Math 54, final exam

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- Write your solutions in the space provided. Do not use your own paper. I can give you extra paper if needed. Indicate clearly where your answer is.

- Explain your solutions as clearly as possible. This will help me find what you did right and what you did wrong, and award partial credit if possible.

- Justify all your steps. (Problem 5 is exempt from this rule.) A correct answer with no justification will be given 0 points. Pictures without explanations are not counted as justification. You may cite a theorem from the book by stating what it says.

- No calculators or notes are allowed on the exam, except for a single two-sided A4/Letter sized sheet of hand-written notes. Cheating will result in academic and/or disciplinary action. Please turn off cellphones and other electronic devices.

1. Find the limit as $t \to +\infty$ of the formal solution to the following initial/boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t), \ 0 < x < \pi, \ t > 0;$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \ t > 0;$$

$$u(x, 0) = \begin{cases} 1, & 0 < x < \pi/2, \\ 0, & \pi/2 \le x < \pi. \end{cases}$$

(You can take the $t \to \infty$ limit inside series without providing justification.)

Let $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ 0, & \pi/2 \le x < \pi \end{cases}$; decompose it into Fourier series: cosine

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx).$$ The solution to the heat equation

is $$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-tk^2} \cos(kx).$$

As $t \to +\infty$, each term $a_k e^{-tk^2} \cos(kx)$ converges

to 0 ; so, $\lim_{t \to +\infty} u(x,t) = \frac{a_0}{2}.$

(Note: we do not need to calculate a_k for $k > 0$!)

Now, $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \cdot \int_0^{\pi/2} 1 \cdot dx = 1$; thus,

$$\boxed{\lim_{t \to +\infty} u(x,t) = \frac{1}{2}.}$$

2. Compute the Fourier sine series for the function

$$f(x) = \pi - x, \ 0 < x < \pi.$$

Your answer should be in the form $f(x) \sim \sum_{...}(\ldots)\sin((\ldots)x)$. Describe the function to which this series converges, and sketch its graph.

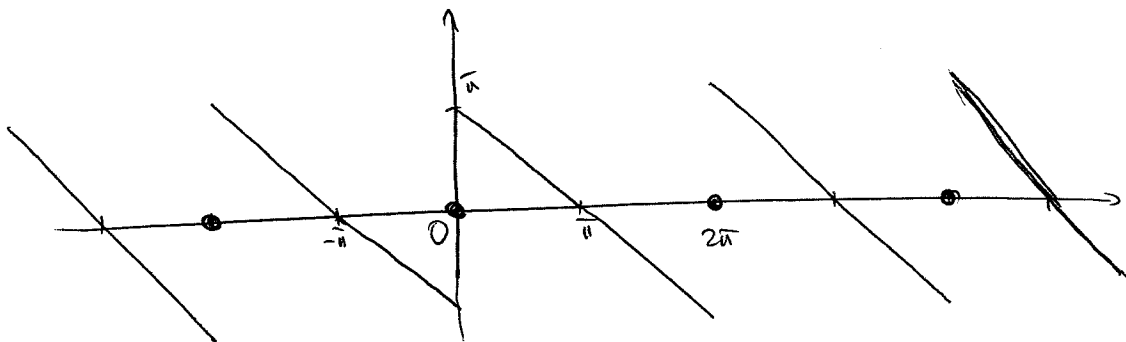$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx), \text{ where}$$

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin(kx)\, dx =$$

$$= -\frac{2}{k\pi} \int_0^{\pi} (\pi - x)\, d(\cos(kx)) = -\frac{2}{k\pi} (\pi - x)\cos(kx)\Big|_{x=0}^{\pi} \, - \,$$

$$- \frac{2}{k\pi} \int_0^{\pi} \cos(kx)\, dx = \frac{2}{k} - \frac{2}{k^2\pi} \sin(kx)\Big|_{x=0}^{\pi} = \frac{2}{k} =)$$

$$=) \quad \boxed{f(x) \sim 2 \sum_{k=1}^{\infty} \frac{\sin(kx)}{k}}$$

The Fourier series converges to the 2π – periodic extension of the function

$$\tilde{f}(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ -\pi - x, & -\pi < x < 0 \\ 0, & x = 0, \pm\pi \end{cases}$$

3. Let V be the vector space of all solutions to the differential equation

$$y'' + 4y = 0.$$

(a) Find a basis for V. (You do not need to prove that what you found is actually a basis for V, as long as you find it correctly.)

(b) Let $T : V \to V$ be the linear transformation given by the formula $T(y) = y'$, $y \in V$. Find the matrix of T in the basis of V from part (a).

(a) Basis for V = fundamental system of solutions

for the equation $y'' + 4y = 0$.

Auxiliary equation: $r^2 + 4 = 0 \to$ roots $r = \pm 2i$.

Basis for V: $\boxed{\{\cos 2x, \sin 2x\}} = \mathcal{B}$

(b) We have $[T]_\mathcal{B} = [\; T(\cos 2x)_\mathcal{B} \quad T(\sin 2x)_\mathcal{B} \;] =$

$= \left[[-2\sin 2x]_\mathcal{B} \quad [2\cos 2x]_\mathcal{B} \right] = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$

4. Find all least squares solutions to the equation

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{b}}.$$

Find the least squares error.

Normal equation: $A^T A \hat{x} = A^T \vec{b}$;

$$A^T A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad A^T \vec{b} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \boxed{\hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \hat{x}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{x}_1 \text{ free}}$$

least squares error: $A\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (does not depend on $\hat{x}_1$).

The error is $\| A\hat{x} - \vec{b} \| = 2.$

5. Given the characteristic polynomial $P(\lambda)$ of some unknown square matrix A, decide:

(a) whether A is certainly diagonalizable, certainly not diagonalizable, or can be diagonalizable or not diagonalizable depending on the dimensions of eigenspaces;

(b) whether A is invertible or not invertible.

No justification is required. For each of the questions (a), you get 2 points if you answer it correctly, -1 point if you answer incorrectly, and 0 points if you decide not to answer. For each of the questions (b), you get 1 point if you answer it correctly, -1 point if you answer it incorrectly, and 0 points if you decide not to answer. If your total score for this problem is negative, it will be replaced by zero.

eigenvalues $\lambda = 0$ (mult. 1)
$\lambda = -1$ (mult. 2)

(1) $P(\lambda) = -\lambda(\lambda^2 + 2\lambda + 1) = -\lambda(\lambda+1)^2$;

(a)   Diagonalizable   Not diagonalizable   [Can be either]

(b)   Invertible   [Not invertible] , since 0 is an eigenvalue

(2) $P(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3)$; eigenvalues $\lambda = -1$ (mult. 1 each)
$\lambda = -3$

(a)   [Diagonalizable]   Not diagonalizable   Can be either

(b)   [Invertible]   Not invertible , since 0 is not an eigenvalue

6. Find the basic solutions to the following boundary value problem for the **damped wave equation** obtained using the method of separation of variables (in other words, solutions that have the form $X(x)T(t)$). You can use the information we have already gathered about the solutions of the corresponding boundary/eigenvalue problem on $X(x)$.

$$\frac{\partial^2 u}{\partial t^2}(x,t) + 2\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \ 0 < x < \pi, \ t > 0; \qquad (\ast)$$

$$u(0,t) = u(\pi, t) = 0, \ t > 0.$$

$$u(x,t) = X(x)\, T(t) \longrightarrow$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t), \quad \frac{\partial u}{\partial x} = X(x)T'(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$u(0,t) = X(0)T(t), \quad u(\pi,t) = X(\pi)T(t).$$

Equation $(\ast) \longrightarrow X \cdot T'' + 2X \cdot T' = X'' \cdot T.$

Divide by $X \cdot T:$ $\quad \dfrac{T'' + T'}{T}(t) = \dfrac{X''}{X}(x).$

So, $\quad \dfrac{T''+2T'}{T}(t) = \dfrac{X''}{X}(x) = -\lambda = \text{const.}$

$$\frac{X''}{X} = -\lambda \longrightarrow \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \text{due to boundary conditions}$$

Solutions to this eigenvalue problem: $\lambda = k^2, \ k \in \mathbb{Z}, \ k > 0;$

$X(x) = \sin(kx).$

$$\frac{T''+2T'}{T} = -\lambda \longrightarrow T'' + 2T' + \lambda T = 0.$$

Put $\lambda = k^2, k \in \mathbb{Z}, k > 0 \longrightarrow T'' + 2T' + k^2 T = 0.$ Auxiliary equation:

$r^2 + 2r + k^2 = 0 \longrightarrow r = -1 \pm \sqrt{1-k^2}.$ Cases:

$k = 1 \longrightarrow r = -1, \text{ mult. 2} \longrightarrow T = c_1 e^{-t} + c_2 t e^{-t} \longrightarrow$

Basic solutions
$e^{-t}\sin x, \ te^{-t}\sin x$

$k > 1 \longrightarrow 2$ complex roots $-1 \pm i\sqrt{k^2-1}$

$T = c_1 e^{-t}\cos(t\sqrt{k^2-1}) + c_2 e^{-t}\sin(t\sqrt{k^2-1}) \longrightarrow$

Basic solutions
$e^{-t}\cos(t\sqrt{k^2-1})\sin x,$
$e^{-t}\sin(t\sqrt{k^2-1})\sin x$

7. Consider the system of linear equations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where a, b, c, d are given integers and $\underbrace{ad - bc}_{"A"} = 1$. Prove that this system has a unique solution (x_1, x_2) and x_1, x_2 are both integers.

$\det A = ad - bc = 1 \neq 0 \rightarrow A$ is invertible $\rightarrow$

$\rightarrow$ the only solution is $\quad A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$

$= \begin{bmatrix} d - 2b \\ 2a - c \end{bmatrix} \rightarrow x_1 = d - 2b$ and $x_2 = 2a - c$ are integers, as a, b, c, d are integers

8. Using Cauchy–Schwarz inequality, prove that for every continuous function f on the interval $[0, 1]$,

$$\left(\int_0^1 x f(x)\, dx \right)^2 \le \frac{1}{3} \int_0^1 f(x)^2\, dx$$

Make sure to specify what functional space and what inner product you are using. (You do not have to verify that the properties of vector space and inner product hold.)

Consider the space $C[0,1]$ of continuous functions $[0,1] \to \mathbb{R}$, with the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx \; ; \; f, g \in V.$$

Now, let $g(x) = x$, $g \in V$, be defined by the formula $g(x) = x$, $0 \le x \le 1$; apply Cauchy–Schwarz to f and g:

$$\langle f, g \rangle \le \|f\| \cdot \|g\|$$

$$\langle f, g \rangle^2 \le \|f\|^2 \cdot \|g\|^2. \qquad (*)$$

Now,
$$\langle f, g \rangle^2 = \left(\int_0^1 x f(x)\, dx \right)^2,$$

$$\|f\|^2 = \int_0^1 f(x)^2\, dx \;;$$

$$\|g\|^2 = \int_0^1 x^2\, dx = \frac{1}{3}.$$

It remains to substitute these into $(*)$.