Worksheet 10: Determinants galore

1. Lay, 3.2.11. (Hint: first, use a row operation to make the element in the fourth row and second column equal to zero. Then, use the cofactor expansion down the second column.)

Solution:

$$\det \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix} \quad (R_4 \leftarrow R_4 - 2R_1)$$

$$= \det \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad \text{(cofactor expansion down column 2)}$$

$$= -5 \det \begin{bmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{bmatrix} \quad (R_2 \leftarrow R_2 + 2R_1)$$

$$= -5 \det \begin{bmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{(cofactor expansion down column 1)}$$

$$= -15 \det \begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix} = 120.$$

2. Assume that A is a 3×3 matrix with $\det A = 5$. Find the determinants of the matrices obtained by:

(a) swapping the second and the third row of A, and then multiplying the first row by 4;

(b) adding the second row to the first row, and then subtracting the second row from the first row;

(c) multiplying each row of A by 2.

Answers: (a) -20 (b) 5 (c) 40

3. Use row operations and cofactor expansions to prove that

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a).$$

(Hint: start by subtracting the first row from the other two rows.) When is this matrix invertible?

Solution: Subtract the first row from the second one and the third one and then do the cofactor expansion down the first column:

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$
$$= \det \begin{bmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{bmatrix} = (b-a)(c-a)\det \begin{bmatrix} 1 & b+a \\ 1 & c+a \end{bmatrix}$$
$$= (b-a)(c-a)(c-b).$$

The matrix is invertible if no two of the numbers a, b, c are equal.

4. Lay, 3.2.31.

Solution: We have $1 = \det I = \det(A \cdot A^{-1}) = \det A \cdot \det(A^{-1})$. Therefore, $\det(A^{-1}) = 1/\det A$.

5. Lay, 3.2.32.

Solution: To get rA from A, we need to multiply each of the n rows of A by r. Each time we multiply a single row by r, the determinant also gets multiplied by r; therefore, $\det(rA) = r^n \det A$.

6. Lay, 3.2.36.

Solution: We have $0 = \det(A^4) = (\det A)^4$. Since $\det A$ is just a number, this implies that $\det A = 0$ and therefore A is not invertible.

7. Define the transformation $T : \mathbb{R}^3 \to \mathbb{R}$ by

$$T(x_1, x_2, x_3) = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Find a formula for $T(x_1, x_2, x_3)$. (Hint: use a cofactor expansion.)

(b) Assuming that T is linear, use (a) to find its standard matrix.

(c)* Use the linearity property of the determinant to conclude that T is linear.

Solution: (a) The cofactor expansion along the first row gives

$$T(x_1, x_2, x_3) = -x_1 - x_2 + x_3.$$

(b) The standard matrix of T is

$$\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}.$$

(c) We have

$$\begin{aligned} T(c\vec{x} + d\vec{y}) &= T(cx_1 + dy_1, cx_2 + dy_2, cx_3 + dy_3) \\ &= \det \begin{bmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} cx_1 & cx_2 & cx_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \det \begin{bmatrix} cy_1 & cy_2 & cy_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ &= c \det \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + d \det \begin{bmatrix} y_1 & y_2 & y_3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = cT(\vec{x}) + dT(\vec{y}). \end{aligned}$$

8. Lay, 3.3.2. (Use Cramer's Rule!)

Solution: The augmented matrix is

$$\begin{bmatrix} 4 & 1 & 6 \\ 5 & 2 & 7 \end{bmatrix}.$$

Cramer's Rule then gives

$$x_1 = \frac{\det \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}}{\det \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}} = \frac{5}{3},$$

$$x_2 = \frac{\det \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}}{\det \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}} = -\frac{2}{3}.$$

9. Lay, 3.3.13. To save time, compute only the element in the second row and the second column of both the adjugate and the inverse.

Solution: See the solution guide.

10. Lay, 3.3.19.

Answer: 8. See the solution guide for details.

100.* Find a geometric explanation why for $\vec{u}, \vec{v} \in \mathbb{R}^2$, the area of the triangle with vertices $0, \vec{u}, \vec{v}$ is the same as that of the triangle with vertices $0, \vec{u}, \vec{v}+3\vec{u}$. Relate this to the way determinants change under row operations combined with the identity $\det A^T = \det A$. (Hint: represent the area of each triangle as the product of the side between 0 and $\vec{u}$ and the corresponding height.)

101.* Find a geometric explanation why for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$, the area of the triangle with vertices $0, \vec{u}, \vec{v} + \vec{w}$ is the sum or the difference of the areas of the triangles with vertices $0, \vec{u}, \vec{v}$ and $0, \vec{u}, \vec{w}$. (Hint: represent the area of each triangle as in the last problem. Think when you get the sum and when the difference.)