Worksheet 13: Bases and coordinates

1. Is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

a basis of $\mathbb{R}^3$? Explain.

Answer: No, as the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

is not invertible.

2. Represent the following set as $\operatorname{Col} A$ for some matrix A and find a basis for it:

$$V = \{(a-b, b-c, c-a) \mid a, b, c \in \mathbb{R}\}.$$

Solution: We have

$$\begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};$$

therefore, $V = \operatorname{Col} A$ with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Row reduction shows that A is row equivalent to

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns are the second and the third one; therefore, a basis for V is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

3. Represent the following set as $\operatorname{Nul} A$ for some matrix A and find a basis for it:

$$V = \{(a, b, c) \mid a + b + c = 0\}.$$

Solution: Writing $a+b+c = 0$ as a matrix equation, we get $V = \operatorname{Nul} A$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

We see that A is already in RREF; the general solution in parametric vector form is

$$b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; b, c \in \mathbb{R}.$$

Therefore, a basis for V is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. Using the definition of a basis, prove that the set $\{1, t, t^2\}$ forms a basis of $\mathbb{P}_2$ (the space of polynomials of degree no more than 2).

Solution: First, we prove that $\operatorname{Span}\{1, t, t^2\} = \mathbb{P}_2$. Indeed, every vector in $\operatorname{Span}\{1, t, t^2\}$ has the form $a + bt + ct^2$ for some $a, b, c \in \mathbb{R}$; this is exactly the form of a general element of $\mathbb{P}_2$.

Next, we prove that $\{1, t, t^2\}$ is linearly independent. Assume that $a + bt + ct^2 = 0$. Then the coefficients of this polynomial are zero, which implies $a = b = c = 0$.

5. Using the definition of a basis, prove that the set $\{t, t^2\}$ forms a basis of the space

$$V = \{f \in \mathbb{P}_2 \mid f(0) = 0\}.$$

Solution: Similar to the previous problem, using the fact that the general element of V has the form $at + bt^2$ for $a, b \in \mathbb{R}$.

6. Given

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, \ \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

find the coordinate vector $[\vec{v}]_{\mathcal{B}}$ of $\vec{v}$ in the basis $\mathcal{B}$.

Solution: We need to find $\vec{x} = (x_1, x_2)$ that solves the coordinate identity

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Solving this vector equation, we find $\vec{x} = (-5/3, 4/3)$.

7. Given the basis $\mathcal{B}$ from problem 6, find the vector $\vec{v}$ such that

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Solution: The coordinate identity gives

$$\vec{v} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

8. Find the coordinate vector of $(1+t)^2$ in the basis of $\mathbb{P}_2$ from problem 4.

Solution: We have $(1+t)^2 = 1 \cdot 1 + 2 \cdot t + 1 \cdot t^2$; therefore, the coordinate vector is $(1, 2, 1)$.

9. Find the coordinate vector of $t(1-t)$ in the basis $\{t, t^2\}$ of the space V from problem 5.

Solution: We have $t(1-t) = 1 \cdot t + (-1) \cdot t^2$; therefore, the coordinate vector is $(1, -1)$.

10. Use the coordinate vectors with respect to the basis from problem 4 to find whether the set

$$\{1, (t-1), (t-1)^2\}$$

is a basis of $\mathbb{P}_2$.

Solution: The coordinate vectors of $1, (t-1), (t-1)^2$ in the basis $\{1, t, t^2\}$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

It remains to verify that these coordinate vectors form a basis of $\mathbb{R}^3$.

11. Find the coordinate vector of $1-2t$ in the basis $\{1+2t, 2+t\}$ of $\mathbb{P}_1$. (You may use either the coordinate identity (1) on page 246 or coordinate vectors with respect to the basis $\{1,t\}$.)

First solution: We are looking for x_1 and x_2 such that

$$1-2t = x_1(1+2t) + x_2(2+t).$$

This is equivalent to

$$1-2t = (x_1+2x_2) + (2x_1+x_2)t.$$

Therefore, we need to solve the system

$$\begin{aligned} x_1 + 2x_2 &= 1, \\ 2x_1 + x_2 &= -2. \end{aligned}$$

The solution is $(x_1, x_2) = (-5/3, 4/3)$.

Second solution: We can replace all the vectors by their coordinate vectors in the basis $\{1,t\}$; then, we need to find the coordinates of the vector $(1,-2)$ in the basis $\{(1,2),(2,1)\}$. This was done in problem 6.

100.* (Caution: doing this problem is not going to directly help you with the exam, and might give a headache. Do it at your own risk.) Assume that we rewrote all definitions so that multiplying a scalar c by a vector $\vec{v}$ is only allowed when $c \in \mathbb{Q}$. Here $\mathbb{Q}$ is the set of all **rational** numbers (i.e., ratios of two integers). Prove that:

(a) The set $\mathbb{R}$ of all real numbers forms a vector space.

(b) The set

$$V = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

is a subspace of $\mathbb{R}$ in the new definition, but not in the old definition.

(c) The set $\{1, \sqrt{2}\}$ is a basis of V.