Math 54

Second midterm

Spring 2010

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This is a closed everything exam. Please put away all books, calculators and other portable electronic devices.

You need to justify every one of your answers. Correct answers without appropriate supporting work will be treated with great skepticism (except for problem number 3). At the conclusion, hand in your exam to your GSI.

Write your name on this exam and on any additional sheets that you hand in. If you need additional paper, get it from me.

Problem	Score
1	
2	
3	
4	
5	
Total	

Your name ______________________________

Your GSI ______________________________

Discussion time ______________________________

Your SID ______________________________

1. Suppose that

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

(a) (5 pts) Find the eigenvalues of A. List them with the smallest one first.

The characteristic polynomial is $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (-\lambda)(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$. The eigenvalues are 1 and 2.

(b) (5 pts) Find corresponding eigenvectors.

For eigenvalue $1:\ A - I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$. An eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For eigenvalue $2:\ A - 2I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$. An eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) (10 pts) Find the limit as $n \to \infty$ of the sequence of matrices $A^n/2^n$.

We can write $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Then $A^n/2^n = P(D^n/2^n)P^{-1}$. Since $D^n/2^n = \begin{bmatrix} 1/2^n & 0 \\ 0 & 1 \end{bmatrix}$, as $n \to \infty$, $P(D^n/2^n)P^{-1}$ approaches

$$P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

2. Below are two examples of matrices A. In each case, find whether A is diagonalizable. If A is diagonalizable, find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$. If A is not diagonalizable, explain clearly how you know that it isn't.

(a) (10 pts)

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

The characteristic polynomial is $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$. The eigenvalues are -2 and 5.

For eigenvalue -2 : $A + 2I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$. An eigenvector is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

For eigenvalue 5 : $A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$. An eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We can take $P = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$.

(b) (10 pts)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic polynomial is $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. The only eigenvalue is 1.

For eigenvalue 1 : $A - I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. The null space is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So $\mathbb{R}^2$ does not have a basis consisting of eigenvectors of A. This means that A is not diagonalizable.

3. True or false? No justification necessary. For each question, if you answer correctly then you get two points. If you answer incorrectly then you lose two points. If you do not answer then you do not gain or lose any points. (If your total score is negative then it will be rounded up to zero.)

$\underline{\text{T}}$ F If a square matrix has orthonormal columns then it also has orthonormal rows.

T $\underline{\text{F}}$ If W is a subspace of $\mathbb{R}^n$ then W and $W^{\perp}$ have no vectors in common.

$\underline{\text{T}}$ F If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors then $\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$.

T $\underline{\text{F}}$ An orthogonal matrix is any square matrix with orthogonal columns.

T $\underline{\text{F}}$ Any triangular matrix is diagonalizable.

$\underline{\text{T}}$ F If A is an $n \times n$ matrix so that $A^2 = A$ then its only possible eigenvalues are zero and one.

T $\underline{\text{F}}$ Every invertible matrix is diagonalizable.

$\underline{\text{T}}$ F If W is a subspace of $\mathbb{R}^n$ then for any vector $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{y}$ is in $W^{\perp}$ if and only if $\operatorname{proj}_W \mathbf{y} = 0$.

$\underline{\text{T}}$ F If A is an $n \times n$ diagonalizable matrix then each vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A.

T $\underline{\text{F}}$ If A and B are diagonalizable $n \times n$ matrices then so is $A + B$.

T $\underline{\text{F}}$ A basis of P_n must contain a polynomial of degree j for each integer j between 0 and n.

T $\underline{\text{F}}$ The nonzero rows of a matrix A form a basis for Row A.

$\underline{\text{T}}$ F Row operations on a matrix cannot change the null space.

4. (a) (10 pts) Suppose that a square matrix A satisfies $(A - I)^2 = 0$. Find an explicit formula for A^{-1}. Your answer should be in terms of A, and not in terms of eigenvalues or eigenvectors.

$A^2 - 2A + I = 0$, so $A(2I - A) = I$. The inverse is $2I - A$.

(b) (10 pts) Find a 3×3 matrix with real entries whose eigenvalues are 1, $2 + 3i$ and $2 - 3i$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix}.$$

5. Recall that P_3 is the vector space of polynomials p of degree at most 3 in a variable t. Define a linear transformation $T : P_3 \to P_3$ by saying that for a polynomial p in P_3, $T(p)$ is the polynomial $\widehat{p}$ given by $\widehat{p}(t) = tp'(t)$.

a. (5 pts) What is the kernel (or null space) of T? Justify your answer. (Hint : do not use matrices.)

Suppose that $T(p)$ is the zero polynomial. Then $\widehat{p} = 0$, so for all t, $tp'(t) = 0$. In particular, for all $t \neq 0$, $p'(t) = 0$. Because p' is a polynomial, it is continuous and so $p'(t) = 0$ for all t. This means that p must be a constant polynomial. So the kernel is the set of constant polynomials.

b. (5 pts) Is T onto? (Equivalently, is the range of T all of P_3?) Justify your answer. (Hint : do not use matrices.)

Suppose that $T(p)$ equals the constant polynomial 1. Then $tp'(t) = 1$. However, no matter what the polynomial $p'(t)$ may be, when we multiply it by t then there is no constant term. This is a contradiction. So 1 is not in the range of T, and T is not onto.

c. (5 pts) Let W be the subspace of P_3 consisting of polynomials p so that $p(1) = 0$. Find a basis of W. (Hint : do not use matrices.)

A basis of P_3 is $\{1, t-1, (t-1)^2, (t-1)^3\}$. A polynomial $p = c_0 + c_1(t-1) + c_2(t-1)^2 + c_3(t-1)^3$ lies in W if and only if $c_0 = 0$. So a basis of W is $\{t-1, (t-1)^2, (t-1)^3\}$.