Math 54, midterm information and review

July 5, 2010

1 General information

The first midterm will take place on Friday, July 9, from 8–10 AM in room 2 Evans. The exam itself will start at 8:10, but I ask you to come at 8 so that I could hand out the exams and everybody would start at the same time. There are no calculators and no materials allowed, except for one two-sided 5"×9" sheet of hand-written notes. Do not bring your own paper — I will provide extra sheets if needed.

The midterm will cover everything we studied, up to and including determinants (Lay, 3.3). At least 60% of the points will be computational problems; the rest will include multiple-choice, true/false, and proof-based problems. The computational problems will have tasks similar to the sample problems listed below (although they can have matrices of bigger size or be a mixture and/or reformulation of the tasks below). If you have any questions about how to do computational problems and/or are not sure whether you are doing them correctly, do not hesitate to ask me.

2 Sample computational problems

The numbers in these problems were picked arbitrarily and the actual calculations can take a long time. The point of the exercises is not to make you calculate, but rather to make you think about what each problem asks and what the method of solution is. You will have to provide reasoning for each problem on the exam (e.g. 'the given system is inconsistent because there is a pivot in the last column of the augmented matrix').

1. You are given the matrix A and the vector $\vec{b}$. Instead of A, you might be

given vectors $\vec{a}_1, \ldots, \vec{a}_p$. Here are some examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix};$$
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix};$$
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}.$$

Here are some questions that you might be asked on the exam:

(a) Write the matrix equation $A\vec{x} = \vec{b}$ as a system of linear equations (and identify its coefficient matrix and augmented matrix) and as a vector equation. (Or, given a system of linear equations or a vector equation, write it in the matrix form.)

(b) Find a row echelon form and the reduced row echelon form of A or the augmented matrix of the equation $A\vec{x} = \vec{b}$. Identify the pivot positions, pivot rows, and pivot columns.

(c) Find the general solution of the equation $A\vec{x} = \vec{b}$. State if there are 0, 1, or infinitely many solutions. Decide if a system of linear equations is consistent. (I might ask you to use Cramer's Rule to solve the system. In this case, A and $\vec{b}$ might depend on some variable λ. Also, I might ask you to solve the equation using the inverse matrix.)

(d) Write the solution set of the equation $A\vec{x} = \vec{b}$ in parametric vector form.

(e) Decide whether $\vec{b} \in \text{Span}\{\vec{a}_1, \ldots, \vec{a}_p\}$.

(f) Write $\vec{b}$ as a linear combination of $\vec{a}_1, \ldots, \vec{a}_p$.

(g) Find whether $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_p\}$ is equal to the whole space $\mathbb{R}^m$ (where the vectors $\vec{a}_1, \ldots, \vec{a}_p$ lie).

(h) Find whether $\vec{a}_1, \ldots, \vec{a}_p$ are linearly independent.

(i) Find a linear dependence relation between the vectors $\vec{a}_1, \ldots, \vec{a}_p$. Express one of these vectors as a linear combination of the other vectors.

(j) Find whether the equation $A\vec{x} = \vec{y}$ has a solution for each $\vec{y}$.

(k) Find whether the equation $A\vec{x} = \vec{y}$ has a unique solution for each $\vec{y}$ for which it is consistent.

(l) Find whether A is an invertible matrix.

(m) Find the inverse of A. (I might ask you to use the formula for 2×2 matrices, the $[A|I] \to [I|A^{-1}]$ method, or the formula from 3.3 involving determinants.)

(n) For the transformation $T : \mathbb{R}^a \to \mathbb{R}^b$ defined by $T(\vec{x}) = A \cdot \vec{x}$, find the values of a and b such that the transformation makes sense. Write the general formula for T (as in $T(x_1, \ldots, x_n) = \ldots$). Find $T(\vec{x})$ for a given $\vec{x}$. Find all $\vec{x}$ such that $T(\vec{x}) = \vec{b}$. Decide whether $\vec{b}$ lies in the range of T.

(o) Find the determinant of A.

(p) If the entries of A depend on some variable λ, use the determinant to find all values of λ for which A is invertible. (I can replace invertibility by any equivalent property listed in the Invertible Matrix Theorem.)

2. Given matrices A and B (one of which may be a vector), compute their sum, their product, or their transposes. If some operation is undefined, explain why. Decide if A and B commute.

3. Given a transformation $T : \mathbb{R}^a \to \mathbb{R}^b$ either described by a formula (e.g. $T(x_1, x_2) = (2x_1, x_2 - x_1)$) or described geometrically (e.g. T rotates points around the origin $\pi/3$ radians clockwise), and assuming that T is linear,

(a) find the standard matrix of T;

(b) Decide if T is 1-to-1 and if it is onto.

(c) Decide if T is invertible and find its inverse transformation.

For some computational problems above, I have provided geometric interpretation in class. This is important because geometric intuition could help you understand why the facts we postulated are true; it could reduce the time you need to learn the theorems and help you with proof based problems. However, the preferred method of solving computational problems on the exam is algebraic. For example, if I give you two vectors and ask whether they span $\mathbb{R}^2$, you should write a certain matrix, row reduce it, and state that it has a pivot in each row. You could draw a picture to illustrate your point, but if you just draw me the two vectors without stating that they are multiples of each other, I will not accept this as a proof of them spanning $\mathbb{R}^2$.

3 Proof-based and multiple choice problems

The multiple choice (or true/false) problems will give a theoretical question and several possible answers. You will be required to pick the correct answer, based on the theory we studied in the course. There could also be proof based problems, which will require you to prove a certain abstract fact. It is impossible to list all questions one can ask here; however, lots of examples of these can be found in past Math 54 exams. (See the exam archive linked from the course website, and also Prof. Lott's midterm 1, which has some true/false questions and a proof based question. We also had some of these things in our homeworks and worksheets.)