Math 54, midterm 2

Instructor: Semyon Dyatlov

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Name: Key SID: —

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- Write your solutions in the space provided. Do not use your own paper. I can give you extra paper if needed. Indicate clearly where your answer is.
- Explain your solutions as clearly as possible. This will help me find what you did right and what you did wrong, and award partial credit if possible.
- Justify all your steps. (Problem 4(a) is exempt from this rule.) A correct answer with no justification will be given 0 points. Pictures without explanations are not counted as justification. You may cite a theorem from the book by stating what it says.
- No calculators or notes are allowed on the exam, except for a single two-sided 5" $\times$9" sheet of hand-written notes. Cheating will result in academic and/or disciplinary action. Please turn off cellphones and other electronic devices.

1. Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ given by the formula

$$T(f) = \begin{bmatrix} f(-1) \\ f'(0) \\ f(1) \end{bmatrix}, \quad f \in \mathbb{P}_2.$$

(a) Find the matrix A of the transformation T relative to the basis $\mathcal{B} = \{1, t, t^2\}$ of $\mathbb{P}_2$ and the standard basis $\mathcal{C} = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$. $1' = 0$, $t' = 1$, $(t^2)' = 2t$

We have $T(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T(t) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; therefore,

$A = \left[[T(1)]_{\mathcal{C}} \quad [T(t)]_{\mathcal{C}} \quad [T(t^2)]_{\mathcal{C}} \right] =$ (since $\mathcal{C}$ is the standard basis; $[\vec{x}]_{\mathcal{C}} = \vec{x}$ for $\vec{x} \in \mathbb{R}^3$)

$= \left[T(1) \quad T(t) \quad T(t^2) \right] =$

$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(b) Find the rank of the matrix A from part (a) and the dimensions of the kernel of T and the range of T.

Row reduce: $A \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$; rank $A = 2$,

dim Ker $T =$ dim Nul $A = 1$,

dim Ran $T =$ dim Col $A = 2$,

since Ker T and Ran T get mapped to Nul A and Col A by the coordinate isomorphisms induced by the bases $\mathcal{B}$ and $\mathcal{C}$, respectively.

2. (a) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

and find a formula for A^k, where k is a nonnegative integer. Your final answer should be written as a single matrix (4 numbers between the square brackets) with the entries depending only on k.

A is upper triangular $\Rightarrow$ the eigenvalues are on the diagonal: 1, 3 (both w/ multiplicity 1)

$\underline{\lambda = 1} \to \text{Nul}(A - I) = \text{Nul}\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = \text{Nul}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;

basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\underline{\lambda = 3} \to \text{Nul}(A - 3I) = \text{Nul}\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$;

basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$\underline{\text{Diagonalization}}$: $A = PDP^{-1}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$,

$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Next, $A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} =$

$= \begin{bmatrix} 1 & 3^k \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}$.

(b) Is the following matrix diagonalizable? Explain. (Do not diagonalize.)

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 5 & 0 & 4 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & -\lambda & 0 \\ 5 & 0 & 4-\lambda \end{vmatrix} = \quad \text{(cofactor expand, } 2^{nd} \text{ row)}$$

$$= -\lambda \begin{vmatrix} 2-\lambda & 3 \\ 5 & 4-\lambda \end{vmatrix} = -\lambda(\lambda^2-6\lambda-7) = -\lambda(\lambda-7)(\lambda+1).$$

Our matrix is 3×3 and has 3 distinct real eigenvalues $0, -1, 7$; therefore, it is diagonalizable.

3. (a) Express the subspace

$$W = \{(a-b+c, b, a+b+c) \mid a, b, c \in \mathbb{R}\}$$

of $\mathbb{R}^3$ as either Col A or Nul A for some matrix A. Find a basis for W and a basis for $W^\perp$.

We have $W = \left\{ \begin{bmatrix} a-b+c \\ b \\ a+b+c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} =$

$= \left\{ a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} =$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} =$ Col A, with

$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Row reduce $A \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

pivot columns are 1^{st} & 2^{nd} $\to$

$\to$ Basis for $W = $ Col A: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Next, $W^\perp = (\text{Col } A)^\perp = \text{Nul } A^T$. Row reduce:

$A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; $\quad$ $x_1 = -x_3$, $x_2 = -2x_3$, x_3 free

Basis for $W^\perp = \text{Nul } A^T = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(b) Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

in $\mathbb{R}^3$. Prove that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal system and find the orthogonal projection of $\vec{v}$ onto $\text{Span}\{\vec{u}_1, \vec{u}_2\}$.

We calculate $\vec{u}_1 \cdot \vec{u}_2 = 1 \cdot 1 + 0 \cdot 2 + 1 \cdot (-1) = 0$;

So, $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal system.

Next, $\text{proj}_{\vec{u}_1, \vec{u}_2} \vec{v} = \dfrac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \dfrac{\vec{v} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 =$

$= \dfrac{4}{2} \vec{u}_1 + \dfrac{2}{6} \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 2/3 \\ 5/3 \end{bmatrix}.$

4. (a) Circle the correct answer for each of the following true/false questions. No justification is required. For each question, you get 1 point if you answer it correctly, 0 points if you choose not to answer, and −1 point if you answer it incorrectly. (If your total score for this part of the problem is negative, it will be changed to zero.)

(1) If the matrix A is invertible, then 0 is not an eigenvalue of A.

True   False

A invertible → no nonzero solution to $A\vec{x}=\vec{0}$ → no eigenvector w/ ~~eigenvalue~~ 0 → 0 is not an eigenvalue

(2) The product of two orthogonal matrices of the same size is an orthogonal matrix.

True   False

U, V orthogonal → $U^{-1}=U^T$, $V^{-1}=V^T$ → $(UV)^{-1} = V^{-1}U^{-1} = V^TU^T = (UV)^T$

(3) If a matrix A has characteristic polynomial $-\lambda^3-\lambda$, then it is diagonalizable.

True   False

$-\lambda^3-\lambda = -\lambda(\lambda^2+1)$. We have only 1 real eigenvalue, $\lambda=0$, w/ multiplicity 1; we need all eigenvalues to be real to diagonalize

(4) If A is row equivalent to B, then the eigenvalues of A and B are the same.

True   False

Counterexample: $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

ev. 1, 1   ev. 1, −1.

(5) If A is row equivalent to B, then $\operatorname{Nul} A = \operatorname{Nul} B$.

True   False

Nul A = the set of all solutions to $A\vec{x}=\vec{0}$. Row reduction does not change the solution set.

(b) Let A be a square matrix such that $A^3 = A^5$. What are the possible eigenvalues of A? Explain.

Assume that λ is an eigenvalue of A and $\vec{x}\neq\vec{0}$ is the corresponding eigenvector. Then $A\vec{x}=\lambda\vec{x}$ → $A^2\vec{x} = AA\vec{x} = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$, ... ~~and~~ similarly, $A^3\vec{x}=\lambda^3\vec{x}$, $A^5\vec{x}=\lambda^5\vec{x}$. So, since $A^3=A^5$, $\lambda^3\vec{x}=\lambda^5\vec{x}$. Since $\vec{x}\neq\vec{0}$, $\lambda^3=\lambda^5$.

$\lambda^3(\lambda^2-1)=0$ → the only possible

Therefore, eigenvalues of A are $0, 1, -1$.

5. Solve **one** of the following two problems. Mark which one you want graded.

(a) Assume that A is a 2×2 orthogonal matrix and $\det A = -1$. Prove that $A^2 = I$. Your proof has to be purely algebraic, starting with the definition of an orthogonal matrix and not appealing directly to the geometric properties of orthogonality that we studied in class. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is orthogonal,

$A^{-1} = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Also, $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} =$

$= \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$ since $\det A = -1$. Therefore,

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \Rightarrow$ ~~a =~~ $d = -a$, $b = c \Rightarrow$

$\Rightarrow A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Now, we see that $A = A^T \Rightarrow$

$\Rightarrow A = A^{-1} \Rightarrow A^2 = I$.

(b) Assume that U, V, W are vector spaces and $T : V \to W$, $S : U \to V$ are linear transformations such that $T \circ S = 0$. (In other words, $T(S(\vec{u})) = \vec{0}$ for each $\vec{u} \in U$.) Prove that the kernel of T contains the range of S.

Assume that ~~∀~~ $\vec{v} \in \operatorname{Ran} S$. Then, by definition of the range, $\exists \vec{u} \in U$: $\vec{v} = S(\vec{u})$. Since $T \circ S = 0$, $T(S(\vec{u})) = \vec{0}$. But then $\vec{0} = T(S(\vec{u})) = T(\vec{v})$; By definition of the kernel, $\vec{v} \in \operatorname{Ker} T$.

We showed that every element of $\operatorname{Ran} S$ lies in $\operatorname{Ker} T$. ~~It~~ Therefore, $\operatorname{Ran} S \subset \operatorname{Ker} T$.