} = 0 &\text{ if and only if} \\ \forall \vec{w} \in \operatorname{Col} A \colon \vec{w} \cdot \vec{v} = 0 &\text{ if and only if} \\ \vec{v} \in (\operatorname{Col} A)^\perp&. \end{aligned}$$

(c) By (a), the matrix $B = (AC)^T$ solves the equation

$$\forall \vec{u}, \vec{v} \colon (AC\vec{u}) \cdot \vec{v} = \vec{u} \cdot (B\vec{v}). \tag{3}$$

However, by (a) applied twice,

$$(AC\vec{u}) \cdot \vec{v} = (C\vec{u}) \cdot (A^T\vec{v}) = \vec{u} \cdot (C^T A^T \vec{v}).$$

Therefore, $C^T A^T$ also solves (3). By uniqueness in (a), $(AC)^T = C^T A^T$.