Math 54, Section 214 Your name: Key

Quiz 11, April 30, 2010

Please write your name on each sheet. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (10 pt) Find the general solution to the differential equation $\vec{x}'(t) = A\vec{x}(t)$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find the solution satisfying the initial condition

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - 4\lambda + 3 = 0 \rightarrow$$

$$\rightarrow \lambda = 1, 3.$$

Eigenvectors:

$$\lambda = 1 \rightarrow \text{Nul} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 3 \rightarrow \text{Nul} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution: $c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If $\vec{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $c_1 = 0$, $c_2 = 2 \rightarrow$

$\rightarrow$ the solution is $2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. (10 pt) Find all $\lambda > 0$ for which the boundary problem

$$y'' + \lambda y = 0, \quad -\pi < x < \pi; \ y'(-\pi) = 0, \ y'(\pi) = 0$$

has nontrivial solutions, and determine these solutions. (Suggestion: first write the general solution, depending on two coefficients a and b. Then represent the boundary conditions as a system of two linear equations on a and b and find when the corresponding matrix is not invertible. Explain why this works.)

General solution: $y = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$.

$y' = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$

$0 = y'(\pi) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)$

$0 = y'(-\pi) = a\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi)$

$A\begin{bmatrix} a \\ b \end{bmatrix} = 0$, where $A = \begin{bmatrix} -\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) & \sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) \\ \sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) & \sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) \end{bmatrix}$.

λ is an eigenvalue $\Leftrightarrow$ Ker A is nontrivial $\Leftrightarrow$

$\Leftrightarrow 0 = \det A = -2\lambda\cos(\sqrt{\lambda}\,\pi)\sin(\sqrt{\lambda}\,\pi) \rightarrow$

$\rightarrow$ either $\cos(\sqrt{\lambda}\,\pi) = 0$, $\sqrt{\lambda} = j + \frac{1}{2}$, $j \in \mathbb{Z}$;

$\lambda = (j+\frac{1}{2})^2$, $A = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \rightarrow$

$\rightarrow a = 0, b = 1$,

$y = \sin(\sqrt{\lambda}\,x)$

or $\sin(\sqrt{\lambda}\,\pi) = 0$, $\sqrt{\lambda} = j$, $j \in \mathbb{Z}$; $\lambda = j^2$, $A = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$;

$a = 1$, $b = 0$, $y = \cos(\sqrt{\lambda}\,x)$.