Worksheet 16: Eigenvalues and eigenvectors

All matrices are assumed to be square.

1. (a) Prove that -1 and 3 are eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

and find the bases for the corresponding eigenspaces. Find one eigenvector $\vec{v}_1$ with eigenvalue -1 and one eigenvector $\vec{v}_2$ with eigenvalue 3.

(b) Let the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = A\vec{x}$. Draw the vectors $\vec{v}_1, \vec{v}_2, T(\vec{v}_1), T(\vec{v}_2)$ on the same set of axes.

(c)* Without doing any computations, write the standard matrix of T in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ and itself. (So, you should apply T to the vectors in $\mathcal{B}$ and find the $\mathcal{B}$-coordinate vectors of the results.)

Solution: (a,b) We have

$$A - (-1)I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

The eigenspace associated to the eigenvalue -1 is $\mathrm{Nul}(A - (-1)I)$; a basis of this space is given by $\{(1, -1)\}$. We can put $\vec{v}_1 = (1, -1)$. Next,

$$A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

The eigenspace associated to the eigenvalue 3 is $\mathrm{Nul}(A - 3I)$; a basis of this space is given by $\{(1, 1)\}$. We can put $\vec{v}_2 = (1, 1)$.

(c) By the definition of an eigenvector,

$$T(\vec{v}_1) = A\vec{v}_1 = -\vec{v}_1,\ T(\vec{v}_2) = A\vec{v}_2 = 3\vec{v}_2.$$

Therefore, the coordinate vectors are

$$[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix};$$

the matrix of T in the basis $\mathcal{B}$ is diagonal:

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

2–4. Find the eigenvalues and the bases of the corresponding eigenspaces of the following matrices: (Use Theorem 5.1.1 to find the eigenvalues.)

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Answers: (2) Eigenvalue 1, eigenspace basis $\{(1,0)\}$ (3) Eigenvalue 1, eigenspace basis $\{(1,0)\}$; eigenvalue 2, eigenspace basis $\{(2,1)\}$ (4) Eigenvalue 1, eigenspace basis $\{(1,0,0),(0,1,0)\}$; eigenvalue 2, eigenspace basis $\{(0,0,1)\}$.

5. Lay, 5.1.25.

Solution: Since λ is an eigenvalue of A, there exists a vector $\vec{x} \neq 0$ such that $A\vec{x} = \lambda\vec{x}$. Multiplying both sides of this equation by A^{-1}, we get $\vec{x} = \lambda A^{-1}\vec{x}$. Since $\vec{x} \neq 0$, it follows that $\lambda \neq 0$; we divide both sides by λ to get $A^{-1}\vec{x} = \lambda^{-1}\vec{x}$; so, $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

6. Show that if $A^2 = I$, then the only possible eigenvalues of A are -1 and 1. (Hint: suppose that $\vec{x} \neq 0$ satisfies $A\vec{x} = \lambda\vec{x}$.)

Solution: Take $\vec{x} \neq 0$ such that $A\vec{x} = \lambda\vec{x}$; we need to prove that $\lambda = 1$ or $\lambda = -1$. We have

$$\vec{x} = A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda^2\vec{x};$$

since $\vec{x} \neq 0$, we get $\lambda^2 = 1$.

7. Lay, 5.1.32.

Solution: Any nonzero vector lying on the axis of rotation will be an eigenvector with eigenvalue 1. There are no other (real) eigenvectors.

8. Assume that P is invertible. Prove that $\vec{x}$ is an eigenvector of $P^{-1}AP$ if and only if $P\vec{x}$ is an eigenvector of A. Prove that the eigenvalues of A and $P^{-1}AP$ are the same.

Solution: $\vec{x}$ is an eigenvector of $P^{-1}AP$ with eigenvalue λ if and only if $P^{-1}AP\vec{x} = \lambda\vec{x}$. Multiplying both sides by P, we see that this is equivalent to $AP\vec{x} = \lambda P\vec{x}$; that is, to $P\vec{x}$ being an eigenvector of A with eigenvalue λ. (We have $P\vec{x} \neq 0$ since $\vec{x} \neq 0$ and P is invertible.) Now, λ is an eigenvalue of A if there is a corresponding eigenvector; the reasoning above shows that there exists an eigenvector of A with eigenvalue λ if and only if there exists an eigenvector of $P^{-1}AP$ with the same eigenvalue; thus, the eigenvalues of A and $P^{-1}AP$ are the same.